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## GROUPS OF MATRICES THAT ACT MONOPOTENTLY\*

JOSHUA D. HEWS<sup>†</sup> AND LEO LIVSHITS<sup>‡</sup>

**Abstract.** In the present article, the authors continue the line of inquiry started by Cigler and Jerman, who studied the separation of eigenvalues of a matrix under an action of a matrix group. The authors consider groups  $\mathcal{G}$  of matrices of the form  $\begin{bmatrix} G & 0 \\ 0 & z \end{bmatrix}$ , where  $z$  is a complex number, and the matrices  $G$  form an irreducible subgroup of  $\mathrm{GL}_n(\mathbb{C})$ . When  $\mathcal{G}$  is not essentially finite, the authors prove that for each invertible  $A$  the set  $\mathcal{G}A$  contains a matrix with more than one eigenvalue.

The authors also consider groups  $\mathcal{G}$  of matrices of the form  $\begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix}$ , where the matrices  $G$  comprise a bounded irreducible subgroup of  $\mathrm{GL}_n(\mathbb{C})$ . When  $\mathcal{G}$  is not finite, the authors prove that for each invertible  $A$  the set  $\mathcal{G}A$  contains a matrix with more than one eigenvalue.

**Key words.** Invertible matrices, Matrix groups, Distinct eigenvalues, Irreducible groups, Unitary group, Monopotent matrices.

**AMS subject classifications.** 15A18.

**1. Introduction.** A classical “inverse multiplicative eigenvalue problem” (see [7]) asks, whether for a given square matrix  $A$  there is a diagonal matrix  $D$  such that  $DA$  has the prescribed spectrum. The question can be stated over  $\mathbb{R}$  or  $\mathbb{C}$ . The default field in our paper is  $\mathbb{C}$ .

In 1958, M.E. Fisher and A.T. Fuller [6] showed that when  $A$  is a real square matrix with positive principal leading minors (of all orders), there is a diagonal matrix  $D$  with a positive diagonal such that all of the (complex) eigenvalues of  $DA$  are positive and algebraically simple. A decade later C.S. Ballantine [1] extended the result to complex matrices under the hypotheses that all principal leading minors of  $A$  are non-zero and  $D$  is allowed to be an invertible complex diagonal matrix. In 1975, S. Friedland [7] improved on Ballantine’s theorem by showing that one can arrange for  $DA$  to have whatever complex spectrum one desires (including the algebraic multiplicities of the eigenvalues) by using general complex diagonal matrices  $D$ .

Fast forward to 2012 and the paper of X.-L. Feng, Z. Li and T.-Z. Huang [5] in which the authors pose the following related question: Is every complex invertible matrix diagonally equivalent to a matrix with distinct eigenvalues? Feng, Li and Huang answer the question in the affirmative for matrices of small size, and later that same year M.-D. Choi, Z. Huang, C.-K. Li and N.-S. Sze [2] settle the general question, also in the affirmative. A formulation of their result states that complex square matrices not diagonally equivalent to a matrix with distinct eigenvalues are exactly the non-invertible matrices whose classical adjoint has zero diagonal.

In 2014, G. Cigler and M. Jerman [4] observed that the question posed by Feng, Li and Huang, and settled by Choi, Huang, Li and Sze, can be considered to be part of a more general inquiry. Given a group

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$\mathcal{G}$  of complex  $n \times n$  matrices, and a matrix  $A \in \mathbb{M}_n$ , one can consider the set  $\mathcal{G}A$  and ask whether this set contains a matrix with a maximal possible number (i.e.,  $\text{rank}(A)$ ) of distinct eigenvalues. When it does, Cigler and Jerman say that  $A$  is  $\mathcal{G}$ -separable.

Cigler and Jerman prove that every matrix is  $\mathcal{G}$ -separable when  $\mathcal{G}$  is the group  $\mathcal{U}_n$  of all unitary matrices, or the group  $\mathcal{M}_n$  of all monomial matrices. We shall express this by saying that  $\mathcal{U}_n$  and  $\mathcal{M}_n$  are *eigenvalue separating groups*. *Monomial matrices*, also known as “weighted permutations”, are the products of invertible diagonal matrices and permutation matrices.

*Irreducible matrix groups* are the groups that do not have common non-trivial invariant subspaces, where the trivial subspaces are  $\{0_n\}$  and the whole space. By the celebrated Burnside’s theorem (in the complex setting) these are exactly the groups that span  $\mathbb{M}_n$ .  $\mathcal{U}_n$  and  $\mathcal{M}_n$  are examples of irreducible subgroups of the general linear group  $\mathbb{GL}_n$ .

Cigler and Jerman give an example of an irreducible subgroup  $\mathcal{G}$  of  $\mathbb{GL}_4$  no member of which has four distinct eigenvalues. In particular,  $I_4$  is not  $\mathcal{G}$ -separable, and so not all irreducible subgroups of  $\mathbb{GL}_n$  are eigenvalue separating groups.

Consequently, it is natural to ask whether irreducible matrix groups can fail at separating eigenvalues in a big way. Is there an irreducible matrix group  $\mathcal{G}$  in dimensions higher than 1, and a non-zero matrix  $A$ , such that every matrix in  $\mathcal{G}A$  has a single eigenvalue? Cigler and Jerman prove in [4] that no such  $\mathcal{G}$  and  $A$  exist. They express this by saying that  $A$  is  $\mathcal{G}$ -semi-separable, which we restate by saying that the irreducible matrix groups (in dimensions greater than 1) are (*eigenvalue*) *semi-separating*.

In their consequent paper [3] from the same year, Cigler and Jerman continue the exploration of eigenvalue semi-separating matrix groups by considering the group  $\mathcal{P}_n$  of all  $n \times n$  permutation matrices (i.e., the symmetric group). Note that  $\mathcal{P}_n$  is not irreducible, but has only one pair of (complementary) non-trivial invariant subspaces: the span of the vector all of whose entries are 1, and the orthogonal complement of that span. Decomposing  $\mathcal{P}_n$  along these subspaces gives a representation of  $\mathcal{P}_n$  of the form  $\mathcal{G} \oplus \{1\}$ , where  $\mathcal{G}$  is a finite irreducible group of unitary matrices.

