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ON ONE-SIDED (B, C)-INVERSES OF ARBITRARY MATRICES*

JULIO BENíTEZ†, ENRICO BOASSO‡, AND HONGWEI JIN§

Abstract. In this article, one-sided (b, c)-inverses of arbitrary matrices as well as one-sided inverses along a (not necessarily square) matrix, will be studied. In addition, the (b, c)-inverse and the inverse along an element will be also researched in the context of rectangular matrices.

Key words. One-sided (b, c)-inverse, One-sided inverse along an element, (b, c)-inverse, Inverse along an element, Matrix.

AMS subject classifications. 15A09, 15A23, 15A60, 65F99.

1. Introduction and notation. Several generalized inverses have been studied in the literature. Recently, two important outer inverses have been introduced: the inverse along an element (see [15]) and the (b, c)-inverse (see [9]). In fact, these two generalized inverses encompass some of the most important outer inverses such as the group inverse, the Drazin inverse and the Moore-Penrose inverse. Furthermore, in the context of semigroups the left and right inverses along an element were defined in [25]; these notions extend the inverse along an element. Similarly, in the frame of rings, left and right (b, c)-invertible elements were introduced in [14]; these definitions extend both the (b, c)-inverse and the left and right inverses along an element.

As it has been said, the aforementioned outer inverses and their extensions were defined in semigroups or rings. However, observe that the set of $n \times m$ complex matrices is not a semigroup (unless $n = m$). The main purpose of this article is to extend the above mentioned (one-sided) inverses as well as the (b, c)-inverse and the inverse along an element to arbitrary matrices and to study their basic properties.

In Section 3, after having recalled the main notions considered in this article in Section 2, the one-sided (b, c)-inverses and the left and right inverses along an element in the context of arbitrary matrices will be thoroughly studied. In Sections 4 and 5, the (b, c)-inverse and the inverse along an element will be introduced and studied in the same frame, respectively. In Section 6, it will be characterized when the generalized inverses introduced in Sections 4 and 5 are inner inverses. In Section 7, the relationships among the notions considered in Sections 4 and 5 and the outer inverse with prescribed range and null space will be studied. In Section 8, the continuity and the differentiability of the notions introduced in Sections 4 and 5 will be considered. Finally, in Section 9, algorithms to compute the (b, c)-inverse in the matrix frame will be given.

Before going on, the definition of several generalized inverses in the context of rings will be given. The corresponding definitions for complex matrices can be obtained making obvious changes. Let $\mathcal{R}$ be a unitary ring and $a \in \mathcal{R}$.

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(i) The element a is said to be *group invertible*, if there exists $x \in \mathcal{R}$ such that $axa = a$, $xax = x$, and $ax = xa$. This x is unique and it is denoted by $a^{\#}$.
(ii) The element a is said to be *Drazin invertible*, if there exists $x \in \mathcal{R}$ such that $xax = x$, $xa = ax$, and $a^{n+1}x = a^n$, for some $n \in \mathbb{N}$. This x is unique and it is denoted by a^d. Note that when $n = 1$, the group inverse is obtained (see [8]).
(iii) Let $\mathcal{R}$ have an involution. The element a is said to be *Moore-Penrose invertible*, if exists $x \in \mathcal{R}$ such that $axa = a$, $xax = x$, $(ax)^* = ax$, and $(xa)^* = xa$. This x is unique and it is denoted by $a^\dagger$ (see [17]).
(iv) Let $\mathcal{R}$ have an involution and let $m, n \in \mathcal{R}$ be invertible Hermitian elements in $\mathcal{R}$. The element $a \in \mathcal{R}$ is said to be *Moore-Penrose invertible with weights* m, n, if there exists $x \in \mathcal{R}$ such that $axa = a$, $xax = x$, $(max)^* = max$, $(nxa)^* = nxa$. This x is unique and it is denoted by $a^\dagger_{m,n}$. In a ring $\mathcal{R}$ with an involution, an element $u \in \mathcal{R}$ is said to be *positive*, if there exists a Hermitian $v \in \mathcal{R}$ such that $u = v^2$.
(v) Let $\mathcal{R}$ have an involution. The element a is said to be *core invertible* (*dual core invertible*, respectively), if there exists $x \in \mathcal{R}$ such that $axa = a$, $x\mathcal{R} = a\mathcal{R}$ and $\mathcal{R}x = \mathcal{R}a^*$ ($axa = a$, $x\mathcal{R} = a^*\mathcal{R}$ and $\mathcal{R}x = \mathcal{R}a$, respectively). This x is unique and it is denoted by $a^{\oplus}$ ($a_{\oplus}$, respectively). (see [1, 19]).

To end this section, some notation is introduced. Let $m, n \in \mathbb{N}$ and denote by $\mathbb{C}_{m,n}$ the set of $m \times n$ complex matrices. The symbol $\mathbb{C}_n$ will stand for $\mathbb{C}_{n,n}$. Any vector of the space $\mathbb{C}^n$ will be considered as a column vector.

Moreover, I_n will mean the identity matrix of order n, $\mathrm{rk}(A)$ the rank of $A \in \mathbb{C}_{m,n}$, and when $n = m$, $\mathrm{tr}(A)$ will stand for the trace of A. Related to a matrix $A \in \mathbb{C}_{m,n}$, there are two linear subspaces, the column space and the null space, which are defined respectively by $\mathcal{R}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{C}^n\} \subseteq \mathbb{C}^m$ and $\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{C}^n$. Given a linear mapping $f : \mathbb{C}^n \to \mathbb{C}^m$, the subsets $\mathcal{R}(f)$ and $\mathcal{N}(f)$ are defined in a similar way. In addition, the conjugate transpose of the matrix A will be denoted by A^*. Two basic equalities are $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp$ and $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp$, for $A \in \mathbb{C}_{n,m}$.

If $\mathcal{M}$ is a subspace of $\mathbb{C}^n$, the symbol $I_\mathcal{M}$ will stand for the identity linear transformation on $\mathcal{M}$ and $P_\mathcal{M}$ for the orthogonal projector onto $\mathcal{M}$. When $\mathcal{N}$ and $\mathcal{M}$ are two subspaces of $\mathbb{C}^n$, $P_{\mathcal{M},\mathcal{N}}$ will stand for the idempotent whose range is $\mathcal{M}$ and whose null space is $\mathcal{N}$.

Recall that given $X \in \mathbb{C}_{n,m}$, $Y \in \mathbb{C}_{m,n}$ is an *inner inverse of* X, if $XYX = X$. In addition, Y is said to be an *outer inverse of* X, if $YXY = Y$. Next the outer inverse with prescribed range and null space will be recalled.

Let $A \in \mathbb{C}_{n,m}$ and consider subspaces $\mathcal{T} \subseteq \mathbb{C}^m$ and $\mathcal{S} \subseteq \mathbb{C}^n$ such that $\dim \mathcal{T} = s \leq \mathrm{rk}(A)$ and $\dim \mathcal{S} = n - s$. Necessary and sufficient for the matrix A to have an outer inverse Z such that $\mathcal{R}(Z) = \mathcal{T}$ and $\mathcal{N}(Z) = \mathcal{S}$ is that $A(\mathcal{T}) \oplus \mathcal{S} = \mathbb{C}^n$, in which case Z is unique and it is denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$ (see for example [21, Lemma 1.1]).

2. The definition of the one-sided (D, E) inverses and their relationship with other inverses.

Firstly the definition of the (b, c)-inverse will be recalled (see [9, Definition 1.3]).

Definition 2.1. Let $\mathcal{S}$ be a semigroup and consider a, b, $c \in \mathcal{S}$. The element $y \in \mathcal{S}$ will be said to be *the* (b, c)*-inverse* of a, if $y \in (b\mathcal{S}y) \cap (y\mathcal{S}c)$, $b = yab$, and $c = cay$.

According to [9, Theorem 2.1], if the element y in Definition 2.1 exists, then it is unique. In this case, this element will be denoted by $a^{\|(b,c)}$. As it was pointed out in [9], this inverse generalizes among others

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the standard inverse, the Drazin inverse, and the Moore-Penrose inverse. To learn more on this inverse, see [5, 6, 9, 10, 13].

The inverse along an element was introduced in [15, Definition 4]. Next its definition will be recalled.

Definition 2.2. Let $\mathcal{S}$ be a semigroup. An element $a \in \mathcal{S}$ is said to be *invertible along* $d \in \mathcal{S}$ if there exists $y \in \mathcal{S}$ such that $yad = d = day$, $y\mathcal{S} \subseteq d\mathcal{S}$, and $\mathcal{S}y \subseteq \mathcal{S}d$.

According to [15, Theorem 6], if the element $y \in \mathcal{S}$ in Definition 2.2 exists, then it is unique. This element is denoted by $a^{\| d}$. It is worth noting that according to [9, Proposition 6.1], the inverse along an element is a particular case of the (b, c)-inverse, i.e., the (d, d)-inverse coincides with the inverse along d. To learn more on this inverse, see [3, 4, 15, 16].

