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ON THE INVERSE OF A CLASS OF WEIGHTED GRAPHS*

S.K. PANDA† AND S. PATI‡

Abstract. In this article, only connected bipartite graphs $G$ with a unique perfect matching $\mathcal{M}$ are considered. Let $G_w$ denote the weighted graph obtained from $G$ by giving weights to its edges using the positive weight function $w : E(G) \to (0, \infty)$ such that $w(e) = 1$ for each $e \in \mathcal{M}$. An unweighted graph $G$ may be viewed as a weighted graph with the weight function $w \equiv 1$ (all ones vector). A weighted graph $G_w$ is nonsingular if its adjacency matrix $A(G_w)$ is nonsingular. The inverse of a nonsingular weighted graph $G_w$ is the unique weighted graph whose adjacency matrix is similar to the inverse of the adjacency matrix $A(G_w)$ via a diagonal matrix whose diagonal entries are either 1 or $-1$. In [S.K. Panda and S. Pati. On some graphs with a unique perfect matching such that $G$ is invertible. That class is denoted by $\mathcal{H}_{nmc}$. It is natural to ask whether $G_w$ is invertible for each invertible graph $G \in \mathcal{H}_{nmc}$ and for each weight function $w \neq 1$. In this article, first an example is given to show that there is an invertible graph $G \in \mathcal{H}_{nmc}$ and a weight function $w \neq 1$ such that $G_w$ is not invertible. Then the weight functions $w$ for each graph $G \in \mathcal{H}_{nmc}$ such that $G_w$ is invertible, are characterized.

Key words. Adjacency matrix, Inverse graph, Weighted graph, Unique perfect matching.

AMS subject classifications. 05C50, 15A09.

1. Introduction. Let $G$ be a simple, undirected graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of $G$, respectively. We use $[i, j]$ to denote an edge between the vertices $i$ and $j$. By $G_w$ we denote the weighted graph obtained from $G$ by assigning weights to its edges using the weight function $w : E(G) \to (0, \infty)$. The unweighted graph $G$ may be viewed as a weighted graph, where each edge has weight 1. Let $G_w$ be a weighted graph on vertices $1, \ldots, n$. The adjacency matrix $A(G_w)$ of $G_w$ is the square symmetric matrix of size $n$ whose $(i, j)$th entry $a_{ij}$ is given by

$$a_{ij} = \begin{cases} \ w([i, j]), & \text{if } [i, j] \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

A perfect matching in a graph $G$ is a spanning forest whose components are paths on two vertices. Note that $G$ can have more than one perfect matching. If $G$ has a unique perfect matching, then we denote it by $\mathcal{M}$. Furthermore, when $v$ is a vertex, we shall always use $v'$ to denote the matching mate for $v$, that is, $v'$ is the vertex for which the edge $[v, v'] \in \mathcal{M}$. Let $G$ be a graph with a unique perfect matching $\mathcal{M}$. An edge $e \in \mathcal{M}$ is called a matching edge, while an edge (of $G$) $e \notin \mathcal{M}$ is called a nonmatching edge. Let $\mathcal{H}$ be the class of connected bipartite graphs $G$ with a unique perfect matching $\mathcal{M}$. A weighted graph $G_w$ is nonsingular if $A(G_w)$ is nonsingular. A weighted bipartite graph with a unique perfect matching is nonsingular.

The inverse of a graph was first introduced by Godsil [3]. The weighted version was supplied in [6].

DEFINITION 1.1. [6] Let $G_w$ be a nonsingular weighted graph. Suppose that there is a signature matrix $S$ (a diagonal matrix with diagonal entries 1 or $-1$) such that $SA(G_w)^{-1}S$ is nonnegative. Consider the weighted graph $H$ such that $A(H) = SA(G_w)^{-1}S$. Then $H$ is called the inverse graph of $G_w$, and it is denoted by $G_w^\dagger$.