Cigler and Jerman show that every complex invertible  $3 \times 3$  matrix is  $\mathcal{P}_3$ -semi-separable, and determine the  $\mathcal{P}_n$ -semi-separable nilpotent matrices. They also show that if the modulus of the sum of the entries of  $A$  does not exceed  $n \sqrt[n]{\det A}$ , then  $A$  is  $\mathcal{P}_n$ -semi-separable.

In the present article, we continue this line of inquiry. We consider groups  $\mathcal{G}$  of matrices of the form  $\begin{bmatrix} G & 0 \\ 0 & z \end{bmatrix}$ , where  $z \in \mathbb{C}$  and the matrices  $G$  form an irreducible subgroup of  $\mathbb{GL}_n$ . We prove that  $\mathcal{G}$  semi-separates the eigenvalues of any invertible  $A$  whenever  $\mathcal{G}$  is not essentially finite. A matrix group  $\mathcal{G}$  is *essentially finite* if there is a finite group  $\mathcal{F}$  such that

$$\mathcal{G} \subset \mathbb{C}\mathcal{F}.$$

Furthermore, we consider groups  $\mathcal{G}$  of matrices of the form  $\begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix}$ , where matrices  $G$  comprise a bounded irreducible subgroup of  $\mathbb{GL}_n$ . We prove that  $\mathcal{G}$  semi-separates the eigenvalues of any invertible  $A$ , whenever  $\mathcal{G}$  is not finite.

Let us mention the following terminology that is used in this paper.

If  $\mathcal{W}$  and  $\mathcal{Z}$  are complementary invariant subspaces for a subgroup  $\mathcal{G}$  of  $\mathbb{GL}_n$ , we say that  $\mathcal{W}$  and  $\mathcal{Z}$  are *reducing subspaces* for  $\mathcal{G}$ .

Let  $\mathbb{M}_n$  denote the algebra of  $n \times n$  complex matrices. A matrix  $A \in \mathbb{M}_n$  is *monopotent* if the (complex) spectrum of  $A$  is a singleton.  $A$  is monopotent exactly when  $A = \alpha I + N$  where  $\alpha \in \mathbb{C}$  and  $N$  is a nilpotent matrix.

We will say a subgroup  $\mathcal{G}$  of  $\mathbb{GL}_n$  *acts monopotently* on a matrix  $A \in \mathbb{M}_n$  if  $GA$  is monopotent for all  $G \in \mathcal{G}$ . We also extend this definition to general collections of matrices, and not just groups.

We denote the multiplicative group of the non-zero complex numbers by  $\mathbb{C}^*$ . For  $z \in \mathbb{C}$  we write  $\Omega_n(z)$  for the set of  $n^{\text{th}}$  roots of  $z$ , and drop the reference to  $z$  when  $z = 1$ .

The normalized trace functional  $\text{Trace}(\cdot)$  on  $\mathbb{M}_n$  is defined by:

$$\text{Trace}(A) \stackrel{\text{def}}{=} \frac{\text{trace}(A)}{n}.$$

We write  $\|A\|$  for the  $\ell^2$ -operator norm of a matrix  $A$ , and  $\|A\|_{\text{t-n}}$  for the trace-norm of  $A$ , defined by

$$\|A\|_{\text{t-n}} \stackrel{\text{def}}{=} \text{trace}(\sqrt{A^*A}),$$

where  $A^*$  is the conjugate-transpose of  $A$ .

## 2. Preliminary results.

*Observation 2.1.* Given matrices  $A, B \in \mathbb{M}_n$  such that

$$\text{trace}(A^p) = \text{trace}(B^p), \quad \text{for } p = 1, 2, 3, \dots, n,$$

it is well-known (see for example Theorem 2.1.16 of [10]) that  $A$  and  $B$  have the same eigenvalues, counting algebraic multiplicity. Consequently,  $B \in \mathbb{M}_n$  is a monopotent matrix with an eigenvalue  $\alpha$  if and only if

$$\text{Trace}(B^p) = \alpha^p, \quad \text{for } p = 1, 2, 3, \dots, n.$$

It follows that  $B \in \mathbb{M}_n$  is monopotent if and only if

$$\left(\text{Trace}(B)\right)^p = \text{Trace}(B^p), \quad \text{for } p = 2, 3, \dots, n.$$

It is also obvious that for a monopotent matrix  $B \in \mathbb{M}_n$ :

$$(2.1) \quad \left(\text{Trace}(B)\right)^n = \text{Trace}(B^n) = \det B.$$

LEMMA 2.2. [8, Proof of Theorem 5] *If  $z_1, z_2, z_3, \dots, z_n$  are complex numbers of modulus 1, and the sequence  $[z_1^p + z_2^p + \dots + z_n^p]_{p=1}^\infty$  converges, then the limit of this sequence is  $n$ .*

PROPOSITION 2.3. *For an invertible  $A \in \mathbb{M}_n$ , the following are equivalent:*

1. For all  $p \in \mathbb{N}$ :  $\left(\text{Trace}(A^p)\right)^n = \det A^p$ .
2. For all  $p \in \mathbb{N}$ :  $\left(\text{Trace}(A^p)\right)^n = \left(\text{Trace}(A)\right)^{pn}$ .
3. For  $p = 1, 2, \dots, n$ :  $\left(\text{Trace}(A^p)\right) = \left(\text{Trace}(A)\right)^p$ .
4.  $A$  is monopotent.

*Proof.* We have established the equivalence of claims 3) and 4) already in Observation 2.1.

4)  $\implies$  1) : Since  $A$  is monopotent, so is every natural power of  $A$ , and 1) follows from (2.1).