The outer inverses recalled in Definition 2.1 and Definition 2.2 encompass several generalized inverses, as the following two theorems show.

Theorem 2.3. ([15, Theorem 11]) *Let $\mathcal{S}$ be a semigroup and let $a \in \mathcal{S}$.*

(i) *If $\mathcal{S}$ has a unity, then a is invertible if and only if a is invertible along 1. In this case, $a^{-1} = a^{\| 1}$.*
(ii) *a is group invertible if and only if a is invertible along a. In this case, $a^{\#} = a^{\| a}$.*
(iii) *a is Drazin invertible if and only if a is invertible along a^m for some $m \in \mathbb{N}$. In this case, $a^D = a^{\| a^m}$.*
(iv) *If $\mathcal{S}$ is a $*$-semigroup, a is Moore-Penrose invertible if and only if a is invertible along a^*. In this case, $a^\dagger = a^{\| a^*}$.*

Theorem 2.4. *Let $\mathcal{R}$ be a ring with an involution and $a \in \mathcal{R}$.*

(i) ([19, Theorem 4.3]) *If a is Moore-Penrose invertible, then a is core invertible (dual core invertible, respectively) if and only if it is invertible along aa^* (a^*a, respectively). In this case, the inverse along aa^* (a^*a, respectively) coincides with $a^{\oplus}$ ($a_{\oplus}$, respectively).*
(ii) ([4, Theorem 3.2]) *If $m, n \in \mathcal{R}$ are invertible and positive, then a is weighted Moore-Penrose invertible with weights m and n if and only if a is invertible along $n^{-1}a^*m$. In this case, the inverse along $n^{-1}a^*m$ coincides with $a^\dagger_{m,n}$.*

Recently, the inverse along an element and the (b, c)-inverse were extended by means of one-sided inverses. Next follow the corresponding definitions. See [14, Definition 2.1] and [25, Definition 2.1].

Definition 2.5. Let $\mathcal{R}$ be a ring and let $b, c \in \mathcal{R}$.

(i) An element $a \in \mathcal{R}$ is said to be *left (b, c)-invertible*, if there exists $y \in \mathcal{R}$ such that $yab = b$ and $\mathcal{R}y \subseteq \mathcal{R}c$. In this case y, is called a *left (b, c)-inverse* of a.
(ii) An element $a \in \mathcal{R}$ is *right (b, c)-invertible*, if there exists $y \in \mathcal{R}$ such that $cay = c$ and $y\mathcal{R} \subseteq b\mathcal{R}$. In this case, y is called a *right (b, c)-inverse* of a.

Recall that given a, b, c elements in a ring $\mathcal{R}$, according to [14, Corollary 3.7], a is (b, c)-invertible if and only if it is both left and right (b, c)-invertible. When in Definition 2.5, $b = c$, the one-sided inverses along an element are obtained.

Definition 2.6. Let $\mathcal{S}$ be a semigroup and $d \in \mathcal{S}$.

(i) An element $a \in \mathcal{S}$ is *left invertible along d*, if there exists $y \in \mathcal{S}$ such that $yad = d$ and $\mathcal{S}y \subseteq \mathcal{S}d$. In this case, y is called a *left inverse of a along d*.

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(ii) An element $a \in \mathcal{S}$ is *right invertible along* d, if there exists $y \in \mathcal{S}$ such that $day = d$ and $y\mathcal{S} \subseteq d\mathcal{S}$. In this case, y is called a *right inverse of* a *along* d.

Recall that given a semigroup $\mathcal{S}$ and $a, d \in \mathcal{S}$, according to [25, Corollary 2.5], a is invertible along d if and only if a is left and right invertible along d.

Naturally, since all the inverses that have been considered in this section up to now have been defined in semigroups and rings, they can not be applied to matrices, unless they are square. However, to extend the aforementioned notions to arbitrary matrices, first it is necessary to recall the following facts. Let $U, V \in \mathbb{C}_{m,n}$. There is $X \in \mathbb{C}_m$ (respectively $Y \in \mathbb{C}_n$) such that $U = XV$ (respectively $U = VY$) if and only if $\mathcal{N}(V) \subseteq \mathcal{N}(U)$ (respectively $\mathcal{R}(U) \subseteq \mathcal{R}(V)$). Now with these facts in mind, the notions in Definition 2.5 and Definition 2.6 can be extended to rectangular matrices.

Definition 2.7. Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$.

(i) The matrix A is said to be *left* (D, E)*-invertible*, if there exists $C \in \mathbb{C}_{m,n}$ such that $CAD = D$ and $\mathcal{N}(E) \subseteq \mathcal{N}(C)$. Any matrix C satisfying these conditions is said to be a left (D, E)*-inverse of* A.
(ii) The matrix A is said to be *right* (D, E)*-invertible*, if there exists $B \in \mathbb{C}_{m,n}$ such that $EAB = E$ and $\mathcal{R}(B) \subseteq \mathcal{R}(D)$. Any matrix B satisfying these conditions is said to be a right (D, E)*-inverse of* A.

The proofs of the following results are straightforward and they are left to the reader.

Remark 2.8. Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. The following statements hold.

(i) The matrix A is left (D, E)-invertible (right (D, E)-invertible, respectively) with a left inverse $C \in \mathbb{C}_{m,n}$ (right inverse $B \in \mathbb{C}_{m,n}$, respectively) if and only if A^* is right (E^*, D^*)-invertible (left (E^*, D^*)-invertible, respectively) and C^* is a right (E^*, D^*)-inverse (B^* is a left (E^*, D^*)-inverse, respectively) of A^*.

Consider $D', E' \in \mathbb{C}_{m,n}$ such that $\mathcal{R}(D') = \mathcal{R}(D)$ and $\mathcal{N}(E') = \mathcal{N}(E)$.

(ii) The matrix A is left (D, E)-invertible (right (D, E)-invertible, respectively) if and only if it is left (D', E')-invertible (right (D', E')-invertible, respectively). In addition, in this case, $C \in \mathbb{C}_{m,n}$ is a left (D, E)-inverse (right (D, E)-inverse, respectively) of A if and only if it is a left (D', E')-inverse (right (D', E')-inverse, respectively) of A.

When the matrices $D, E \in \mathbb{C}_{m,n}$ in Definition 2.7 coincide, the notions of left and right inverse along a matrix can be introduced.

Definition 2.9. Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$.

(i) The matrix A is said to be *left invertible along* D, if there exists $C \in \mathbb{C}_{m,n}$ such that $CAD = D$ and $\mathcal{N}(D) \subseteq \mathcal{N}(C)$. Any matrix C satisfying these conditions is said to be a *left inverse of* A *along* D.
(ii) The matrix A is said to be *right invertible along* D, if there exists $B \in \mathbb{C}_{m,n}$ such that $DAB = D$ and $\mathcal{R}(B) \subseteq \mathcal{R}(D)$. Any matrix B satisfying these conditions is said to be a *right inverse of* A *along* D.

Note that similar results to the ones in Remark 2.8 for the case $D = E \in \mathbb{C}_{m,n}$ hold for left and right invertible matrices along a matrix. The details are left to the reader.

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Recall that given a ring $\mathcal{R}$ and $a, b, c \in \mathcal{R}$, in [14, Definition 2.3] the left and right annihilator (b, c)-inverses of the element a were introduced. However, in the case of matrices, as under the conditions of [14, Proposition 2.5], these notions coincide with the ones in Definition 2.7.

3. Characterizations of the one-sided (D, E)-invertibility. Firstly, matrices that satisfy Definition 2.7 will be characterized.

THEOREM 3.1. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *The matrix A is right (D, E)-invertible.*
(ii) $\mathcal{R}(E) = \mathcal{R}(EAD)$.
(iii) $\mathrm{rk}(E) = \mathrm{rk}(EAD)$.
(iv) $\dim \mathcal{N}(E) = \dim \mathcal{N}(EAD)$.
(v) $\mathbb{C}^n = \mathcal{R}(AD) + \mathcal{N}(E)$.
(vi) $\mathbb{C}^m = \mathcal{R}(D) + \mathcal{N}(EA)$ *and* $\mathrm{rk}(EA) = \mathrm{rk}(E)$.

Proof. Firstly, it will be proved that statement (i) implies statement (ii). Assume that there exists a matrix $B \in \mathbb{C}_{m,n}$ such that $EAB = E$ and $\mathcal{R}(B) \subseteq \mathcal{R}(D)$. Recall that the latter condition is equivalent to the fact that there exists $M \in \mathbb{C}_n$ such that $B = DM$. Therefore, $E = EAB = EADM$. In particular, $\mathcal{R}(E) = \mathcal{R}(EAD)$.

Suppose that statement (ii) holds. Thus, there exists $X \in \mathbb{C}_n$ such that $EADX = E$. To prove statement (i), it is enough to define $B = DX$.

Statements (ii), (iii), and (iv) are equivalent. In fact, since $\mathcal{R}(EAD) \subseteq \mathcal{R}(E)$, statements (ii) and (iii) are equivalent. In addition, since $\dim \mathcal{N}(E) + \mathrm{rk}(E) = n = \dim \mathcal{N}(EAD) + \mathrm{rk}(EAD)$, statements (iii) and (iv) are equivalent.