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Let $H = \{G \in \mathcal{H} \mid G/M$ is bipartite$\}$, where $G/M$ is the graph obtained from $G$ by contracting each matching edge to a vertex. In [3], Godsil showed that if $G \in H$, then $G^+$ exists. He posed the problem of characterizing the graphs in $\mathcal{H}$ which possess inverses. In [1], Akbari and Kirkland characterized the unicyclic graphs $G \in \mathcal{H}$ which possess inverses. In [8], Tifenbach and Kirkland supplied necessary and sufficient conditions for graphs in $\mathcal{H}$ to possess inverses, utilizing constructions derived from the graph itself. In [5], Panda and Pati characterized a class of bipartite graphs $G$ with a unique perfect matching such that $G$ is invertible. This class contains the class $H$ and the unicyclic graphs. In [6], Panda and Pati extended the notion of an inverse graph to positively weighted graphs. They showed that for each $G \in H$, the inverse graph $G_w^+$ exists for each weight function $w$ such that $w(e) = 1$ for each $e \in M$.

Graphs $G$ and $H$ are isomorphic ($G \cong H$) if one can be obtained by relabeling the vertices of the other. An invertible graph $G$ is said to be a self-inverse graph if $G$ is isomorphic to its inverse graph. Characterizing self-inverse graphs in $\mathcal{H}$ is also a challenging problem. This question, for the class $H$, was asked by Godsil in 1985 and has already been answered by Simion and Cao in [7]. In [8], Tifenbach and Kirkland supplied necessary and sufficient conditions for a unicyclic graph $G \in \mathcal{H}$ to be self-inverse. In [9], Tifenbach supplied a necessary and sufficient condition for a graph $G \in \mathcal{H}$ to satisfy $G \cong G^+$ via a particular isomorphism. In [6], the authors have proved many different characteristics of the inverse graphs of the graphs $G \in H$.

To proceed further we need the following known definitions.

**Definition 1.2.** [5] Consider a graph $G$ with a unique perfect matching $\mathcal{M}$. A path $P = [u_1, u_2, \ldots, u_{2k}]$ is called an alternating path if the edges on $P$ are alternately matching and nonmatching edges, that is, for each $i$, if $[u_i, u_{i+1}]$ is a matching (resp., nonmatching) edge and $[u_{i+1}, u_{i+2}] \in E(G)$, then $[u_{i+1}, u_{i+2}]$ is a nonmatching (resp., matching) edge. Let $P = [u_1, u_2, \ldots, u_{2k}]$ be an alternating path. We say $P$ is an $mm$-alternating path (matching-matching-alternating path) if $[u_1, u_2, u_{2k-1}, u_{2k}] \in \mathcal{M}$. We say $P$ is an $nn$-alternating path (nonmatching-nonmatching-alternating path) if $[u_1, u_2, u_{2k-1}, u_{2k}] \notin \mathcal{M}$.

**Example 1.3.** Consider the graph $G$ shown in Figure 1. The graph $G$ has a unique perfect matching $\mathcal{M} = \{[1, 1'], [2, 2'], [3, 3'], [4, 4']\}$. The alternating paths $[1, 1', 2, 2', 3, 3']$, $[2', 4, 4', 5, 5', 3]$ and $[1, 1', 2, 2', 3]$ in $G$ are examples of $mm$-alternating path, $nn$-alternating path, $nm$-alternating path and $mn$-alternating path, respectively.

![Diagram](http://repository.uwyo.edu/ela)  
**Figure 1.** Here, the solid edges are the matching edges.

**Definition 1.4.** [5] Let $G$ be a connected graph with a unique perfect matching $\mathcal{M}$ and $[u, v] \notin \mathcal{M}$. An extension at $[u, v]$ is an $nn$-alternating $u-v$-path other than $[u, v]$. An extension at $[u, v]$ is called even type (resp., odd type) if the number of nonmatching edges on that extension is even (resp., odd). For example, in the graph $G$ shown in Figure 1, the path $[2', 4, 4', 5, 5', 3]$ is an extension at $[2', 3]$.
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**Definition 1.5.** [5] The nonmatching edge \([u, v]\) is said to be an *odd type* edge, if either there are no extensions at \([u, v]\) or each extension at \([u, v]\) is odd type. An odd type nonmatching edge \([u, v]\) is said to be *simple odd type* if there is no extension at \([u, v]\). We say \([u, v]\) is an *even type* edge, if each extension at \([u, v]\) is even type. We say \([u, v]\) is *mixed type*, if it has an even type extension and an odd type extension. Let \(\mathcal{E}\) be the set of all even type edges of \(G\).