1)  $\implies$  2) : When  $p = 1$ , claim 1) states that  $(\text{Trace}(A))^n = \det A$ . Thus, assuming 1) holds, we get:

$$(\text{Trace}(A^p))^n = \det A^p = (\det A)^p = (\text{Trace}(A))^{pn},$$

which yields 2).

2)  $\implies$  4) : The argument is essentially that of [8, Proof of Theorem 5], with just a few modifications. After replacing  $A$  with a non-zero scalar multiple of  $A$ , which of course does not affect the equalities in 2), we may assume that  $\text{Trace}(A) = 1$ .

Hence, we can rewrite the equalities in 2) as

$$(\text{Trace}(A^p))^n = 1, \quad \text{for all } p \in \mathbb{N};$$

from where we see that

$$(2.2) \quad \text{Trace}(A^p) \in \Omega_n, \quad \text{for all } p \in \mathbb{N}.$$

Let us write  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  for the eigenvalues of  $A$ , listed with algebraic multiplicity and in order of the decreasing modulus. Then (2.2) states that

$$(2.3) \quad s(p) \stackrel{\text{def}}{=} \sum_{i=1}^n \lambda_i^p \in n \cdot \Omega_n,$$

for all natural  $p$ , and obviously  $|s(p)| = n$  for all such.

Suppose that  $\rho = |\lambda_1| = \dots = |\lambda_r|$ , and all other  $\lambda_i$  have strictly smaller modulus than  $\rho$ . Then

$$(2.4) \quad \left( \frac{s(p)}{\rho^p} - \sum_{i=1}^r \left( \frac{\lambda_i}{\rho} \right)^p \right) \rightarrow 0, \quad \text{as } p \rightarrow \infty.$$

If  $\rho > 1$ , then we would have

$$\left( \frac{s(p)}{\rho^p} \right) \rightarrow 0, \quad \text{as } p \rightarrow \infty,$$

and therefore,

$$\sum_{i=1}^r \left( \frac{\lambda_i}{\rho} \right)^p \rightarrow 0, \quad \text{as } p \rightarrow \infty,$$

which would dictate (via Lemma 2.2) that  $r = 0$ , leading to a contradiction.

If  $\rho < 1$ , then we would have

$$n = |s(p)| \leq \sum_{i=1}^n |\lambda_i|^p \leq n \cdot \rho^p < n,$$

causing a contradiction again. Therefore, it must be that  $\rho = 1$ , and (2.4) becomes:

$$(2.5) \quad s(p) - \sum_{i=1}^r \lambda_i^p \rightarrow 0, \quad \text{as } p \rightarrow \infty.$$

Yet  $|s(p)| = n$  and  $|\sum_{i=1}^r \lambda_i^p| \leq r$ , and so, we can deduce from (2.5) that  $r = n$ . This shows that all eigenvalues of  $A$  have modulus 1.

Yet we also have

$$n = |s(p)| = \left| \sum_{i=1}^n \lambda_i^p \right| \leq \sum_{i=1}^n |\lambda_i^p| = n.$$

The equality in the complex triangle inequality holds exactly when all of the summands are positive multiples of a fixed complex number of modulus 1. Since all  $\lambda_i$  have modulus 1, this means that they are all equal 1, and so  $A$  is monopotent.  $\square$

PROPOSITION 2.4. *If  $\mathcal{G}$  is a subgroup of  $\mathbb{C}^*$  and  $I_n \oplus \mathcal{G}$  acts monopotently on an invertible matrix  $A \in \mathbb{M}_{n+1}$ , then  $\mathcal{G} \subset \{-1, 1\}$ .*

*Proof.* Let us write

$$\mathcal{H} = I_n \oplus \mathcal{G} = \left\{ \begin{bmatrix} I_n & 0 \\ 0 & \alpha \end{bmatrix} \mid \alpha \in \mathcal{G} \right\},$$

and express  $A$  as  $\begin{bmatrix} B_o & x_o \\ y_o & a_o \end{bmatrix}$  with respect to the same direct sum decomposition of the underlying space.

Let us prove that  $\mathcal{G}$  has at most two elements. Suppose, for the sake of contradiction, that  $\mathcal{G}$  has at least three distinct elements.

By Proposition 2.3,

$$(n + 1) \text{trace}(C_\alpha^2) = (\text{trace}(C_\alpha))^2,$$

where  $\alpha \in \mathcal{G}$  and

$$C_\alpha \stackrel{\text{def}}{=} \begin{bmatrix} I_n & 0 \\ 0 & \alpha \end{bmatrix} A = \begin{bmatrix} B_o & x_o \\ \alpha y_o & \alpha a_o \end{bmatrix}.$$

This leads to the equality

$$(2.6) \quad (n + 1) (\text{trace}(B_o^2) + 2\alpha y_o x_o + \alpha^2 a_o^2) = (\text{trace}(B_o) + \alpha a_o)^2,$$

for all  $\alpha \in \mathcal{G}$ . Since  $\mathcal{G}$  contains at least three distinct  $\alpha$ 's, the equality (2.6) is equivalent to the equality of the corresponding coefficients of the powers of  $\alpha$  :

$$\begin{cases} (n + 1) \text{trace}(B_o^2) = (\text{trace}(B_o))^2; \\ (n + 1) y_o x_o = a_o \text{trace}(B_o); \\ (n + 1) a_o^2 = a_o^2. \end{cases}$$

The third equality yields  $a_o = 0$ , which leads to a contradiction, because in such a case, according to (2.1), for any  $\alpha \in \mathcal{G}$  :

$$(2.7) \quad \alpha \det A = \det C_\alpha = \left( \frac{\text{trace}(C_\alpha)}{n + 1} \right)^{n+1} = \left( \frac{\text{trace}(B_o)}{n + 1} \right)^{n+1} = \left( \frac{\text{trace}(A)}{n + 1} \right)^{n+1} = \det A,$$

which indicates that  $\mathcal{G} = \{1\}$ , contradicting our hypothesis that  $\mathcal{G}$  has at least three elements.  $\square$

EXAMPLE 2.5. Proposition 2.4 is not vacuous. For example, the group

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

acts monopotently on the invertible matrix  $\begin{bmatrix} 1+1 & 1 \\ 1 & 1-i \end{bmatrix}$ , and the group

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

acts monopotently on the invertible matrix  $\begin{bmatrix} 3 & 3 & 2 \\ -4 & -3 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ .

*Observation 2.6.* If a collection  $\mathcal{F}$  in  $\mathbb{M}_n$  acts monopotently on a matrix  $A$ , then so does the closure  $\overline{\mathbb{C}\mathcal{F}}$  of  $\mathbb{C}\mathcal{F}$ . Indeed, it follows immediately from Proposition 2.3, and from the continuity of power functions and of trace, that the set of monopotent matrices is closed. Hence,

$$(\overline{\mathbb{C}\mathcal{F}})A \subset \overline{\mathbb{C}\mathcal{F}A} \subset \overline{\{\text{monopotent matrices}\}} = \{\text{monopotent matrices}\},$$

and the claim follows.

### 3. Main results.

**THEOREM 3.1.** *If  $\mathcal{G}$  is an irreducible subgroup of  $\mathbb{GL}_n$  and  $\mathcal{G} \oplus 1$  acts monopotently on some  $A \in \mathbb{GL}_{n+1}$ , then  $\mathcal{G}$  is finite.*

*Proof.* Let us write

$$\mathcal{H} = \mathcal{G} \oplus 1 = \left\{ \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix} \mid G \in \mathcal{G} \right\},$$

and express  $A$  as  $\begin{bmatrix} B_\alpha & x_\alpha \\ y_\alpha & a_\alpha \end{bmatrix}$  with respect to the same direct sum decomposition of the underlying space. After multiplying  $A$  by a scalar, we can assume without loss of generality that  $\det A = 1$ .

Since  $\mathbb{C}^*\mathcal{H}$  acts monopotently on  $A$ , the same can be said of the group  $\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1}$ .

Now

$$\begin{aligned} \mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1} &= \left\{ \begin{bmatrix} T & 0 \\ 0 & \alpha \end{bmatrix} \mid \alpha \in \mathbb{C}^*, T \in \alpha\mathcal{G}, \alpha \det T = 1 \right\} \\ &= \left\{ \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} \mid \det(T)T \in \mathcal{G} \right\}. \end{aligned}$$

Let

$$(3.8) \quad \tilde{\mathcal{G}} \stackrel{\text{def}}{=} \{ T \in \mathbb{M}_n \mid \det(T)T \in \mathcal{G} \},$$

so that

$$\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1} = \left\{ \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} \mid T \in \tilde{\mathcal{G}} \right\}.$$

It is easy to check that  $\tilde{\mathcal{G}}$  is a group. Let us demonstrate that

$$(3.9) \quad \tilde{\mathcal{G}} = \left\{ \frac{G}{\alpha} \mid G \in \mathcal{G}, \alpha \in \Omega_{n+1}(\det G) \right\}.$$

( $\supset$ ) Suppose  $G \in \mathcal{G}$  and  $\alpha \in \Omega_{n+1}(\det G)$ . Then

$$(3.10) \quad \det \left( \frac{G}{\alpha} \right) \frac{G}{\alpha} = \frac{\det G}{\alpha^{n+1}} G = G \in \mathcal{G}.$$

( $\subset$ ) If  $T \in \tilde{\mathcal{G}}$ , then  $T = \frac{1}{\det T} G$  for some  $G \in \mathcal{G}$ . Thus,

$$G = \det(T) T = \det \left( \frac{G}{\det T} \right) \frac{G}{\det T} = \frac{\det G}{(\det T)^{n+1}} G,$$

from where it follows that

$$(\det T)^{n+1} = \det G.$$

Therefore,  $T = \frac{1}{\alpha} G$ , where  $\alpha \in \Omega_{n+1}(\det G)$ .

Using equalities (3.8)–(3.10), we can now write:

$$\mathcal{G} = \left\{ (\det T)T \mid T \in \tilde{\mathcal{G}} \right\},$$

and conclude that  $\mathbb{C}^*\mathcal{G} = \mathbb{C}^*\tilde{\mathcal{G}}$ , and that  $\mathcal{G}$  is finite if and only if  $\tilde{\mathcal{G}}$  is finite. Since a group  $\mathcal{K}$  is irreducible if and only if  $\mathbb{C}^*\mathcal{K}$  is irreducible, it must be that  $\tilde{\mathcal{G}}$  is an irreducible group.

By Burnside's theorem  $\mathbb{M}_n$  has no proper irreducible subalgebras, and hence,  $\text{Span}(\tilde{\mathcal{G}})$ , being an irreducible subalgebra of  $\mathbb{M}_n$ , must equal  $\mathbb{M}_n$ . Thus,  $\tilde{\mathcal{G}}$  contains a basis  $T_1, \dots, T_{n^2}$  of  $\mathbb{M}_n$ .

Recall that  $\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1}$  acts monopotently on  $A$ , and

$$(\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1}) A = \left\{ \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} A \mid T \in \tilde{\mathcal{G}} \right\} = \left\{ C_T \mid T \in \tilde{\mathcal{G}} \right\},$$

where

$$C_T \stackrel{\text{def}}{=} \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} A.$$

Clearly, for every  $i \leq n^2$ :

$$\begin{bmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{bmatrix} C_T \in (\mathbb{C}^*\mathcal{H} \cap \mathbb{SL}_{n+1}) A,$$

and so, each  $\begin{bmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{bmatrix} C_T$  has a sole eigenvalue which we denote by  $\omega_i(T)$ , and which is an  $(n+1)$ -st root of unity, since both  $\begin{bmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{bmatrix} C_T$  and  $C_T$  have determinant 1.