Statements (i) and (v) are equivalent. In fact, according to what has been proved, if statement (i) holds, then there is $M \in \mathbb{C}_n$ such that $E = EADM$. In particular, $\mathcal{R}(I_n - ADM) \subseteq \mathcal{N}(E)$. Since $\mathbf{x} = ADM\mathbf{x} + (\mathbf{x} - ADM\mathbf{x})$ for any $\mathbf{x} \in \mathbb{C}^n$, statement (v) holds. On the other hand, statement (v) implies statement (ii), since $\mathcal{R}(E) = E(\mathbb{C}^n) = E[\mathcal{R}(AD) + \mathcal{N}(E)] = \mathcal{R}(EAD)$.

In this paragraph, it will be proved that statement (i) implies statement (vi). Assume that statement (i) holds. Then there exists a matrix $B \in \mathbb{C}_{m,n}$ such that $E = EAB$ and $\mathcal{R}(B) \subseteq \mathcal{R}(D)$. Since any $\mathbf{y} \in \mathbb{C}^m$ can be written as $\mathbf{y} = BA\mathbf{y} + (\mathbf{y} - BA\mathbf{y})$, the equality $\mathbb{C}^m = \mathcal{R}(D) + \mathcal{N}(EA)$ is obtained. To prove the rank equality, according to statement (iii), $\mathrm{rk}(E) = \mathrm{rk}(EAD) \leq \mathrm{rk}(EA) \leq \mathrm{rk}(E)$.

Finally, it will be proved that statement (vi) implies statement (iii). In fact, if statement (vi) holds, then

$$\mathcal{R}(EA) = EA(\mathbb{C}^m) = EA(\mathcal{R}(D) + \mathcal{N}(EA)) = \mathcal{R}(EAD).$$

Therefore, $\mathrm{rk}(E) = \mathrm{rk}(EA) = \mathrm{rk}(EAD)$. □

THEOREM 3.2. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *The matrix A is left (D, E)-invertible.*
(ii) $\mathcal{N}(D) = \mathcal{N}(EAD)$.
(iii) $\dim \mathcal{N}(D) = \dim \mathcal{N}(EAD)$.
(iv) $\mathrm{rk}(D) = \mathrm{rk}(EAD)$.

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(v) $\mathcal{N}(EA) \cap \mathcal{R}(D) = 0$.
(vi) $\mathcal{R}(AD) \cap \mathcal{N}(E) = 0$ *and* $\mathrm{rk}(D) = \mathrm{rk}(AD)$.

Proof. Recall that according to Remark 2.8 (i), A is left-(D,E)-invertible if and only if A^* is right (E^*, D^*)-invertible. In addition, recall that $\mathcal{N}(X) = \mathcal{R}(X^*)^\perp$, $\mathcal{R}(X) = \mathcal{N}(X^*)^\perp$, and $\mathrm{rk}(X) = \mathrm{rk}(X^*)$ for any matrix X. To conclude the proof, apply Theorem 3.1 to A^*, E^* and D^*, use the above identities and note that since $\mathcal{N}(EAD) \subseteq \mathcal{N}(D)$, statements (ii) and (iii) are equivalent. In addition, note that since $\dim \mathcal{N}(D) + \mathrm{rk}(D) = n = \dim \mathcal{N}(EAD) + \mathrm{rk}(EAD)$, statements (iii) and (iv) are equivalent. ∎

Next given $D, E \in \mathbb{C}_{m,n}$, left and right (D,E)-invertible matrices will be characterized using a particular map.

THEOREM 3.3. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. Let $\mathcal{X}$ be any subspace of $\mathbb{C}^n$ such that $\mathbb{C}^n = \mathcal{N}(E) \oplus \mathcal{X}$. Consider $\phi : \mathcal{R}(D) \to \mathcal{X}$ the map defined by $\phi(\mathbf{x}) = P_{\mathcal{X},\mathcal{N}(E)}(A\mathbf{x})$, for $\mathbf{x} \in \mathcal{R}(D)$. The following statements hold.*

(i) *The matrix A is left (D,E)-invertible if and only if ϕ is injective.*
(ii) *The matrix A is right (D,E)-invertible if and only if ϕ is surjective.*

Proof. First statement (i) will be proved. Observe that $\mathcal{N}(\phi) = \mathcal{R}(D) \cap \mathcal{N}(EA)$. Thus, according to Theorem 3.2, $\mathcal{N}(\phi) = 0$ if and only if A is (D,E)-left invertible.

The assertion (ii) will be proved in this paragraph. Note that $\mathbf{x} \in \mathcal{R}(EAD)$ if and only if exists $\mathbf{y} \in \mathbb{C}^n$ such that

$$\mathbf{x} = E[P_{\mathcal{X},\mathcal{N}(E)}(AD\mathbf{y}) + P_{\mathcal{N}(E),\mathcal{X}}(AD\mathbf{y})].$$

Since $E[P_{\mathcal{X},\mathcal{N}(E)}(AD\mathbf{y}) + P_{\mathcal{N}(E),\mathcal{X}}(AD\mathbf{y})] = E(\phi(D\mathbf{y}))$, the equality $\mathcal{R}(EAD) = E(\mathcal{R}(\phi))$ is obtained. In addition, the linear mapping $f : \mathcal{R}(\phi) \to \mathbb{C}^m$ given by $f(\mathbf{x}) = E\mathbf{x}$ is injective (because $\mathcal{R}(\phi) \subseteq \mathcal{X}$ and $\mathcal{X} \oplus \mathcal{N}(E) = \mathbb{C}^n$), therefore, $\dim \mathcal{R}(\phi) = \dim E(\mathcal{R}(\phi)) = \mathrm{rk}(EAD)$, and thus, ϕ is surjective (which is equivalent to $\dim \mathcal{R}(\phi) = \dim \mathcal{X}$) if and only if $\mathrm{rk}(EAD) = \mathrm{rk}(E)$. However, according to Theorem 3.1, this latter condition is equivalent to the fact that A is right (D,E)-invertible. ∎

In the following theorem matrices satisfying simultaneously Theorem 3.1 and Theorem 3.2 will be studied.

THEOREM 3.4. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *A is left and right (D,E)-invertible.*
(ii) $\mathcal{R}(EAD) = \mathcal{R}(E)$ *and* $\mathcal{N}(EAD) = \mathcal{N}(D)$.
(iii) $\mathcal{R}(AD) \oplus \mathcal{N}(E) = \mathbb{C}^n$ *and* $\mathrm{rk}(D) = \mathrm{rk}(AD)$.
(iv) $\mathcal{R}(D) \oplus \mathcal{N}(EA) = \mathbb{C}^m$ *and* $\mathrm{rk}(E) = \mathrm{rk}(EA)$.
(v) *The map $\phi : \mathcal{R}(D) \to \mathcal{X}$ defined in* Theorem 3.3 *is bijective.*

Furthermore, in this case, $\mathrm{rk}(E) = \mathrm{rk}(D)$.

Proof. Apply Theorem 3.1, Theorem 3.2 and Theorem 3.3. Note also that $\mathrm{rk}(E) = \mathrm{rk}(D)$. Actually, this equality can be derived from the fact that $n = \mathrm{rk}(EAD) + \dim \mathcal{N}(EAD) = \mathrm{rk}(E) + \dim \mathcal{N}(D)$. ∎

Next the left and right (D,E)-inverses of a matrix A satisfying Theorem 3.4 will be characterized.

PROPOSITION 3.5. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ be such that A is both left and right (D,E)-invertible. Then, there exist only one left (D,E)-inverse of A and only one right (D,E)-inverse of A. Moreover, these*

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inverses coincide with the unique matrix $R \in \mathbb{C}_{m,n}$ *satisfying*

$$\mathcal{N}(R) = \mathcal{N}(E), \qquad R\mathbf{y} = f^{-1}(\mathbf{y}), \quad \forall \mathbf{y} \in \mathcal{R}(AD),$$

where $f : \mathcal{R}(D) \to \mathcal{R}(AD)$ *is the isomorphism defined by* $f(\mathbf{x}) = A\mathbf{x}$*.*

Proof. Consider $C \in \mathbb{C}_{m,n}$ a left (D, E)-inverse of A. Then, $CAD = D$ and $\mathcal{N}(E) \subseteq \mathcal{N}(C)$. Let $f : \mathcal{R}(D) \to \mathcal{R}(AD)$ and $g : \mathcal{R}(AD) \to \mathcal{R}(D)$ be given by $f(\mathbf{x}) = A\mathbf{x}$ and $g(\mathbf{y}) = C\mathbf{y}$. If $\mathbf{x} \in \mathcal{R}(D)$, then $gf(\mathbf{x}) = CA\mathbf{x}$ and $\mathbf{x} = D\mathbf{u}$ for some $\mathbf{u} \in \mathbb{C}^n$. From $CAD = D$, it is obtained that $gf(\mathbf{x}) = \mathbf{x}$. In a similar way, $fg = I_{\mathcal{R}(AD)}$ can be proved, and therefore, $g = f^{-1}$.

