2017

A new kind of companion matrix

Eunice Y. S. Chan  
*Western University, echan295@uwo.ca*

Robert Corless  
*University of Western Ontario, rcorless@uwo.ca*

Follow this and additional works at: [http://repository.uwyo.edu/ela](http://repository.uwyo.edu/ela)

**Recommended Citation**  
DOI: [https://doi.org/10.13001/1081-3810.3400](https://doi.org/10.13001/1081-3810.3400)

This Article is brought to you for free and open access by Wyoming Scholars Repository. It has been accepted for inclusion in Electronic Journal of Linear Algebra by an authorized editor of Wyoming Scholars Repository. For more information, please contact scholcom@uwyo.edu.
A NEW KIND OF COMPANION MATRIX*

EUNICE Y.S. CHAN† AND ROBERT M. CORLESS†

Abstract. A new kind of companion matrix is introduced, for polynomials of the form \( c(\lambda) = \lambda a(\lambda) b(\lambda) + c_0 \), where upper Hessenberg companions are known for the polynomials \( a(\lambda) \) and \( b(\lambda) \). This construction can generate companion matrices with smaller entries than the Fiedler or Frobenius forms. This generalizes Piers Lawrence’s Mandelbrot companion matrix. The construction was motivated by use of Narayana-Mandelbrot polynomials, which are also new to this paper.

Key words. Companion matrix, Eigenvalue, Narayana’s cows sequence, Mandelbrot polynomials, Mandelbrot matrices.

AMS subject classifications. 15A18, 15A23, 65F15, 65F50.

1. Introduction. Recently, we generalized the Mandelbrot polynomials

\[ p_{n+1} = z p_n^2 + 1, \quad p_0 = 0 \]

to the Fibonacci-Mandelbrot polynomials

\[ q_{n+1} = z q_n q_{n-1} + 1, \quad q_0 = 0, q_1 = 1 \]

and generalized Piers Lawrence’s supersparse\(^1\) companion matrix for \( p_n \) \(^8\) to an analogous one for \( q_n \). See \(^4\), \(^5\) and \(^7\) for details, though we summarize these constructions below.

If \( p_n = \det (zI - M_n) \) for the Mandelbrot polynomials, then the subdiagonals of \( M_n \) are all \(-1\) and the matrices are the same size, which gives

\[
M_{n+1} = \begin{bmatrix}
M_n & -c_n r_n \\
-r_n & 0 \\
-c_n & M_n
\end{bmatrix},
\]

where \( r_n = [ 0 \ 0 \cdots 1 ] \) and \( c_n = [ 1 \ 0 \cdots 0 ]^T \) are both of length \( d_n \). This is Piers Lawrence’s original construction \(^8\). These are remarkable matrices: They contain only \(-1\) or 0, and therefore are Bohemian matrices\(^2\); yet the characteristic polynomial contains coefficients that grow exponentially in the degree \( d_n \) (doubly exponentially in \( n \)).

For the Fibonacci-Mandelbrot polynomials, the degree of \( q_n = F_n - 1 \) and the construction contains matrices of different size. We begin with

\[
M_3 = \begin{bmatrix}
-1
\end{bmatrix}
\]

---

*Received by the editors on September 13, 2016. Accepted for publication on August 8, 2017. Handling Editor: Dario Bini. This work was funded by NSERC and The University of Western Ontario, aka Western University.

†Ontario Research Centre for Computer Algebra, Department of Applied Mathematics, Western University, London, ON (echan295@uwo.ca, rcorless@uwo.ca).

\(^1\)A matrix is supersparse if it is sparse and its nonzero elements are drawn from a small set, e.g. \([-1, 1]\).

\(^2\)The name “Bohemian” is an acronym for Bounded height matrix of integers. See example OEIS A272658.
and
\[ \mathbf{M}_4 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \]
to construct our recursive companion matrix:
\[ \mathbf{M}_{n+1} = \begin{bmatrix} \mathbf{M}_n & (-1)^{d_n+1} \mathbf{c}_n \mathbf{r}_{n-1} \\ -\mathbf{r}_n & 0 \\ -\mathbf{c}_{n-1} & \mathbf{M}_{n-1} \end{bmatrix}, \]
where \( \mathbf{r}_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \) and \( \mathbf{c}_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T \) are, as before, the row and column vectors of length \( d_n \). This gives a matrix of slightly greater height than (1.1) because the entries may be \( \{-1, 0, 1\} \).