**Definition 1.6.** [5] By \(\mathcal{H}_{\text{nmc}}\) we denote the class of graphs \(G\) in \(\mathcal{H}\) such that \(G\) has no mixed type edges and \(G\) satisfies the condition C: The extensions at two distinct even type edges never have an odd type edge in common. Here ‘nmc’ is an abbreviation of ‘no mixed type edges and a condition’. Thus,

\[
\mathcal{H}_{\text{nmc}} = \{G \in \mathcal{H} \mid G \text{ has no mixed type edges and } G \text{ satisfies condition } C\}.
\]

**Definition 1.7.** [5] Let \(G \in \mathcal{H}_{\text{nmc}}\) and \(E\) be the set of all even type edges. Then by \((G - E) / M\) denote the graph obtained by deleting all the even type edges and then contracting each matching edge to a single vertex.

**Theorem 1.8.** [5] Let \(G \in \mathcal{H}_{\text{nmc}}\). Then the inverse \(G^+\) exists if and only if \((G - E) / M\) is bipartite.

**Definition 1.9.** [6] Let \(G \in \mathcal{H}\). We shall consider weight functions \(w\) such that \(w(e) = 1\) for each matching edge \(e\). Let \(W_G\) be the class of such weight functions on \(G\).

The following result can be found in [6].

**Theorem 1.10.** Let \(G \in \mathcal{H}_g\) and \(w \in W_G\). Then the inverse \(G^+_w\) exists.

Having considered Theorems 1.8 and 1.10, it is natural to ask the following questions.

a) Does \(G^+_w\) exist for each invertible graph \(G \in \mathcal{H}_{\text{nmc}}\) and for each \(w(\neq 1)\) in \(W_G\) (see, Example 2.3)?

b) If the answer of question a) is negative, then characterize all the weight functions \(w\) for each graph \(G \in \mathcal{H}_{\text{nmc}}\) such that \(G^+_w\) exists (see, Theorem 2.11).

We supply answers to both these questions in Section 2. Finally, we show that if \(G \in \mathcal{H}_{\text{nmc}}\) and \(G^+_w\) exists for some weight function \(w \in W_G\), then \((G - E) / M\) is bipartite. That is, there is no weight function \(w \in W_G\) such that \(G^+_w\) exists for some \(G \in \{G \in \mathcal{H}_{\text{nmc}} \mid (G - E) / M\text{ is nonbipartite}\}\).

**2. Inverses of weighted graphs.** To state the next result, we need the following definition.

**Definition 2.1.** Let \(G\) be a graph. Assume that \(P\) is a path in \(G\). We use \(w(P)\) to mean the weight of \(P\), which is the product of the weights of the edges on \(P\). That is \(w(P) = \prod_{e \in E(P)} w(e)\).

The following is essentially contained in [2, Theorem 1] and [4, Lemma 2.1]. We note that the mm-alternating paths have been termed as alternating paths in [2, 4].

**Lemma 2.2.** Consider \(G_w\), where \(G \in \mathcal{H}\) and \(w \in W_G\). Let \(B = [b_{ij}]\), where

\[
b_{ij} = \sum_{P(i,j) \in \mathcal{P}(i,j)} (-1)^{||P(i,j)||-1/2} w(P),
\]

where \(\mathcal{P}(i,j)\) is the set of mm-alternating i-j-paths in \(G_w\) and \(||P(i,j)||\) is the number of edges in the i-j-path \(P(i,j)\). Then \(B = A(G_w)^{-1}\).
The following example tells us that there is an invertible graph \( G \) in \( \mathcal{H}_{nmc} \) such that \( G^+_w \) does not exist for some \( w \in W_G \).