In this paragraph, it will be proved that $\mathcal{N}(C) = \mathcal{N}(E)$. Since $\mathcal{N}(E) \subseteq \mathcal{N}(C)$ is already known, it is enough to prove the opposite inclusion. Let $\mathbf{x} \in \mathcal{N}(C)$, by Theorem 3.4 (iii), $\mathbf{x}$ can be written as $\mathbf{x} = AD\mathbf{y}+\mathbf{w}$, where $\mathbf{y} \in \mathbb{C}^n$ and $\mathbf{w} \in \mathcal{N}(E)$. Now, $\mathbf{0} = C\mathbf{x} = CAD\mathbf{y} + C\mathbf{w} = D\mathbf{y}$ because $\mathbf{w} \in \mathcal{N}(E) \subseteq \mathcal{N}(C)$. Finally, $\mathbf{x} = AD\mathbf{y} + \mathbf{w} = \mathbf{w} \in \mathcal{N}(E)$.

Now consider $B \in \mathbb{C}_{m,n}$ a right (D, E)-inverse of A. In particular, $EAB = E$ and $\mathcal{R}(B) \subseteq \mathcal{R}(D)$. Let $\mathbf{x} \in \mathcal{N}(E)$. Then, $B\mathbf{x} \in \mathcal{R}(D) \cap \mathcal{N}(EA) = 0$ (Theorem 3.4 (iv)). Thus, $\mathcal{N}(E) \subseteq \mathcal{N}(B)$. The inclusion $\mathcal{N}(B) \subseteq \mathcal{N}(E)$ is evident from $EAB = E$. Therefore, $\mathcal{N}(B) = \mathcal{N}(E)$.

Let $h : \mathcal{R}(AD) \to \mathcal{R}(D)$ and $k : \mathcal{R}(D) \to \mathcal{R}(E)$ defined by $h(\mathbf{y}) = B\mathbf{y}$ and $k(\mathbf{y}) = E\mathbf{y}$. The mapping k is an isomorphism because it is simple to prove in view of Theorem 3.4 that $\mathcal{N}(k) = 0$. Furthermore, $EAB = E$ leads to $kfh = k$, and using that k is an isomorphism, $fh = I_{\mathcal{R}(AD)}$, i.e., $h = f^{-1}$. ∎

Now the relationship between the notions given in Definition 2.9 will be studied. To this end, a characterization of left invertibility along a matrix will be given.

THEOREM 3.6. *Let* $A \in \mathbb{C}_{n,m}$ *and* $D \in \mathbb{C}_{m,n}$*. The following statements are equivalent.*

(i) A *is right invertible along* D*.*
(ii) $\mathcal{R}(D) = \mathcal{R}(DAD)$*.*
(iii) $\mathrm{rk}(D) = \mathrm{rk}(DAD)$*.*
(iv) $\mathbb{C}^n = \mathcal{N}(D) \oplus \mathcal{R}(AD)$*.*
(v) $\mathbb{C}^m = \mathcal{R}(D) \oplus \mathcal{N}(DA)$*.*

Proof. Suppose that statement (i) holds. Then, according to Theorem 3.1 applied to the case $D = E$, statements (ii) and (iii) hold, $\mathrm{rk}(DA) = \mathrm{rk}(D)$, $\mathbb{C}^n = \mathcal{N}(D) + \mathcal{R}(AD)$ and $\mathbb{C}^m = \mathcal{R}(D) + \mathcal{N}(DA)$. Now, since

$$n = \mathrm{rk}(AD) + \dim \mathcal{N}(D) - \dim[\mathcal{R}(AD) \cap \mathcal{N}(D)] = \mathrm{rk}(D) + \dim \mathcal{N}(D) - \dim[\mathcal{R}(AD) \cap \mathcal{N}(D)],$$

$\mathcal{R}(AD) \cap \mathcal{N}(D) = 0$ and statement (iv) holds.

Similarly, since

$$m = \mathrm{rk}(D) + \dim \mathcal{N}(DA) - \dim[\mathcal{R}(D) \cap \mathcal{N}(DA)] = \mathrm{rk}(DA) + \dim \mathcal{N}(DA) - \dim[\mathcal{R}(D) \cap \mathcal{N}(DA)],$$

$\mathcal{R}(D) \cap \mathcal{N}(DA) = 0$ and statement (v) holds.

On the other hand, note that statement (ii) of this theorem (respectively (iii), (iv), (v)) implies statement (ii) (respectively (iii), (iv), (v)) of Theorem 3.1 applied to the case D = E. For statement (v), note also that since $\mathbb{C}^m = \mathcal{R}(D) \oplus \mathcal{N}(DA)$, the equality $\mathrm{rk}(D) = \mathrm{rk}(DA)$ can be obtained. ∎

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It is possible to obtain similar statements for right invertible elements along a matrix, however, as the following theorem shows, left and right inverses along a matrix are equivalent notions.

THEOREM 3.7. *Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *A is right invertible along D.*
(ii) *A is left invertible along D.*
(iii) $\mathcal{N}(D) = \mathcal{N}(DAD)$.
(iv) $\dim \mathcal{N}(D) = \dim \mathcal{N}(DAD)$.
(v) *$AD \in \mathbb{C}_n$ is group invertible and* $\dim \mathcal{N}(D) = \dim \mathcal{N}(AD)$.
(vi) *$DA \in \mathbb{C}_m$ is group invertible and* $\operatorname{rk}(DA) = \operatorname{rk}(D)$.
(vii) *The map $\phi : \mathcal{R}(D) \to \mathcal{X}$ defined in* Theorem 3.3 *for the case $D = E$ is bijective.*

Proof. According to Theorem 3.2 applied to the case $D = E$, statements (ii), (iii) and (iv) are equivalent. In addition, note that statement (iv) is equivalent to Theorem 3.6 (iii). In particular, statements (i) and (ii) are equivalent.

Suppose that statement (i) holds. Then according to Theorem 3.6 (iv), $\mathbb{C}^n = \mathcal{N}(D) \oplus \mathcal{R}(AD)$. Moreover, according to Theorem 3.2 (v), $\operatorname{rk}(D) = \operatorname{rk}(AD)$. However, the latter equality is equivalent to $\mathcal{R}(D) = \mathcal{R}(AD)$, which in turn is equivalent to $\mathcal{N}(D) = \mathcal{N}(AD)$. In particular, $\mathbb{C}^n = \mathcal{N}(AD) \oplus \mathcal{R}(AD)$, i.e., AD is group invertible, and $\dim \mathcal{N}(D) = \dim \mathcal{N}(AD)$.

On the other hand, if statement (v) holds, then $\mathbb{C}^n = \mathcal{N}(AD) \oplus \mathcal{R}(AD)$ and $\mathcal{N}(D) = \mathcal{N}(AD)$ ($\mathcal{N}(AD) \subseteq \mathcal{N}(D)$). Consequently, Theorem 3.6 (iv) holds.

The equivalence between statements (i) and (vi) can be proved by means of a similar argument, using in particular Theorem 3.6 (v) and Theorem 3.1 (vi).

Since statements (i) and (ii) are equivalent, according to Theorem 3.4, statements (i) and (vii) are equivalent. ∎

In the following corollary, the left and the right inverses of a matrix $A \in \mathbb{C}_{n,m}$ that are left or right invertible along $D \in \mathbb{C}_{m,n}$ will be presented.

COROLLARY 3.8. *Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$ such that A is left or right invertible along D. Then, there exists only one left inverse of A along D and only one right inverse of A along D. Moreover, these inverses coincide with the matrix $R \in \mathbb{C}_{m,n}$ satisfying*

$$\mathcal{N}(R) = \mathcal{N}(D), \qquad R\mathbf{y} = f^{-1}(\mathbf{y}), \quad \forall \mathbf{y} \in \mathcal{R}(AD),$$

where $f : \mathcal{R}(D) \to \mathcal{R}(AD)$ is given by $f(\mathbf{x}) = A\mathbf{x}$.

Proof. Apply Theorem 3.7 and Proposition 3.5. ∎

Now the existence of left and right (D, E)-inverses will be studied. To this end, the sets of left and right (D, E)-invertible matrices will be characterized. First of all some notation will be given.

Consider $D, E \in \mathbb{C}_{m,n}$. Let $(\mathbb{C}_{n,m})^{\|D,E}_{left}$ and $(\mathbb{C}_{n,m})^{\|D,E}_{right}$ be the sets of left and right (D, E)-invertible matrices, respectively, i.e.,

$$(\mathbb{C}_{n,m})^{\|D,E}_{left} = \{A \in \mathbb{C}_{n,m} : A \text{ is left } (D, E)\text{-invertible}\},$$
$$(\mathbb{C}_{n,m})^{\|D,E}_{right} = \{A \in \mathbb{C}_{n,m} : A \text{ is right } (D, E)\text{-invertible}\}.$$

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When $D = E \in \mathbb{C}_{m,n}$, the sets of left and right invertible matrices along D, $(\mathbb{C}_{n,m})^{\|D}_{left}$ and $(\mathbb{C}_{n,m})^{\|D}_{right}$ respectively, are introduced.

$$(\mathbb{C}_{n,m})^{\|D}_{left} = (\mathbb{C}_{n,m})^{\|D,D}_{left} = \{A \in \mathbb{C}_{n,m} : A \text{ is left invertible along } D\},$$
$$(\mathbb{C}_{n,m})^{\|D}_{right} = (\mathbb{C}_{n,m})^{\|D,D}_{right} = \{A \in \mathbb{C}_{n,m} : A \text{ is right invertible along } D\}.$$

Next conditions under which the sets $(\mathbb{C}_{n,m})^{\|D,E}_{left}$ and $(\mathbb{C}_{n,m})^{\|D,E}_{right}$ are nonempty will be given.