Consequently, for every  $i = 1, \dots, n^2$ ,

$$\text{Trace} \left( \begin{bmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{bmatrix} C_T \right) = \omega_i(T) \in \Omega_{n+1}.$$

Therefore, for every  $T \in \tilde{\mathcal{G}}$ ,  $C_T$  can be interpreted as a solution of a system

$$(3.11) \quad \left\{ \text{Trace} \left( \begin{bmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{bmatrix} Z \right) = \omega_i(T); \quad i = 1, \dots, n^2, \right.$$



of  $n^2$  linearly independent equations in  $(n+1)^2$  variable entries of  $Z \in \mathbb{M}_{n+1}$ .

Writing  $Z$  as  $\begin{bmatrix} K & u \\ v & t \end{bmatrix}$  allows us to express the linear system (3.11) as

$$(3.12) \quad \left\{ \text{trace}(T_i K) = (n+1)\omega_i(T) - \frac{t}{\det T_i}; \quad i = 1, \dots, n^2, \right.$$

making it apparent that the only relevant variables are the complex scalar  $t$  and the  $n^2$  entries of  $K$ .

Since  $T_1, \dots, T_{n^2}$  are linearly independent, for each choice of a complex  $t$  there is a unique  $K_{(t)} \in \mathbb{M}_n$  that is a solution to system (3.12). We shall write  $K_T$  for  $K_{(t)}$ , and observe that  $\begin{bmatrix} K_T & 0 \\ 0 & 0 \end{bmatrix}$  is a solution to system (3.11).

Since  $\left\{ (n+1)\omega_i(T) \mid T \in \tilde{\mathcal{G}} \right\}$  is a subset of a finite set  $(n+1)\Omega_{n+1}$ , the set  $\left\{ K_T \mid T \in \tilde{\mathcal{G}} \right\}$  must be finite.

Furthermore,  $C_T - \begin{bmatrix} K_T & 0 \\ 0 & 0 \end{bmatrix}$  is a solution of the homogenous system

$$(3.13) \quad \left\{ \text{trace} \left( \begin{bmatrix} T_i & 0 \\ 0 & \frac{1}{\det T_i} \end{bmatrix} Z \right) = 0; \quad i = 1, \dots, n^2, \right.$$

of  $n^2$  linearly independent equations in  $(n+1)^2$  variable entries of  $Z \in \mathbb{M}_{n+1}$ .

Let us compute:

$$(3.14) \quad C_T - \begin{bmatrix} K_T & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} A - \begin{bmatrix} K_T & 0 \\ 0 & 0 \end{bmatrix}$$

$$(3.15) \quad = \begin{bmatrix} T & 0 \\ 0 & \frac{1}{\det T} \end{bmatrix} \begin{bmatrix} B_\circ & x_\circ \\ y_\circ & a_\circ \end{bmatrix} - \begin{bmatrix} K_T & 0 \\ 0 & 0 \end{bmatrix}$$

$$(3.16) \quad = \begin{bmatrix} TB_\circ - K_T & Tx_\circ \\ \frac{y_\circ}{\det T} & \frac{a_\circ}{\det T} \end{bmatrix}.$$

Using the matricial form of  $Z$ , system (3.13) can be expressed as

$$(3.17) \quad \left\{ \frac{t}{\det T_i} + \text{trace}(T_i K) = 0; \quad i = 1, \dots, n^2. \right.$$

The dimension of the solution space of system (3.13) is  $(n+1)^2 - n^2$ , i.e.,  $2n+1$ . Using (3.17), we can see that

$$\left\{ \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix} \mid u, v^T \in \mathbb{M}_{n \times 1} \right\}$$

is a  $2n$ -dimensional subspace of the solution space of system (3.13), and therefore, the system has a non-zero solution of the form  $\begin{bmatrix} K_* & 0 \\ 0 & t_* \end{bmatrix}$ .

If  $t_* = 0$ , then  $K_*$  satisfies the equations

$$\text{trace}(T_i K_*) = 0; \quad i = 1, \dots, n^2,$$

which can only happen if  $K_* = 0$ , because  $T_1, \dots, T_{n^2}$  are linearly independent. Therefore,  $t_* \neq 0$ , and so after scaling we see that system (3.13) has a solution of the form  $\begin{bmatrix} K_o & 0 \\ 0 & 1 \end{bmatrix}$ , so that

$$\left\{ \begin{bmatrix} \gamma K_o & u \\ v & \gamma \end{bmatrix} \mid \gamma \in \mathbb{C}, u, v^T \in \mathbb{M}_{n \times 1} \right\}$$

is the general solution of system (3.13).

It follows from (3.16) that

$$\begin{bmatrix} TB_o - K_T & Tx_o \\ \frac{y_o}{\det T} & \frac{a_o}{\det T} \end{bmatrix} \in \left\{ \begin{bmatrix} \gamma K_o & u \\ v & \gamma \end{bmatrix} \mid \gamma \in \mathbb{C}, u, v^T \in \mathbb{M}_{n \times 1} \right\},$$

so that

$$(3.18) \quad TB_o = \frac{a_o}{\det T} K_o + K_T, \text{ for every } T \in \tilde{\mathcal{G}}.$$

Letting  $T = I$ , we obtain

$$B_o = a_o K_o + K_I,$$

so that

$$a_o K_o = B_o - K_I.$$

Thus, (3.18) can be rewritten as:

$$(3.19) \quad TB_o = K_T + \frac{B_o - K_I}{\det T}, \text{ for every } T \in \tilde{\mathcal{G}}.$$

Using equation (3.19) we arrive at the following identity for all  $S, T \in \tilde{\mathcal{G}}$ :

$$\begin{aligned} K_{ST} + \frac{B_o - K_I}{\det ST} &= (ST)B_o = S(TB_o) = SK_T + \frac{SB_o - SK_I}{\det T} \\ &= SK_T + \frac{1}{\det T} \left( K_S + \frac{B_o - K_I}{\det S} - SK_I \right) \\ &= SK_T + \frac{K_S - SK_I}{\det T} + \frac{B_o - K_I}{\det ST} \end{aligned}$$

from which it follows that

$$(3.20) \quad K_{ST} = SK_T + \frac{K_S - SK_I}{\det T},$$

for all  $S, T \in \tilde{\mathcal{G}}$ . There are two possibilities.