**Example 2.3.** Consider the graph \( G \) shown in Figure 2. Notice that \( G \in \mathcal{H}_{nmc} \). By Theorem 1.8, \( G^+ \) exists. We consider the weight function \( w : E(G) \to (0, \infty) \) such that \( w(e) = 1 \) for each edge in \( G - [1', 3] \) and \( w([1', 3]) = 2 \). Suppose that \( G^+_w \) exists. Then there is a signature matrix \( S \) such that \( SA(G_w)^{-1}S \geq 0 \). We use \( s_i \) and \( A(G_w)^{-1}_{ij} \) to denote the \( i \)th diagonal entry of \( S \) and \( ij \)th entry of the matrix \( A(G_w)^{-1} \), respectively. Notice that \( A(G_w)^{-1}_{1,1'} = 1 = A(G_w)^{-1}_{1,2} = A(G_w)^{-1}_{1,3} \) and \( A(G_w)^{-1}_{2,2'} = 1 \). Then we have \( s_1s'_2 = s_2s'_3 = s_1s'_3 = -1 \) and \( s_2s'_2 = 1 \). Therefore, \( s_1^2s'_2s'_3s_3^2 = -1 \) which is not possible. Hence, the answer to question a) is negative.

![Figure 2. Here the solid edges are the matching edges.](image)

Next we give an unexpected combinatorial answer to the following question. Let \( G \in \{ G \in \mathcal{H}_{nmc} \mid (G - \mathcal{E})/\mathcal{M} \text{ is bipartite} \} \). What are those weight functions \( w \in W_G \) for which \( G^+_w \) exists? In order to answer this we need the following definition and results from the literature.

**Definition 2.4.** Let \( G \in \mathcal{H} \) and suppose that \( u \) and \( v \) are two distinct vertices in \( G \). Following [5], we call an mm-alternating \( u-v \)-path a minimal path, if this path does not contain any even type extensions (of any nonmatching edge in \( G \)).

To proceed further we need the following three known results.

**Lemma 2.5.** [5] Let \( G \in \mathcal{H}_{nmc} \). Let \( P(i, j) \) be an mm-alternating \( i-j \)-path. Then there exists a unique minimal \( i-j \)-path \( P_m(i, j) \) and a set \( F \) of even type edges on \( P_m(i, j) \) such that \( P(i, j) \) is created from \( P_m(i, j) \) by replacing each edge \( f \in F \) with an even type extension \( Q_f \) at \( f \).

**Lemma 2.6.** [5] Let \( G \in \mathcal{H}_{nmc} \) with \( (G - \mathcal{E})/\mathcal{M} \) is bipartite. Then \( G \) does not contain a cycle which has an odd number of odd type edges. In particular, if one path from \( u \) to \( v \) contains an odd (resp., even) number odd type edges, then each path from \( u \) to \( v \) must contain an odd (resp., even) number odd type edges.

**Lemma 2.7.** [5] Let \( G \in \mathcal{H} \) and \( P(i, j) \) be an mm-alternating \( i-j \)-path. Let \( [u, v] \) be a nonmatching edge on \( P(i, j) \) and \( Q(u, v) \) be an extension at \( [u, v] \). Then \( Q(u, v) \) contains no vertex of \( P(i, j) \) other than \( u \) and \( v \). That is, \( V(P(i, j)) \cap V(Q(u, v)) = \{u, v\} \).

We need another definition which is given below.

**Definition 2.8.** Let \( G \in \mathcal{H} \), \( w \in W_G \) and \( e \) be an even type edge in \( G \). We define \( W(e) = \sum_{Q(e)} w(Q(e)) \), where the sum is taken over all extensions at \( e \). That is, \( W(e) \) is the sum of the weights of all extensions at \( e \).
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The following result supplies a necessary condition on weight functions \( w \) such that \( G_w^+ \) exists for each \( G \in \{ G \in \mathcal{H}_{\text{nmc}} \mid (G - \mathcal{E})/\mathcal{M} \text{ is bipartite} \} \).

**Theorem 2.9.** Let \( G \in \mathcal{H}_{\text{nmc}} \) for which \((G - \mathcal{E})/\mathcal{M}\) is bipartite and let \( w \in \mathcal{W}_G \). Assume that \( G_w^+ \) exists. Then \( w(e) \leq W(e) \) for each \( e \in \mathcal{E} \).