THEOREM 3.9. *Let $D, E \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) $(\mathbb{C}_{n,m})^{\|D,E}_{left} \neq \emptyset$.
(ii) $\mathrm{rk}(D) \leq \mathrm{rk}(E)$.
(iii) $\dim \mathcal{N}(E) \leq \dim \mathcal{N}(D)$.
(iv) $\dim \mathcal{N}(E) + \mathrm{rk}(D) \leq n$.

Proof. Here, it will be proved that statement (i) implies statement (ii). Suppose that there exist $A \in \mathbb{C}_{n,m}$ and $C \in \mathbb{C}_{m,n}$ such that C is a left (D, E)-inverse of A. In particular, $CAD = D$ and $\mathcal{N}(E) \subseteq \mathcal{N}(C)$. Thus, $\mathrm{rk}(D) \leq \mathrm{rk}(C)$ and $\dim \mathcal{N}(E) \leq \dim \mathcal{N}(C)$. As a result, $\mathrm{rk}(D) \leq \mathrm{rk}(C) \leq \mathrm{rk}(E)$.

Suppose that statement (ii) holds. Let $r = \mathrm{rk}(E)$ and $s = \mathrm{rk}(D)$. Let $X = [\mathbf{x}_1 \cdots \mathbf{x}_n] \in \mathbb{C}_n$ and $Y = [\mathbf{y}_1 \cdots \mathbf{y}_m] \in \mathbb{C}_m$ be two nonsingular matrices such that the last $n - r$ columns of X span $\mathcal{N}(E)$ and the first s columns of Y span $\mathcal{R}(D)$. Define

$$A = X \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} Y^{-1} \in \mathbb{C}_{n,m} \quad \text{and} \quad C = Y \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix} X^{-1} \in \mathbb{C}_{m,n}.$$

If $\mathbf{x} \in \mathcal{N}(E)$, then $\mathbf{x} = \sum_{i=r+1}^{n} \alpha_i \mathbf{x}_i$ for some scalars α_i. Since $s = \mathrm{rk}(D) \leq \mathrm{rk}(E) = r$, the vector $\mathbf{v} = [0 \cdots 0 \ \alpha_{r+1} \cdots \alpha_n]^T \in \mathbb{C}^{n-s}$ can be defined (the superscript T means the transposition). Hence, $\mathbf{x} = X \left[\begin{smallmatrix} \mathbf{0} \\ \mathbf{v} \end{smallmatrix}\right]$, and thus, $C\mathbf{x} = \mathbf{0}$. If $\mathbf{y} \in \mathcal{R}(D)$, then exists $\mathbf{w} \in \mathbb{C}^s$ such that $\mathbf{y} = Y \left[\begin{smallmatrix} \mathbf{w} \\ \mathbf{0} \end{smallmatrix}\right]$ (because the first s columns of Y span $\mathcal{R}(D)$). Now, it is trivial to prove $CA\mathbf{y} = \mathbf{y}$, which implies $CAD = D$ since $\mathbf{y} \in \mathcal{R}(D)$ is arbitrary. Hence, (i) holds.

The remaining equivalences are clear. $\square$

THEOREM 3.10. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) $(\mathbb{C}_{n,m})^{\|D,E}_{right} \neq \emptyset$.
(ii) $\mathrm{rk}(E) \leq \mathrm{rk}(D)$.
(iii) $\dim \mathcal{N}(D) \leq \dim \mathcal{N}(E)$.
(iv) $n \leq \dim \mathcal{N}(E) + \mathrm{rk}(D)$.

Proof. Recall that according to Remark 2.8 (ii), A is right (D, E)-invertible if and only if $A^* \in \mathbb{C}_{m,n}$ is left (E^*, D^*)-invertible (E^*, $D^* \in \mathbb{C}_{n,m}$). Consequently, statement (i) is equivalent to $(\mathbb{C}_{m,n})^{\|E^*,D^*}_{left} \neq \emptyset$, which in turn is equivalent to $\mathrm{rk}(E^*) \leq \mathrm{rk}(D^*)$. However, the latter inequality coincides with statement (ii).

The remaining equivalences are clear. $\square$

Next the case of the left and right inverses along a matrix will be considered.

COROLLARY 3.11. *Let $D \in \mathbb{C}_{m,n}$. Then, $(\mathbb{C}_{m,n})^{\|D}_{left} = (\mathbb{C}_{n,m})^{\|D}_{right} \neq \emptyset$.*

Proof. Apply Theorem 3.7 and Theorem 3.9 or Theorem 3.10 for $D = E$. $\square$

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Next the results in Corollary 3.11 will be extended to the case $\dim \mathcal{N}(E) + \mathrm{rk}(D) = n$ $(D, E \in \mathbb{C}_{m,n})$. Note that this condition is equivalent to $\mathrm{rk}(D) = \mathrm{rk}(E)$, which in turn is equivalent to $\dim \mathcal{N}(D) = \dim \mathcal{N}(E)$, which is also equivalent to $\dim \mathcal{N}(D) + \mathrm{rk}(E) = n$.

Corollary 3.12. *Let $D, E \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) $\mathrm{rk}(E) = \mathrm{rk}(D)$.
(ii) $(\mathbb{C}_{n,m})^{\|D,E}_{left} \neq \emptyset$ *and* $(\mathbb{C}_{n,m})^{\|D,E}_{right} \neq \emptyset$.
(iii) $(\mathbb{C}_{n,m})^{\|D,E}_{left} = (\mathbb{C}_{n,m})^{\|D,E}_{right} \neq \emptyset$.

Proof. To prove the equivalence between statements (i) and (ii), apply Theorem 3.9 and Theorem 3.10.

Suppose that statement (i) holds and consider $A \in (\mathbb{C}_{n,m})^{\|D,E}_{right}$. According to Theorem 3.1 (iii), $\mathrm{rk}(D) = \mathrm{rk}(EAD)$. Thus, $\dim \mathcal{N}(D) = \dim \mathcal{N}(EAD)$. Consequently, according to Theorem 3.2 (iii), $A \in (\mathbb{C}_{n,m})^{\|D,E}_{left}$. A similar argument, using in particular that $\dim \mathcal{N}(D) = \dim \mathcal{N}(E)$, proves that $(\mathbb{C}_{n,m})^{\|D,E}_{left} \subseteq (\mathbb{C}_{n,m})^{\|D,E}_{right}$. On the other hand, if statement (iii) holds, then consider $A \in (\mathbb{C}_{n,m})^{\|D,E}_{left} = (\mathbb{C}_{n,m})^{\|D,E}_{right}$. According to Theorem 3.1 and Theorem 3.2, $\mathrm{rk}(E) = \mathrm{rk}(EAD)$ and $\mathcal{N}(D) = \mathcal{N}(EAD)$. Therefore, $\dim \mathcal{N}(D) + \mathrm{rk}(E) = n$. ◻

Now the case $\mathrm{rk}(D) \neq \mathrm{rk}(E)$ will be presented.

Corollary 3.13. *Let $D, E \in \mathbb{C}_{m,n}$ such that $\mathrm{rk}(D) \neq \mathrm{rk}(E)$. Then, the following statements hold.*

(i) *If* $\mathrm{rk}(D) < \mathrm{rk}(E)$, *then* $(\mathbb{C}_{n,m})^{\|D,E}_{left} \neq \emptyset$ *and* $(\mathbb{C}_{n,m})^{\|D,E}_{right} = \emptyset$.
(ii) *If* $\mathrm{rk}(E) < \mathrm{rk}(D)$, *then* $(\mathbb{C}_{n,m})^{\|D,E}_{right} \neq \emptyset$ *and* $(\mathbb{C}_{n,m})^{\|D,E}_{left} = \emptyset$.

Proof. Apply Theorem 3.9 and Theorem 3.10. ◻

Next representations of the sets $(\mathbb{C}_{n,m})^{\|D,E}_{left}$ and $(\mathbb{C}_{n,m})^{\|D,E}_{right}$ will be given. Before, some notation needs to be introduced.

Let $D, E \in \mathbb{C}_{m,n}$ and consider $A \in (\mathbb{C}_{n,m})^{\|D,E}_{left}$. Then, *the set of all left (D, E)-inverses of A* will be denoted by $\mathcal{I}(A)^{\|D,E}_{left}$. Similarly, when $A \in (\mathbb{C}_{n,m})^{\|D,E}_{right}$, $\mathcal{I}(A)^{\|D,E}_{right}$ will stand for *the set of all right (D, E)-inverses of A*.