*Case 1.* There exists an  $S_o \in \tilde{\mathcal{G}}$  such that  $K_{S_o} \neq S_o K_I$ .

In this case, for all  $T \in \tilde{\mathcal{G}}$ :

$$(3.21) \quad K_{S_o T} - S_o K_T = \frac{K_{S_o} - S_o K_I}{\det T} \neq 0.$$

Recall that the set  $\{ K_P \mid P \in \tilde{\mathcal{G}} \}$  is finite, and consequently, so is the set

$$\left\{ K_{S_o T} - S_o K_T \mid T \in \tilde{\mathcal{G}} \right\}.$$

Thus, from (3.21) we see that  $\left\{ \det T \mid T \in \tilde{\mathcal{G}} \right\}$  is also finite, and the same is true for  $\left\{ TB_{\circ} \mid T \in \tilde{\mathcal{G}} \right\}$ , since

$$TB_{\circ} = K_T + \frac{B_{\circ} - K_I}{\det T},$$

by (3.19).

When  $B_{\circ} \neq 0$ , which certainly holds true when  $n > 1$  (since  $A$  is invertible),  $\tilde{\mathcal{G}}$  is finite by Corollary 5 in [10], and so  $\mathcal{G}$  is finite as well.

If  $n = 1$  and  $B_{\circ} = 0$ , then (since  $\det A = 1$ ):

$$A = \begin{bmatrix} 0 & x_{\circ} \\ -\frac{1}{x_{\circ}} & a_{\circ} \end{bmatrix},$$

and the characteristic polynomial of  $A$  is

$$p(z) = z^2 - \text{trace}(A)z + \det A = z^2 - a_{\circ}z + 1.$$

Since  $A$  is monopotent,  $a_{\circ} = \pm 2$ . The characteristic polynomial of  $\begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & x_{\circ} \\ -\frac{1}{x_{\circ}} & a_{\circ} \end{bmatrix}$  is

$$p_g(z) = z^2 - (\pm 2)z + g,$$

which is a perfect square (for the sake of monopotency of the matrix) if and only if  $g = 1$ . Thus,  $\mathcal{G} = \{1\}$ , and we are done with the first case.

*Case 2.* For all  $S \in \tilde{\mathcal{G}} : K_S = SK_I$ .

In this case, (3.19) states that for all  $T \in \tilde{\mathcal{G}}$ :

$$TB_{\circ} = TK_I + \frac{B_{\circ} - K_I}{\det T},$$

and therefore,

$$(3.22) \quad (\det(T)T)(B_{\circ} - K_I) = (B_{\circ} - K_I).$$

Since  $\mathcal{G} = \left\{ \det(T)T \mid T \in \tilde{\mathcal{G}} \right\}$ , equation (3.22) states that

$$G(B_{\circ} - K_I) = (B_{\circ} - K_I), \quad \text{for all } G \in \mathcal{G}.$$

Thus, the non-zero columns of  $B_{\circ} - K_I$  are common eigenvectors of the elements of the irreducible group  $\mathcal{G}$ , whose elements obviously have no common eigenvectors. Thus,  $B_{\circ} = K_I$ , so that for all  $T \in \tilde{\mathcal{G}}$ :

$$TB_{\circ} = K_T \in \left\{ K_P \mid P \in \tilde{\mathcal{G}} \right\},$$

and the rightmost set is finite. Again, when  $B_{\circ} \neq 0$ , which holds true when  $n > 1$ ,  $\tilde{\mathcal{G}}$  is finite by Corollary 5 in [10], so that  $\mathcal{G}$  is finite as well, and we have already dealt with the case of  $n = 1$  and  $B_{\circ} = 0$  above. This completes the proof of the present case and hence of the theorem.  $\square$

**COROLLARY 3.2.** *Suppose that all elements of a subgroup  $\mathcal{H}$  of  $\mathbb{GL}_{n+1}$  have the form  $\begin{bmatrix} G & 0 \\ 0 & * \end{bmatrix}$ , where the matrices  $G$  comprise an irreducible subgroup  $\mathcal{G}$  of  $\mathbb{GL}_n$ .*

*If  $\mathcal{H}$  acts monopotently on an invertible matrix  $A$ , then  $\mathcal{H}$  is essentially finite.*

*Proof.* By our hypothesis,

$$\mathcal{G} \stackrel{\text{def}}{=} \left\{ G \mid \begin{bmatrix} G & 0 \\ 0 & \alpha \end{bmatrix} \in \mathcal{H}, \text{ for some } \alpha \in \mathbb{C} \right\}$$

is an irreducible group.

Let us define

$$\mathcal{H}_1 \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix} \mid \begin{bmatrix} \alpha T & 0 \\ 0 & \alpha \end{bmatrix} \in \mathcal{H}, \text{ for some } \alpha \in \mathbb{C}^* \right\}$$

It is easy to check that  $\mathcal{H}_1$  is a group, and that  $\mathbb{C}^*\mathcal{H} = \mathbb{C}^*\mathcal{H}_1$ . In particular,  $\mathcal{H}_1$  acts monopotently on  $A$ .

Let  $\mathcal{G}_1$  be the compression of  $\mathcal{H}_1$  to  $\mathcal{M}$ ; i.e.,

$$\mathcal{G}_1 = \left\{ T \mid \begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{H}_1 \right\}.$$

Then  $\mathbb{C}^*\mathcal{G} = \mathbb{C}^*\mathcal{G}_1$ , and consequently,  $\mathcal{G}_1$  is an irreducible group.

By Theorem 3.1,  $\mathcal{G}_1$  is finite, and so  $\mathcal{H}_1$  is finite. Since

$$\mathcal{H} \subset \mathbb{C}^*\mathcal{H} = \mathbb{C}^*\mathcal{H}_1,$$

our proof is complete. □

#### 4. A case of block-upper triangular groups.

LEMMA 4.1. *Suppose that  $\mathcal{G}$  is a subgroup of  $\mathbb{GL}_n$  and  $\varphi : \mathcal{G} \rightarrow \mathbb{C}^n$ . Then the following are equivalent:*

1.  $\mathcal{H} \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} G & \varphi(G) \\ 0 & 1 \end{bmatrix} \mid G \in \mathcal{G} \right\}$  is a subgroup of  $\mathbb{GL}_{n+1}$ ;
2.  $\varphi(AB) = A\varphi(B) + \varphi(A)$ , for any  $A, B \in \mathcal{G}$ .