The surprising analogy between these two families of supersparse companions led us to conjecture and prove the following.

2. Main result.

**Theorem 2.1.** Suppose \( a(z) = \det(z \mathbf{I} - \mathbf{A}) \), \( b(z) = \det(z \mathbf{I} - \mathbf{B}) \), and both \( \mathbf{A} \) and \( \mathbf{B} \) are upper Hessenberg matrices with nonzero subdiagonal entries, and
\[ \alpha = \frac{1}{\prod_{j=1}^{d_a-1} a_{j+1,j} \prod_{j=1}^{d_b-1} b_{j+1,j}} \]
is the reciprocal of the product of the subdiagonal entries of \( \mathbf{A} \) and \( \mathbf{B} \), and \( d_a = \deg_z a \) and \( d_b = \deg_z b \), so the dimension of \( \mathbf{A} \) is \( d_a \times d_a \) and the dimension of \( \mathbf{B} \) is \( d_b \times d_b \). Suppose both \( d_a \) and \( d_b \) are at least 1. Then if
\[ \mathbf{C} = \begin{bmatrix} \mathbf{A} & -\alpha \mathbf{c}_0 \mathbf{c}_a \mathbf{r}_b \\ -\mathbf{r}_a & 0 \\ -\mathbf{c}_b & \mathbf{B} \end{bmatrix}, \]
where \( \mathbf{r}_a = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \) of length \( d_a \) and \( \mathbf{c}_b = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T \) of length \( d_b \), we have
\[ c(z) = \det(z \mathbf{I} - \mathbf{C}) = z \cdot a(z)b(z) + c_0. \]

**Remark 2.2.** Proving this theorem automatically proves the validity of the constructions of the supersparse companion matrices for \( p_n, q_n, \) and \( r_n \).

**Remark 2.3.** Starting with a polynomial \( c(z) \), we see that there are potentially many such \( a(z) \) and \( b(z) \). This freedom may be quite valuable or, it may be an obstacle.

**Proof.** Partition
\[ z \mathbf{I} - \mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}, \]
where \( \mathbf{C}_{22} = z \mathbf{I} - \mathbf{B} \) is nonsingular if \( z \) is not an eigenvalue of \( \mathbf{B} \), i.e., \( b(z) \neq 0 \). Later we will remove this...
A New Kind of Companion Matrix

restriction. Also,

$$C_{21} = \begin{bmatrix} 1 \end{bmatrix}$$

is $d_b \times (d_a + 1)$ and has only one nonzero element, which is a $1$ in the upper right corner. Next,

$$C_{12} = \begin{bmatrix} \alpha c_0 \end{bmatrix}$$

is $(1 + d_a) \times d_b$ and again has only one nonzero element, $\alpha c_0$ in the upper right corner. (In fact, $c_0$ can be zero.) This leaves

$$C_{11} = \begin{bmatrix} zI - A & 0 \\ \vdots & \vdots \\ 0 & 0 \\ -I & z \end{bmatrix},$$

which is $d_a + 1$ by $d_a + 1$.

The Schur factoring is

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} I & C_{12} \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} C_{11} - C_{12}C_{22}^{-1}C_{21} & 0 \\ C_{22}^{-1}C_{21} & I \end{bmatrix},$$

with the computation of the Schur complement $C_{11} - C_{12}C_{22}^{-1}C_{21}$ going to do most of the work in the proof. The Schur determinantal formula [10, Chapter 12] is then

$$\det C = \det(C_{22}) \det(C_{11} - C_{12}C_{22}^{-1}C_{21}).$$

We have the following propositions.