In the following theorems representations of $(\mathbb{C}_{n,m})^{\|D,E}_{left}$, $(\mathbb{C}_{n,m})^{\|D,E}_{right}$, $\mathcal{I}(A)^{\|D,E}_{left}$ and $\mathcal{I}(A)^{\|D,E}_{right}$ will be given.

Theorem 3.14. *Let $D, E \in \mathbb{C}_{m,n}$ be such that $(\mathbb{C}_{n,m})^{\|D,E}_{left} \neq \emptyset$. Let $s = \mathrm{rk}(D) \leq \mathrm{rk}(E) = r$. Then*

(i) $A \in (\mathbb{C}_{n,m})^{\|D,E}_{left}$ *if and only if there exist two nonsingular matrices $X \in \mathbb{C}_n$ and $Y \in \mathbb{C}_m$ such that*

$$A = X \begin{array}{c} \\ s \\ r-s \\ n-r \end{array} \begin{array}{c} \begin{array}{cc} s & m-s \end{array} \\ \begin{bmatrix} A_1 & * \\ 0 & * \\ 0 & * \end{bmatrix} \end{array} Y^{-1}, \tag{3.1}$$

where the last $n - r$ columns of X are a basis of $\mathcal{N}(E)$, the first s columns of Y are a basis of $\mathcal{R}(D)$, and A_1 is nonsingular.

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(ii) *Under the conditions in statement* (i), $C \in \mathcal{I}(A)_{left}^{\|D,E}$ *if and only if*

$$C = Y \begin{array}{c} \\ s \\ m-s \end{array} \begin{array}{c} \begin{array}{ccc} s & r-s & n-r \end{array} \\ \left[\begin{array}{ccc} A_1^{-1} & * & 0 \\ 0 & * & 0 \end{array} \right] \end{array} X^{-1}. \tag{3.2}$$

Proof. Assume that $A \in (\mathbb{C}_{n,m})_{left}^{\|D,E}$. Let $X \in \mathbb{C}_n$ be any nonsingular matrix such that the last $n-r$ columns span $\mathcal{N}(E)$. Let $Y \in \mathbb{C}_m$ be any nonsingular matrix such that the first s columns span $\mathcal{R}(D)$. Let us decompose the matrix A as follows:

$$A = X \begin{array}{c} \\ r \\ n-r \end{array} \begin{array}{c} \begin{array}{cc} s & m-s \end{array} \\ \left[\begin{array}{cc} B_1 & * \\ B_2 & * \end{array} \right] \end{array} Y^{-1}. \tag{3.3}$$

Observe that if $\mathbf{y}_i$ is the i-th column of Y, then $A\mathbf{y}_i \in \mathcal{R}(AD)$ for $i = 1, \ldots, s$, because the first s columns of Y span $\mathcal{R}(D)$. From (3.3), it is obtained that $B_2 = 0$, because $\mathcal{R}(AD) \cap \mathcal{N}(E) = 0$ (Theorem 3.2 (vi)).

Let $\mathcal{X}$ be the subspace spanned by the first r columns of X (this subspace satisfies $\mathcal{X} \oplus \mathcal{N}(E) = \mathbb{C}^n$). It is evident that the matrix of the linear mapping $\phi : \mathcal{R}(D) \to \mathcal{X}$ defined in Theorem 3.3 with respect to the considered basis of $\mathcal{R}(D)$ and $\mathcal{X}$ is B_1. By Theorem 3.3 (i), ϕ is injective, thus, $\mathcal{X}$ can be decomposed as $\mathcal{X} = \phi(\mathcal{R}(D)) \oplus \mathcal{X}_1$, and without loss of generality, the first r columns of X can be rearranged so that the first s are a basis of $\phi(\mathcal{R}(D))$. In this way, the decomposition written in the statement (i) is obtained.

Assume that $A \in \mathbb{C}_{n,m}$ is decomposed as in (3.1). As in the previous paragraph, let $\mathcal{X}$ be the subspace spanned by the first r columns of X and consider the mapping $\phi : \mathcal{R}(D) \to \mathcal{X}$ defined in Theorem 3.3. The first s columns of Y is a basis of $\mathcal{R}(D)$ and the first r columns of X is a basis of $\mathcal{X}$. The matrix of ϕ with respect to the aforementioned basis is $\left[\begin{smallmatrix} A_1 \\ 0 \end{smallmatrix}\right]$. Since A_1 is nonsingular, the mapping ϕ is injective, and according to Theorem 3.3 (i), A is left (D,E)-invertible.

Now, it will be proved statement (ii). Let $\mathbf{y}_j$ be the j-th column of Y and $\mathbf{x}_i$ be the i-th column of X.

Assume that $C \in \mathbb{C}_{m,n}$ is a left (D,E)-inverse of A. Let us decompose C as

$$C = Y \begin{array}{c} \\ s \\ m-s \end{array} \begin{array}{c} \begin{array}{ccc} s & r-s & n-r \end{array} \\ \left[\begin{array}{ccc} C_1 & C_2 & C_3 \\ C_4 & C_5 & C_6 \end{array} \right] \end{array} X^{-1}. \tag{3.4}$$

Since $\mathbf{y}_j = Y\left[\begin{smallmatrix} \mathbf{e}_j \\ \mathbf{0} \end{smallmatrix}\right]$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_s\}$ is the standard basis of $\mathbb{C}^s$, from (3.1) and (3.4), it is obtained that $CA\mathbf{y}_j = Y\left[\begin{smallmatrix} C_1A_1\mathbf{e}_j \\ C_4A_1\mathbf{e}_j \end{smallmatrix}\right]$ for any $j = 1, \ldots, s$. From $CAD = D$, it follows that $C_1A_1\mathbf{e}_j = \mathbf{e}_j$ and $C_4A_1\mathbf{e}_j = \mathbf{0}$, for any $j = 1, \ldots, s$. Therefore, $C_1 = A_1^{-1}$ and $C_4 = 0$. Having in mind that the last $n-r$ columns of X is a basis of $\mathcal{N}(E)$ and $\mathcal{N}(E) \subseteq \mathcal{N}(C)$, the decomposition (3.4) yields that C_3 and C_6 are zero matrices.

Assume that C is written as in (3.2). The following two relations will be proved: $CAD = D$ and $\mathcal{N}(E) \subseteq \mathcal{N}(C)$. To prove that $CAD = D$, it is enough to check that $CA\mathbf{y}_j = \mathbf{y}_j$ for $j = 1, \ldots, s$ (because $\mathcal{R}(D)$ is spanned by $\mathbf{y}_1, \ldots, \mathbf{y}_s$), and this trivially follows from (3.1), (3.2), and $\mathbf{y}_j = Y\left[\begin{smallmatrix} \mathbf{e}_j \\ \mathbf{0} \end{smallmatrix}\right]$. To prove that $\mathcal{N}(E) \subseteq \mathcal{N}(C)$, it is enough to prove that $C\mathbf{x}_i = \mathbf{0}$ for $i = r+1, \ldots, n$ (because $\mathcal{N}(E)$ is spanned by the last $n-r$ columns of X). This is trivial in view of (3.2). ∎

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THEOREM 3.15. *Let $D, E \in \mathbb{C}_{m,n}$ be such that $(\mathbb{C}_{n,m})^{\|D,E}_{right} \neq \emptyset$. Let $r = \operatorname{rk}(E) \leq \operatorname{rk}(D) = s$. Then*

(i) *$A \in (\mathbb{C}_{n,m})^{\|D,E}_{right}$ if and only if there exist two nonsingular matrices $X \in \mathbb{C}_n$ and $Y \in \mathbb{C}_m$ such that*

$$A = X \begin{array}{c c} & \begin{array}{ccc} s-r & r & m-s \end{array} \\ \begin{array}{c} r \\ n-r \end{array} & \left[\begin{array}{ccc} 0 & A_2 & * \\ * & * & * \end{array}\right] \end{array} Y^{-1}, \tag{3.5}$$

where the last $n-r$ columns of X are a basis of $\mathcal{N}(E)$, the first s columns of Y are a basis of $\mathcal{R}(D)$, and A_2 is nonsingular.