**Proof.** As \( G \in \mathcal{H}_{\text{nmc}} \) and \((G - \mathcal{E})/\mathcal{M}\) is bipartite, by Theorem 1.8, \( G^+ \) exists. Now suppose that \( w \in \mathcal{W}_G \) is a weight function such that \( G_w^+ \) exists. We shall show that \( w(e) \leq W(e) \) holds for each even type edge \( e \) in \( G \). Proceeding by the way of contradiction, let if possible, \( w(e) > W(e) \) hold for some even type edge \( e = [u, v] \). Let \( Q(u, v) = [u, u_1, u'_1, u_2, u'_2, \ldots, u_{2k-1}, u'_{2k-1}, v] \) be a maximum length even type extension at \([u, v]\). Let \([x, y]\) be a nonmatching edge on \( Q(u, v) \). The edge \([x, y]\) is odd type.

**Claim.** The edge \([x, y]\) is simple odd type.

**Proof of the Claim.** Suppose that the edge \([x, y]\) is not simple odd type. Then there is an extension \( Q(x, y) \) at \([x, y]\), and by Lemma 2.7, \( x \) and \( y \) are the only common points on the paths \( Q(x, y) \) and \([u', Q(u, v), v']\). In that case, by replacing \([x, y]\) with \( Q(x, y) \) in \( Q(u, v) \), we get a larger length even type extension at \([u, v]\), which is a contradiction. So the claim is justified.

Thus, each nonmatching edge on \( Q(u, v) \) is simple odd type. Consider \( B = A(G_w)^{-1} \). By using Lemma 2.2, we see that

\[
\begin{align*}
  i) & \quad b_{u_i, u'_i} = 1 \text{ for all } i = 1, \ldots, 2k - 1; \\
  ii) & \quad -w([u'_i, u'_i]) = b_{u'_i, u'_i}, b_{u'_i, u_{2k-1}} = -w([v', u'_{2k-1}]) \text{ and } b_{u_i, u'_{i+1}} = -w([u_i, u'_{i+1}]) \text{ for all } i = 1, \ldots, 2k - 2, \\
  iii) & \quad b_{u_i, v'} = -w(e) + W(e) < 0. 
\end{align*}
\]

Since \( G_w^+ \) exists, there is a signature matrix \( S \) such that \( SA(G_w)^{-1}S \geq 0 \). Then we have

\[
\begin{align*}
  i) & \quad s_{u_i} s_{u'_i} = 1 \text{ for all } i = 1, \ldots, 2k - 1; \\
  ii) & \quad -1 = s_{u_i} s_{u'_i} = s_{v'} s_{u_{2k-1}} = s_{u_i} s_{u'_{i+1}} \text{ for all } i = 1, \ldots, 2k - 2; \text{ and} \\
  iii) & \quad s_{u_i} s_{v'} = -1. 
\end{align*}
\]

Therefore, \( s_{u_1} s_{u'_{1}} s_{u_2} s_{u'_{2}} \cdots s_{u_{2k-1}} s_{u'_{2k-1}} = -1 \) which is not possible. This is a contradiction to our hypothesis that \( G_w^+ \) exists. Hence, \( w(e) \leq W(e) \) for all \( e \in \mathcal{E} \).

The following result tells us that the above necessary condition on weight functions \( w \) is also sufficient for the existence of \( G_w^+ \) for a graph \( G \in \{ G \in \mathcal{H}_{\text{nmc}} \mid (G - \mathcal{E})/\mathcal{M} \text{ is bipartite} \} \).

**Theorem 2.10.** Let \( G \in \mathcal{H}_{\text{nmc}} \) for which \((G - \mathcal{E})/\mathcal{M}\) is bipartite and \( w \in \mathcal{W}_G \). Assume that \( w(e) \leq W(e) \) for each \( e \in \mathcal{E} \). Then \( G_w^+ \) exists.

**Proof.** Suppose that \( G \in \mathcal{H}_{\text{nmc}} \) with \((G - \mathcal{E})/\mathcal{M}\) bipartite. Take a weight function \( w \in \mathcal{W}_G \) such that \( w(e) \leq W(e) \) holds for each \( e \in \mathcal{E} \). We shall show that \( G_w^+ \) exists. Let \( S \) be the signature matrix defined by \( s_1 = 1 \) and \( s_i = (-1)^z \), where \( z \) is the number of odd type edges on a \( i \)-1-path. This matrix is well defined, in view of Lemma 2.6. Suppose that \( SA(G_w)^{-1}S \geq 0 \). That is, there exist \( i \) and \( j \) such that \( s_i A(G_w)^{-1} s_j < 0 \).