*Proof.* The forward implication is trivial. To see the reverse implication, observe that under hypothesis (2)  $\mathcal{H}$  is a semigroup of invertible matrices, and hence, it is sufficient to show that  $\mathcal{H}$  is closed under inversion. Clearly,

$$\begin{bmatrix} G & \varphi(G) \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} G^{-1} & -G^{-1}\varphi(G) \\ 0 & 1 \end{bmatrix},$$

and we would like to demonstrate that

$$\varphi(G^{-1}) = -G^{-1}\varphi(G),$$

for  $G \in \mathcal{G}$ . Let  $A = G^{-1}$  and  $B = G$  in our hypothesis (2), and arrive at

$$\varphi(I) = G^{-1}\varphi(G) + \varphi(G^{-1}).$$

The proof will be complete as soon as we show that  $\varphi(I) = 0$ . This is accomplished by letting  $A = I = B$  in our hypothesis (2). □

LEMMA 4.2. *Suppose that all elements of a subgroup  $\mathcal{H}$  of  $\mathbb{GL}_{n+1}$  have the form  $\begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix}$ , where the matrices  $G$  comprise a subgroup  $\mathcal{G}$  of  $\mathbb{GL}_n$ . Let*

$$\mathcal{N}_\circ = \left\{ x \in \mathbb{C}^n \mid \begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix} \in \mathcal{H} \right\}.$$

*Then  $\mathcal{N}_\circ$  contains zero, is closed under addition and subtraction, and is invariant (as a set) under  $\mathcal{G}$ .*

*Consequently, since the span of  $\mathcal{N}_\circ$  is invariant under  $\mathcal{G}$ , if  $\mathcal{G}$  is irreducible, then either  $\mathcal{N}_\circ = \{0\}$ , or  $\mathcal{N}_\circ$  spans  $\mathbb{C}^n$ .*

*Proof.* The first part of the claim follows immediately from the observations that

$$\begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & x+y \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} I & -x \\ 0 & 1 \end{bmatrix}.$$

Let us also note that

$$\begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} G & Gy+x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} G^{-1} & -G^{-1}x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & Gy \\ 0 & 1 \end{bmatrix},$$

which shows that  $\mathcal{N}_\circ$  is invariant under  $\mathcal{G}$ . □

LEMMA 4.3. *Suppose that a group  $\mathcal{H}$  satisfying the hypotheses of Lemma 4.2 acts monopotently on an invertible matrix  $A \in \mathbb{M}_{n+1}$ , where  $n \geq 2$ .*

*If  $\mathcal{G}$  is irreducible then there is a function  $\varphi : \mathcal{G} \rightarrow \mathbb{C}^n$  such that*

$$\mathcal{H} = \left\{ \begin{bmatrix} G & \varphi(G) \\ 0 & 1 \end{bmatrix} \mid G \in \mathcal{G} \right\}.$$

*Proof.* Scaling  $A$  if necessary, we can assume without loss of generality that  $\det A = 1$ . Thus,  $A = \omega I + N$ , for some nilpotent  $N$  and some  $\omega \in \Omega_{n+1}$ . The same holds true for every  $\begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix} A$ , where  $x \in \mathcal{N}_\circ$ . Let us write

$$A = \begin{bmatrix} B_\circ & z_\circ \\ v_\circ & a_\circ \end{bmatrix},$$

so that

$$\text{trace} \left( \begin{bmatrix} I & x \\ 0 & 1 \end{bmatrix} A \right) = \text{trace}(A) + \text{trace}(xv_\circ) = \text{trace}(A) + v_\circ x.$$

It follows that for every  $x \in \mathcal{N}_\circ$ :

$$v_\circ x \in (n+1)(\Omega_{n+1} - \Omega_{n+1}),$$

and the set on the right is finite and independent of  $x$ .

Yet as we have seen in Lemma 4.2, since  $\mathcal{N}_\circ$  is closed under addition, either  $\mathcal{N}_\circ = \{0\}$  or  $\mathcal{N}_\circ$  spans  $\mathbb{C}^n$ . Hence, either  $\mathcal{N}_\circ = \{0\}$  or  $v_\circ = 0$ .

If  $v_\circ = 0$ , then the irreducible group  $\mathcal{G}$  acts monopotently on the invertible matrix  $B_\circ$ , which is not possible by Corollary 4.6 of [4]. Hence, it must be that  $\mathcal{N}_\circ = \{0\}$ .

In this case, the identity

$$\begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} G & y \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} G^{-1} & -G^{-1}y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & x-y \\ 0 & 1 \end{bmatrix}$$

demonstrates that for each  $G \in \mathcal{G}$  there is a unique  $x$  such that  $\begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix} \in \mathcal{H}$ , which is equivalent to the desired conclusion.  $\square$

**THEOREM 4.4.** *Suppose that all elements of a subgroup  $\mathcal{H}$  of  $\mathbb{GL}_{n+1}$  have the form  $\begin{bmatrix} G & x \\ 0 & 1 \end{bmatrix}$ , where the matrices  $G$  comprise a bounded irreducible subgroup  $\mathcal{G}$  of  $\mathbb{GL}_n$ , and  $n \geq 2$ .*

*If  $\mathcal{H}$  acts monopotently on an invertible matrix, then  $\mathcal{H}$  is finite; (equivalently: a simultaneous similarity applied to  $\mathcal{H}$  produces  $U \oplus 1$ , where  $U$  is a finite unitary group).*

*Proof.* A well-known theorem of Auerbach (see, for example, Theorem 3.1.5 in [9]) states that every bounded subgroup of  $\mathbb{M}_n(\mathbb{C})$  is simultaneously similar to a group of unitary matrices. After applying to  $\mathcal{H}$  a similarity of the form

$$\begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix}^{-1} ( ) \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix},$$

we may assume that  $\mathcal{G}$  is an irreducible unitary group.