(ii) *Under the conditions in statement* (i), *$B \in \mathcal{I}(A)^{\|D,E}_{right}$ if and only if*

$$B = Y \begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} s-r \\ r \\ m-s \end{array} & \left[\begin{array}{cc} * & * \\ A_2^{-1} & 0 \\ 0 & 0 \end{array}\right] \end{array} X^{-1}. \tag{3.6}$$

Proof. Assume that $A \in (\mathbb{C}_{n,m})^{\|D,E}_{right}$. Let $X \in \mathbb{C}_n$ be any nonsingular matrix such that the last $n-r$ columns span $\mathcal{N}(E)$ and let $\mathcal{X}$ be the subspace spanned by the first r columns of X. The mapping $\phi\colon \mathcal{R}(D) \to \mathcal{X}$ defined in Theorem 3.3 is surjective, and therefore, $s = \dim \mathcal{R}(D) = \dim \mathcal{N}(\phi) + \dim \mathcal{X} = \dim \mathcal{N}(\phi) + r$. In Theorem 3.3, it was proved that $\mathcal{N}(\phi) = \mathcal{R}(D) \cap \mathcal{N}(EA)$. Let $Y \in \mathbb{C}_m$ be any nonsingular matrix such that the first $s-r$ columns of Y are a basis of $\mathcal{R}(D) \cap \mathcal{N}(EA)$ and the first s columns of Y are a basis of $\mathcal{R}(D)$. Decompose

$$A = X \begin{array}{c c} & \begin{array}{ccc} s-r & r & m-s \end{array} \\ \begin{array}{c} r \\ n-r \end{array} & \left[\begin{array}{ccc} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \end{array}\right] \end{array} Y^{-1}.$$

Let $\mathbf{y}_j$ be the j-th column of Y. For $j = 1, \ldots, s-r$, it is obtained that $EA\mathbf{y}_j = \mathbf{0}$, because the first $s-r$ columns of Y belong to $\mathcal{N}(EA)$, in other words, $A\mathbf{y}_j \in \mathcal{N}(E)$, and thus, A_1 is a zero matrix. The matrix of ϕ with respect to the considered basis of $\mathcal{R}(D)$ and $\mathcal{X}$ is $[A_1\ A_2] = [0\ A_2]$. Since $\phi : \mathcal{R}(D) \to \mathcal{X}$ is surjective, $r = \dim \mathcal{X} = \operatorname{rk}[0\ A_2] = \operatorname{rk}(A_2)$. By recalling that A_2 is an $r \times r$ matrix, A_2 is nonsingular.

Assume that $A \in \mathbb{C}_{n,m}$ is decomposed as in (3.5). As in the previous paragraph, let $\mathcal{X}$ be the subspace spanned by the first r columns of X. The matrix of the mapping $\phi : \mathcal{R}(D) \to \mathcal{X}$ with respect to the considered basis is

$$\begin{array}{c c} & \begin{array}{cc} s-r & r \end{array} \\ r & \left[\begin{array}{cc} 0 & A_2 \end{array}\right] \end{array}.$$

Since the rank of this latter matrix is r (because $A_2 \in \mathbb{C}_r$ is nonsingular) and $\dim \mathcal{X} = r$, the mapping ϕ is surjective. According to Theorem 3.3 (ii), $A \in (\mathbb{C}_{n,m})^{\|D,E}_{right}$.

Assume that $B \in \mathbb{C}_{m,n}$ is a right (D,E)-inverse of A. Decompose

$$B = Y \begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} s-r \\ r \\ m-s \end{array} & \left[\begin{array}{cc} B_1 & B_2 \\ B_3 & B_4 \\ B_5 & B_6 \end{array}\right] \end{array} X^{-1}.$$

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By $\mathcal{R}(B) \subseteq \mathcal{R}(D)$, it is obtained that $B\mathbf{x} \in \mathcal{R}(D)$ for any $\mathbf{x} \in \mathbb{C}^n$. Therefore,

$$Y \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \\ B_5 & B_6 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \in \mathcal{R}(D), \quad \forall\, (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{C}^r \times \mathbb{C}^{n-r}.$$

Recall that the first s columns of Y span $\mathcal{R}(D)$, and thus, $B_5\mathbf{x}_1 + B_6\mathbf{x}_2 = \mathbf{0}$, for all $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{C}^r \times \mathbb{C}^{n-r}$, which implies that B_5 and B_6 are zero matrices. The equality $EAB = E$ is equivalent to $\mathcal{R}(AB - I_n) \subseteq \mathcal{N}(E)$. But

$$AB - I_n = X \begin{bmatrix} 0 & A_2 & * \\ * & * & * \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \\ 0 & 0 \end{bmatrix} X^{-1} - XX^{-1} = X \begin{bmatrix} A_2B_3 - I_r & A_2B_4 \\ * & * \end{bmatrix} X^{-1}.$$

Therefore, $X \left[\begin{smallmatrix} A_2B_3 - I_r & A_2B_4 \\ * & * \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{smallmatrix}\right] \in \mathcal{N}(E)$, for all $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{C}^r \times \mathbb{C}^{n-r}$. Recall that the last $n - r$ columns of X span $\mathcal{N}(E)$, which implies that $A_2B_3 = I_r$ and $B_4 = 0$.

Assume that $B \in \mathbb{C}_{m,n}$ is written as in (3.6). It will be proved now that $\mathcal{R}(B) \subseteq \mathcal{R}(D)$. If $\mathbf{x} \in \mathbb{C}^n$, then

$$B\mathbf{x} = Y \begin{bmatrix} * & * \\ A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} X^{-1}\mathbf{x} \in \mathcal{R}(D),$$

because the first s columns of Y belong to $\mathcal{R}(D)$. Now it will be proved that $EAB = B$. Since

$$AB - I_n = X \begin{bmatrix} 0 & A_2 & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ A_2^{-1} & 0 \\ 0 & 0 \end{bmatrix} X^{-1} - XX^{-1} = X \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix} X^{-1},$$

it is obtained that $E(AB - I_n)\mathbf{x} = \mathbf{0}$ for any $\mathbf{x} \in \mathbb{C}^n$ because the last $n - r$ columns of X belong to $\mathcal{N}(E)$. Thus, $\mathcal{R}(AB - I_n) \subseteq \mathcal{N}(E)$, which is equivalent to $EAB = E$. ∎

Next the situation $\operatorname{rk}(E) = \operatorname{rk}(D)$ will be studied, i.e., when $(\mathbb{C}_{n,m})^{\|D,E}_{left} = (\mathbb{C}_{n,m})^{\|D,E}_{right} \neq \emptyset$ $(D, E \in \mathbb{C}_{m,n})$. Compare with Proposition 3.5.

Corollary 3.16. *Let $D, E \in \mathbb{C}_{m,n}$ be such that $\operatorname{rk}(E) = \operatorname{rk}(D) = r$. Let $X \in \mathbb{C}_n$ be any nonsingular matrix such that its last $n - r$ columns span $\mathcal{N}(E)$ and $Y \in \mathbb{C}_m$ be any nonsingular matrix such that its first r columns span $\mathcal{R}(D)$. If $A \in \mathbb{C}_{n,m}$ is written as*

$$A = X \begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ \begin{array}{c} r \\ m-r \end{array} & \begin{bmatrix} A_1 & * \\ * & * \end{bmatrix} \end{array} Y^{-1},$$

then $A \in (\mathbb{C}_{n,m})^{\|D,E}_{left} = (\mathbb{C}_{n,m})^{\|D,E}_{right}$ if and only if A_1 is nonsingular. Furthermore, under this equivalence, the unique left and the unique right (D, E)-inverse of A is

$$Y \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} X^{-1}.$$

Proof. Let $\mathcal{X}$ be the subspace spanned by the first r columns of X. The matrix of the mapping $\phi : \mathcal{R}(D) \to \mathcal{X}$ defined in Theorem 3.3 with respect to the considered basis is A_1. Therefore, according to Theorem 3.4, the matrix A_1 is nonsingular if and only if A is left and right (D, E)-invertible. The proof of Theorem 3.14 (ii) shows that the unique left and right (D, E)-inverse of A has the form of the statement of this corollary. ∎

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REMARK 3.17. Let $D, E \in \mathbb{C}_{m,n}$ be such that $\mathrm{rk}(E) = \mathrm{rk}(D)$ and let $A \in (\mathbb{C}_{n,m})^{\|D,E}_{left} = (\mathbb{C}_{n,m})^{\|D,E}_{right}$. Let $\mathbb{C}^n = \mathcal{N}(E) \oplus \mathcal{X}$ be any decomposition. From Corollary 3.16, the unique left and right (D,E)-inverse of A, say R, satisfies

$$\mathcal{N}(R) = \mathcal{N}(E), \qquad R\mathbf{y} = \phi^{-1}(\mathbf{y}), \quad \forall \mathbf{y} \in \mathcal{X}.$$

Next the case of left and right inverses along a fixed matrix will be considered.

COROLLARY 3.18. *Let $D \in \mathbb{C}_{m,n}$ and $r = \mathrm{rk}(D)$. Let $X \in \mathbb{C}_n$ be any nonsingular matrix such that its last $n-r$ columns span $\mathcal{N}(D)$ and $Y \in \mathbb{C}_m$ be any nonsingular matrix such that its first r columns span $\mathcal{R}(D)$. If $A \in \mathbb{C}_{n,m}$ is written as*

$$A = X \begin{bmatrix} A_1 & * \\ * & * \end{bmatrix} Y^{-1}, \quad A_1 \in \mathbb{C}_r,$$

then $A \in (\mathbb{C}_{n,m})^{\|D}_{left} = (\mathbb{C}_{n,m})^{\|D}_{right}$ if and only if A_1 is nonsingular. Furthermore, under this equivalence, the unique left and the unique right inverse of A along D is

$$Y \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} X^{-1}.$$

Proof. Apply Corollary 3.11 and Corollary 3.16. ∎

In the following theorem, the sets $\mathcal{I}(A)^{\|D,E}_{left}$ and $\mathcal{I}(A)^{\|D,E}_{right}$ will be represented using the Moore-Penrose inverse.