We have two possibilities.

**Case I.** The entry \( A(G_w)^{-1}_{i,j} < 0 \). Then \( s_i = s_j \). By Lemma 2.6, the parity of the number of odd type edges on any path from \( i \) to \( j \) is the same with that of any path from \( i \) to \( j \). It follows that any path from \( i \) to \( j \) must contain an even number of odd type edges.
Let \( P_m^1(i, j), P_m^2(i, j), \ldots, P_m^t(i, j) \) be the minimal paths from \( i \) to \( j \). Let \( P^r(i, j) \) be the set of all mm-alternating \( i-j \)-paths which are created from \( P_m^r(i, j) \), for \( r = 1, \ldots, t \). Using Lemma 2.5, we have \(|\mathcal{P}(i, j)| = \sum_{r=1}^{t} |\mathcal{P}^r(i, j)|\). Using Lemma 2.2, we have

\[
A(G_w)_{i,j}^{-1} = \sum_{r=1}^{t} \sum_{P(i,j) \in \mathcal{P}^r(i,j)} (-1)^{\frac{|P(i,j)|-1}{2}} w(P(i,j)),
\]

where \( \sum_{P(i,j) \in \mathcal{P}^r(i,j)} (-1)^{\frac{|P(i,j)|-1}{2}} w(P(i,j)) \) is the contribution to \( A(G_w)_{i,j}^{-1} \) coming from the \( r \)th minimal path \( P_m^r(i, j) \).

Assume first that \( P_m^r(i, j) \) contains an odd number of nonmatching edges. As any \( i-j \)-path contains an even number of odd type edges, we must have an odd number of even type edges on \( P_m^r(i, j) \). Let \( e_1, e_2, \ldots, e_k \) be the odd type edges on the \( r \)th minimal path \( P_m^r(i, j) \), where \( k \) is odd. Let \( m_l \geq 1 \) be the number of extensions (these are even type) at the edge \( e_l \), for \( l = 1, \ldots, k \). Suppose that we choose the even type edges \( e_{l_1}, \ldots, e_{l_r} \) from \( e_1, e_2, \ldots, e_k \) and create an mm-alternating \( i-j \)-path by using one extension for each of the chosen even type edges. Then we can create \( m_{l_1} \cdots m_{l_r} \) many such mm-alternating \( i-j \)-paths and each such path has an odd (resp., even) number of nonmatching edges if \( p \) is even (resp., odd). Thus, the contribution of the mm-alternating paths that are created from \( P_m^r(i, j) \) by choosing \( p \) many edges out of \( e_1, e_2, \ldots, e_k \), to \( A(G_w)_{i,j}^{-1} \) is

\[
(-1)^{p+1} w(P_m^r(i, j)) \sum_{\{e_{l_1}, \ldots, e_{l_p}\} \subseteq \{e_1, e_2, \ldots, e_k\}} \frac{W(e_{l_1})W(e_{l_2}) \cdots W(e_{l_p})}{w(e_{l_1})w(e_{l_2}) \cdots w(e_{l_p})}.
\]

Hence, the total contribution of \( \mathcal{P}^r(i, j) \), the set of mm-alternating \( i-j \)-paths that are created from \( P_m^r(i, j) \), to \( A(G_w)_{i,j}^{-1} \) is

\[
\sum_{p=0}^{k} (-1)^{p+1} w(P_m^r(i, j)) \sum_{\{e_{l_1}, \ldots, e_{l_p}\} \subseteq \{e_1, e_2, \ldots, e_k\}} \frac{W(e_{l_1})W(e_{l_2}) \cdots W(e_{l_p})}{w(e_{l_1})w(e_{l_2}) \cdots w(e_{l_p})}.
\]

\[
= -w(P_m^r(i, j)) \sum_{p=0}^{k} \sum_{\{e_{l_1}, \ldots, e_{l_p}\} \subseteq \{e_1, e_2, \ldots, e_k\}} \prod_{T} \frac{-W(e_{l_i})}{w(e_{l_i})},
\]