1. $zI - A$ and $zI - B$ are upper Hessenberg because $A$ and $B$ are.
2. The first $d_a$ columns of $C_{22}^{-1}C_{21}$ are zero.
3. The final column of $C_{22}^{-1}C_{21}$ is the solution, say $\vec{v}$, of $(zI - B) \vec{v} = e_1$. Again, $zI - B$ is nonsingular.
4. By Cramer’s rule, the final entry in $\vec{v}$, say $v$, is

$$v = \frac{\det \left( C_{22} \leftrightarrow e_1 \right)}{\det(C_{22})},$$

where the notation $M \leftarrow_k \vec{v}$ means replace the $k$th column of $M$ with the vector $\vec{v}$ [3].
5. Since $C_{22} = zI - B$ is upper Hessenberg,

$$C_{22} \leftarrow_{d_a} e_1 = \begin{bmatrix}
* & * & * & \cdots & * & 1 \\
-b_{21} & * & * & \cdots & * & 0 \\
-b_{32} & * & \vdots & \ddots & \vdots & \\
-b_{43} & \ddots & \ddots & \ddots & \ddots & \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & * & 0 & \\
& & & & & -b_{d_a,d_a-1} \\
\end{bmatrix}. $$

Laplace expansion about the final column gives

$$\det \left( C_{22} \leftarrow_{d_a} e_1 \right) = (-1)^{d_a-1} (-1)^{d_a-1} \prod_{j=1}^{d_a-1} b_{j+1,j}$$

$$= \prod_{j=1}^{d_a-1} b_{j+1,j}.$$

Therefore,

$$v = \prod_{j=1}^{d_a-1} b_{j+1,j} / b(z)$$

because $\det C_{22} = \det (zI - B) = b(z)$ by hypothesis.

6. Now

$$C_{12}C_{22}^{-1}C_{21} = \begin{bmatrix}
\alpha c_0 \\
\vdots \\
* \\
v
\end{bmatrix}
\begin{bmatrix}
* \\
\vdots \\
* \\
v
\end{bmatrix}
= \begin{bmatrix}
\alpha c_0 v \\
\vdots \\
0 \\
v
\end{bmatrix}$$

is $d_a + 1$ by $d_a + 1$ and has its only nonzero entry, $\alpha c_0 v$, in the upper right corner.

7. The Schur complement is therefore

$$\begin{bmatrix}
* & -\alpha c_0 v \\
\vdots & 0 \\
\vdots & \vdots \\
0 & 0 \\
\end{bmatrix},$$

$$\begin{bmatrix}
zI - A \\
\vdots \\
\vdots \\
0 & 1 \\
\end{bmatrix}.$$
A New Kind of Companion Matrix

and we compute \( \det \left( C_{11} - C_{12}C_{22}^{-1}C_{21} \right) \) by Laplace expansion on the last column:

\[
\det \left( C_{11} - C_{12}C_{22}^{-1}C_{21} \right) = -(-1)^{d_{a} \alpha c_{0} v} \det \left[ \begin{array}{cccc}
-a_{21} & * & * & * \\
-a_{32} & * & * & \\
& -a_{43} & & \\
& & & \ddots \\
& & & -a_{d_{a},d_{a}-1}
\end{array} \right]
+ z \det (zI - A)
\]

\[
= -(-1)^{d_{a} \alpha c_{0} v} \prod_{j=1}^{d_{a}-1} (-a_{j+1,j}) + z \cdot a(z)
\]

\[
= \alpha v \prod_{j=1}^{d_{a}-1} a_{j+1,j} \cdot c_{0} + z \cdot a(z)
\]

\[
= \alpha \cdot \left( \prod_{j=1}^{d_{a}-1} b_{j+1,j} \right) \cdot \left( \prod_{j=1}^{d_{a}-1} a_{j+1,j} \right) \cdot c_{0} + z \cdot a(z)
\]

\[
= \frac{c_{0}}{b(z)} + z \cdot a(z)
\]

by the definition of \( \alpha \).

Therefore, by the Schur determinantal formula,

\[
\det (zI - C) = \det (C_{22}) \det \left( C_{11} - C_{12}C_{22}^{-1}C_{21} \right)
\]

\[
= b(z) \left( \frac{c_{0}}{b(z)} + z \cdot a(z) \right)
\]

\[
= z \cdot a(z)b(z) + c_{0}.
\]

Since the left hand side is a polynomial as is the right hand side, the formula will be true even if \( b(z) = 0 \), by continuity.

\[ \square \]

3. Applications and examples. Sequence A000930 of the Online Encyclopedia of Integer Sequences, Narayana’s cows sequence, begins

\[ 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, \ldots \]

and is generated by \( R_{n} = R_{n-1} + R_{n-3} \) [13]. The connection to cows is that an ideal cow produces a calf every year, starting in its fourth year. Narayana was a mathematician in 14th century India. Various facts are known for this sequence, which is similar to the Fibonacci sequence: For instance, the generating function is \( 1/(1 - x - x^{3}) \). Many references are given in the OEIS, but see also [12].