Suppose that  $\mathcal{H}$  acts monopotently on a matrix  $A_o$ , and we express  $A_o$  as  $\begin{bmatrix} B_o & z_o \\ v_o & a_o \end{bmatrix}$ , assuming without loss of generality that  $\det A_o = 1$ .

Since we know from [4] that irreducible subgroups of  $\mathbb{GL}_n$  do not act monopotently on invertible matrices (for  $n \geq 2$ ), we see that  $v_o \neq 0$  in our case.

By Lemmas 4.1 and 4.3, there is a function  $\varphi : \mathcal{G} \rightarrow \mathbb{C}^n$  such that

$$(4.23) \quad \mathcal{H} = \left\{ \begin{bmatrix} U & \varphi(U) \\ 0 & 1 \end{bmatrix} \mid U \in \mathcal{G} \right\},$$

and

$$(4.24) \quad \varphi(UW) = U \varphi(W) + \varphi(U),$$

for any  $U, W \in \mathcal{G}$ .

Let us write

$$H_U \stackrel{\text{def}}{=} \begin{bmatrix} U & \varphi(U) \\ 0 & 1 \end{bmatrix}, \quad D_U \stackrel{\text{def}}{=} \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad N_U \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \varphi(U) \\ 0 & 0 \end{bmatrix}.$$

Clearly,

$$(4.25) \quad \text{trace}(H_U A_o) - \text{trace}(D_U A_o) = \text{trace}(N_U A_o) = \text{trace}(\varphi(U)v_o) = v_o \varphi(U).$$

Let us denote by  $\lambda_U$  the eigenvalue of the monopotent matrix  $H_U A_o$ . Since

$$|\det H_U| = |\det U| = 1 = \det A_o,$$

we see that

$$|\lambda_U| = 1 \quad \text{and} \quad |\text{trace}(H_U A_o)| \leq n + 1.$$

Since

$$|\text{trace}(D_U A_o)| \leq \|D_U\| \cdot \|A_o\|_{t-n} = \text{trace}\left(\sqrt{A_o^* A_o}\right) \leq (n + 1),$$

we see that  $\{ v_o\varphi(U) \mid U \in \mathcal{G} \}$  is a bounded set. Of course

$$\begin{aligned} \{ v_o\varphi(U) \mid U \in \mathcal{G} \} &= \{ v_o\varphi(WU) \mid U, W \in \mathcal{G} \} \\ &\stackrel{\text{via (4.24)}}{=} \left\{ v_o \left( W\varphi(U) + \varphi(W) \right) \mid U, W \in \mathcal{G} \right\} \\ &= \{ (v_oW)\varphi(U) + v_o\varphi(W) \mid U, W \in \mathcal{G} \}, \end{aligned}$$

and since  $\{ v_o\varphi(W) \mid W \in \mathcal{G} \}$  is bounded, we conclude that

$$(4.26) \quad \{ (v_oW)\varphi(U) \mid U, W \in \mathcal{G} \} \text{ is a bounded set.}$$

Since  $\mathcal{G}$  is irreducible and  $v_o \neq 0$ , and

$$\text{span} \{ (v_oW)^* \} = \text{span} \{ W^* v_o^* \} = \text{span} \{ W^{-1} v_o^* \},$$

which is a non-zero invariant subspace of  $\mathcal{G}$ , we can conclude that

$$\text{span} \{ (v_oW)^* \} = \mathbb{C}^n,$$

and therefore,  $\{ v_oW \mid W \in \mathcal{G} \}$  contains a basis  $R_1, \dots, R_n$  of  $\mathbb{M}_{1 \times n}$ . Let  $T \in \mathbb{M}_n$  be the invertible matrix with rows  $R_1, \dots, R_n$ . Then, by (4.26),

$$\{ T\varphi(U) \mid U \in \mathcal{G} \} \text{ is a bounded set.}$$

Since

$$\|\varphi(U)\| = \|T^{-1}T\varphi(U)\| \leq \|T^{-1}\| \cdot \|T\varphi(U)\|,$$

it follows that  $\{ \varphi(U) \mid U \in \mathcal{G} \}$  is a bounded set, and therefore,  $\mathcal{H}$  is bounded (see (4.23)).

Applying Auerbach's theorem to  $\mathcal{H}$ , we see that after an application of a similarity, we can take  $\mathcal{H}$  to be a group of some unitary matrices in  $\mathbb{M}_{n+1}$ . On the other hand, an application of a similarity does not change the fact that  $\mathcal{H}$  has an invariant subspace  $\mathcal{M}$  of dimension  $n$  that is minimal among non-trivial invariant subspaces of  $\mathcal{H}$ . Since  $\mathcal{H}$  is a group of unitaries, every invariant subspace  $\mathcal{L}$  of  $\mathcal{H}$  is reducing (with  $\mathcal{L}^\perp$  also being invariant). It follows that we can take  $\mathcal{H}$  to have a block-diagonal form with respect to the decomposition

$$\mathbb{C}^{n+1} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

Recalling that  $\mathcal{H}$  has a common eigenvector for the eigenvalue 1, i.e., a "fixed" non-zero vector, and that  $\mathcal{H}|_{\mathcal{M}}$  is irreducible, and that  $n \geq 2$ , we deduce that  $\mathcal{H} = \mathcal{H}|_{\mathcal{M}} \oplus 1$ .

To complete the proof we now simply apply Theorem 3.1 to  $\mathcal{H}$ . □

Notice that the example of  $\mathcal{G} = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in \mathbb{C} \right\}$  and  $A = I_2$  shows that the hypothesis " $n \geq 2$ " cannot be removed from Lemma 4.3 and Theorem 4.4.

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