\[
= -w(P_m^r(i, j)) \prod_{i=1}^{k} \left[ 1 - \frac{W(e_i)}{w(e_i)} \right] \geq 0,
\]

as \( k \) is odd. Similarly, if \( P_m^r(i, j) \) contains an even number of nonmatching edges, then also the contribution \( P_m^r(i, j) \), to \( A(G_w)_{i,j}^{-1} \) is nonnegative. Hence, \( A(G_w)_{i,j}^{-1} \geq 0 \), by (2.1). This contradicts the hypothesis that \( A(G_w)_{i,j}^{-1} < 0 \).

**Case II.** The entry \( A(G_w)_{i,j}^{-1} > 0 \). Carrying the arguments in a way similar to **Case I**, we get a contradiction to our hypothesis that \( A(G_w)_{i,j}^{-1} > 0 \).

Hence, we conclude that \( S A(G_w)^{-1} S \geq 0 \). That is, \( G_w^{++} \) exists.

The main theorem of this section is the following.

**Theorem 2.11.** Let \( G \in \mathcal{H}_{mm} \) for which \((G - \mathcal{E})/M\) is bipartite and \( w \in \mathcal{W}_G \). Then \( G_w^{++} \) exists if and only if \( w(e) \leq W(e) \) for each \( e \in \mathcal{E} \), where \( \mathcal{E} \) is the set of all even type edges.
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Remark 2.12. It is clear that Theorems 1.8 and 1.10 are particular cases of Theorem 2.11.

Instead of looking at \( \{ G \in \mathcal{H}_{nmc} \mid (G - E)/M \text{ is bipartite} \} \), let us look at the larger class \( \mathcal{H}_{nmc} \) itself. Suppose that for \( G \in \mathcal{H}_{nmc} \) and \( w \in \mathcal{W}_G \), the inverse \( G_w^+ \) exists. What can be said about such a graph \( G \)? The following result says that in that case the graph \( G \) must belong to \( \{ G \in \mathcal{H}_{nmc} \mid (G - E)/M \text{ is bipartite} \} \).

In other words, these are the only graphs in \( \mathcal{H}_{nmc} \) which have inverses for some weight functions.

Proposition 2.13. Let \( G \in \mathcal{H}_{nmc} \) and \( w \in \mathcal{W}_G \). If \( G_w^+ \) exists, then \( (G - E)/M \) is bipartite.

Proof. Let \( G \in \mathcal{H}_{nmc} \), \( w \in \mathcal{W}_G \) for which \( G_w^+ \) exists. Let \( S \) be the signature matrix such that \( SA(G_w)^{-1}S \geq 0 \). As \( G \in \mathcal{H}_{nmc} \), deleting the even type edges, we see that \( (G - E) \) has no even type edges. Then by using Lemma 2.2, we have

i) \( A((G_w)^{-1})_{u'v'} < 0 \) for any nonmatching edge \( [u, v] \in (G - E) \) and

ii) \( A(G_w)^{-1}x,s \) for any matching edge \( [x, x'] \in (G - E) \).

Let \( [u, v] \in (G - E) \) be a nonmatching edge. So \( A(G_w)^{-1}u,v < 0 \). Since \( s_wA(G_w)^{-1}u',v' \geq 0 \), we have that \( s_ws_w' = -1 \). Let \( [x, x'] \) be a matching edge in \( (G - E) \). By similar arguments, we have \( sxs'_x = 1 \). Taking \( X = \{ u \in (G - E)/M \mid s_u > 0 \} \) and \( Y = \{ u \in (G - E)/M \mid s_u < 0 \} \), we get a bipartition.

We summarize our observation of this article by the following result which also addresses question b).

Theorem 2.14. Let \( G \in \mathcal{H}_{nmc} \) and \( w \in \mathcal{W}_G \).

i) If \( (G - E)/M \) is bipartite, then \( G_w^+ \) exists if and only if \( w(e) \leq W(e) \) for each \( e \in \mathcal{E} \).

ii) If \( (G - E)/M \) is not bipartite, then \( G_w^+ \) does not exist.

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REFERENCES