We define the Narayana-Mandelbrot polynomials by \( r_{0} = 1, r_{1} = r_{2} = 1 \) and

\[
r_{n+1} = 2r_{n}r_{n-2} + 1
\]
for $n \geq 2$. We construct a recursive family of companion matrices $R_n$, i.e., such that

$$r_n(z) = \det(zI - R_n).$$

Just as the Fibonacci-Mandelbrot polynomials, the construction contains matrices of different sizes. However, for this family, we start with

$$R_3 = \begin{bmatrix} -1 \end{bmatrix},$$

$$R_4 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},$$

and

$$R_5 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}.$$

Our construction is then

$$R_{n+1} = \begin{bmatrix} R_n & (-1)^{d_n+1}c_n r_{n-2} \\ -r_n & 0 \\ -c_{n-2} & R_{n-2} \end{bmatrix},$$

where $r_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$ and $c_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$ are, as before, the row and column vectors of length $d_n = \deg r_n = R_{n+1} - 1$.

This construction also allows new matrix families. For instance, suppose $s_0 = 0$, $s_{n+1} = z^3 s_n^4 + 1$. Then if $S_n$ is an upper Hessenberg companion for $s_n$ (with all $-1$ on the subdiagonal) the matrix

$$S_{n+1} = \begin{bmatrix} S_n & -c_n r_n \\ -r_n & 0 \\ -c_n & S_n \\ -r_n & 0 \\ -c_n & S_n \end{bmatrix}$$

is an upper Hessenberg companion for $s_{n+1}$.

4. Concluding remarks. This is a genuinely new kind of companion matrix. We demonstrate this on Newton’s example polynomial $x^3 - 2x - 5$. We see that $x^3 - 2x - 5 = x(x^2 - 2) - 5 = x(x - \sqrt{2})(x + \sqrt{2}) - 5$, and companion matrices for $x - \sqrt{2}$ and $x + \sqrt{2}$ are just $[+\sqrt{2}]$ and $[-\sqrt{2}]$ respectively. Thus, a companion matrix for Newton’s polynomial is

$$\begin{bmatrix} \sqrt{2} & 5 \\ -1 & -\sqrt{2} \end{bmatrix}.$$
This matrix contains $\sqrt{5}$, unlike any previously recorded companion matrix. For unimodular polynomials, such companion matrices may be of lower height than the Frobenius or Fiedler [9] companions, and may offer better numerical condition.

We have now established that if $c(z) = z \cdot a(z)b(z) + c_0$ and $A$ and $B$ are upper Hessenberg companion matrices for the polynomials $a(z)$ and $b(z)$ respectively, then

$$C = \begin{bmatrix} A & -\alpha c_0 c_a r_b \\ -r_a & 0 \\ -c_b & B \end{bmatrix}$$

is a companion matrix for $c(z)$. One wonders immediately about a corresponding linearization, $L_C$, strong or otherwise, for the matrix polynomial

$$C(z) = zA(z)B(z) + C_0,$$

if $L_A$ is a linearization for $A$, $L_B$ for $B$. Some very preliminary experiments, where $L_A$ and $L_B$ were block
upper Hessenberg with all blocks I, so \( \alpha = 1 \), find that indeed

\[
L_C = \begin{bmatrix}
L_A & 0 & -C_0 \\
-I & 0 & -I \\
-L_B & -I & 0
\end{bmatrix}
\]

is a (strong) linearization for \( c(z) \), in the examples we tried.

In a paper to be submitted soon, we have now proved that this construction can be extended to matrix polynomials; see [6].

A referee pointed out that Robol et al. [11] use a similar construction to linearize polynomials of the form \( p(z) = a(z)b(z) + zc(z)d(z) \) to find the roots of rational functions, which can also be applied to matrix polynomials.

We leave these extensions to future work.

Acknowledgments. We thank Neil J.A. Sloane for introducing us to the Narayana’s cows sequence, Dario Bini [1, 2] for teaching us about Mandelbrot polynomials and the Schur complement, and Donald E. Knuth. We also thank the referees for their remarks.

REFERENCES