THEOREM 3.19. *Let $D, E \in \mathbb{C}_{m,n}$. The following statements hold.*

(i) *If $A \in (\mathbb{C}_{n,m})^{\|D,E}_{right}$, then*

$$\mathcal{I}(A)^{\|D,E}_{right} = \left\{ D\left[(EAD)^\dagger E + \left(I_n - (EAD)^\dagger EAD\right) Z\right] : Z \in \mathbb{C}_n \right\}.$$

(ii) *If $A \in (\mathbb{C}_{n,m})^{\|D,E}_{left}$, then*

$$\mathcal{I}(A)^{\|D,E}_{left} = \left\{ \left[D(EAD)^\dagger + Z(I_m - EAD(EAD)^\dagger)\right] E : Z \in \mathbb{C}_m \right\}.$$

Proof. Consider $A \in (\mathbb{C}_{n,m})^{\|D,E}_{right}$. If $B \in \mathbb{C}_{m,n}$ satisfies $EAB = E$ and $\mathcal{R}(B) \subseteq \mathcal{R}(D)$, then according to the proof of Theorem 3.1, there exists a matrix M such that $B = DM$ and $EADM = E$. Notice that the general solution of the equation $EADX = E$ is

$$X = (EAD)^\dagger E + (I_n - (EAD)^\dagger EAD)Z,$$

where $Z \in \mathbb{C}_{n,n}$ is arbitrary (see [17, Theorem 2]). Hence, $M = (EAD)^\dagger E + (I_n - (EAD)^\dagger EAD)Z$, for some $Z \in \mathbb{C}_{n,n}$. Therefore, $B = D(EAD)^\dagger E + D(I_n - (EAD)^\dagger EAD)Z$. Thus,

$$\mathcal{I}(A)^{\|D,E}_{right} \subseteq \{D(EAD)^\dagger E + D(I_n - (EAD)^\dagger EAD)Z : Z \in \mathbb{C}_{n,n}\}.$$

To prove the opposite inclusion, let $Z \in \mathbb{C}_{n,n}$ be arbitrary and consider

$$Y = D\left[(EAD)^\dagger E + (I_n - (EAD)^\dagger EAD)Z\right].$$

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It is evident that $\mathcal{R}(Y) \subseteq \mathcal{R}(D)$. Furthermore, according to Theorem 3.1 (ii), $\mathcal{R}(E) = \mathcal{R}(EAD)$. Now

$$EAY = EAD\left[(EAD)^{\dagger}E + (I_n - (EAD)^{\dagger}EAD)Z\right] = EAD(EAD)^{\dagger}E = E.$$

In fact, if $\mathbf{e}$ is any column of E, then $\mathbf{e} \in \mathcal{R}(E) = \mathcal{R}(EAD)$ and since $EAD(EAD)^{\dagger}$ is the orthogonal projection onto $\mathcal{R}(EAD)$, $EAD(EAD)^{\dagger}\mathbf{e} = \mathbf{e}$.

To prove statement (ii), apply Remark 2.8 (i) and what has been proved. ∎

Let $D, E \in \mathbb{C}_{m,n}$. If $A \in \mathbb{C}_{n,m}$ is left (respectively right) (D, E)-invertible, the case in which $\mathcal{I}(A)^{\|D,E}_{left}$ (respectively $\mathcal{I}(A)^{\|D,E}_{right}$) is a singleton will be studied.

Theorem 3.20. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *The matrix A has a unique left (D, E)-inverse.*
(ii) *The matrix A has a unique right (D, E)-inverse.*
(iii) $\operatorname{rk}(D) = \operatorname{rk}(E) = \operatorname{rk}(EAD)$.

Furthermore, in this case, $\mathcal{I}(A)^{\|D,E}_{left} = \mathcal{I}(A)^{\|D,E}_{right} = \{D(EAD)^{\dagger}E\}$.

Proof. Note that according to Theorem 3.2 (iv) and Theorem 3.14 (ii), statement (i) and statement (iii) are equivalent. To prove the equivalence between statements (ii) and (iii), apply Theorem 3.1 (iii) and Theorem 3.15 (ii).

Now, according to Proposition 3.5, the unique left and the unique right (D, E)-inverse of A coincide. To conclude the proof, notice that according to Theorem 3.19, $D(EAD)^{\dagger}E \in \mathcal{I}(A)^{\|D,E}_{left} \cap \mathcal{I}(A)^{\|D,E}_{right}$. ∎

Due to Theorem 3.20, another representation of the left and the right inverses along a matrix can be given.

Corollary 3.21. *Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$. The matrix A is left or right invertible along D if and only if* $\operatorname{rk}(D) = \operatorname{rk}(DAD)$. *Moreover, in this case, the unique left inverse of A along D and the unique right inverse of A along D coincide with the matrix $D(DAD)^{\dagger}D$.*

Proof. Apply Theorem 3.7, Corollary 3.8 and Theorem 3.20. ∎

It is known that a nonzero matrix can be expressed as the product of a matrix of full column rank and a matrix of full row rank. This factorization is known as a full-rank factorization and these factorizations turn out to be a powerful tool in the study of generalized inverses. Recall that given $H \in \mathbb{C}_{m,n}$ such that $\operatorname{rk}(H) = r > 0$, the matrix H is said to have a *full rank factorization*, if there exist $F \in \mathbb{C}_{m,r}$ and $G \in \mathbb{C}_{r,n}$ such that $H = FG$. Such a factorization always exists but it is not unique ([18, Theorem 2]). Moreover, $F^{\dagger}F = I_r = GG^{\dagger}$ ([18, Theorem 1]). Note in particular that $\operatorname{rk}(F) = \operatorname{rk}(G) = \operatorname{rk}(H)$. To learn more results on this topic, see [2, 18]. In the following theorem, given D, $E \in \mathbb{C}_{m,n}$, matrices $A \in \mathbb{C}_{n,m}$ that are left or right (D, E)-invertible will be characterized using a full rank factorization. In addition, the sets $\mathcal{I}(A)^{\|D,E}_{left}$ and $\mathcal{I}(A)^{\|D,E}_{right}$ will be represented using a full rank factorization and the Moore-Penrose inverse.

Theorem 3.22. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. Consider $D = D_1D_2$ and $E = E_1E_2$ two full rank factorizations of D and E, respectively.*

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(i) *The matrix A has a right (D,E)-inverse if and only if* $\mathrm{rk}(E_2)=\mathrm{rk}(E_2AD_1)$*. In addition,*

$$\mathcal{I}(A)^{\|D,E}_{right}=\{D_1\left[(E_2AD_1)^\dagger E_2+(I_r-(E_2AD_1)^\dagger E_2AD_1)Y\right]:Y\in\mathbb{C}_{r,n}\},$$

where $r=\mathrm{rk}(D)$.

(ii) *The matrix A has a left (D,E)-inverse if and only if* $\mathrm{rk}(D_1)=\mathrm{rk}(E_2AD_1)$*. Moreover,*

$$\mathcal{I}(A)^{\|D,E}_{left}=\{\left[D_1(E_2AD_1)^\dagger+Y(I_s-(E_2AD_1)(E_2AD_1)^\dagger)\right]E_2:Y\in\mathbb{C}_{m,s}\},$$

where $s=\mathrm{rk}(E)$.

Proof. Let $r=\mathrm{rk}(D)$ and $s=\mathrm{rk}(E)$. According to [18, Theorem 1], $D_2D_2^\dagger=I_r$, $E_2E_2^\dagger=I_s$, $D_1^\dagger D_1=I_r$ and $E_1^\dagger E_1=I_s$. In addition, since

$$\mathrm{rk}(EAD)=\mathrm{rk}(E_1E_2AD_1D_2)\le\mathrm{rk}(E_2AD_1)\le\mathrm{rk}(E_1^\dagger E_1E_2AD_1D_2D_2^\dagger)\le\mathrm{rk}(EAD),$$

$\mathrm{rk}(EAD)=\mathrm{rk}(E_2AD_1)$. According to Theorem 3.1 (iii), A has a right (D,E)-inverse if and only if $\mathrm{rk}(E_2)=\mathrm{rk}(E)=\mathrm{rk}(EAD)=\mathrm{rk}(E_2AD_1)$.

Let $A\in(\mathbb{C}_{n,m})^{\|D,E}_{right}$ and consider $B\in\mathcal{I}(A)^{\|D,E}_{right}$, i.e., $EAB=E$ and $\mathcal{R}(B)\subseteq\mathcal{R}(D)$. Since $E_1E_2AB=E_1E_2$, multiplying by $E_1^\dagger$ on the left hand side of this equation, $E_2AB=E_2$. In addition, since $\mathcal{R}(B)\subseteq\mathcal{R}(D)$, there exists $M\in\mathbb{C}_n$ such that $B=DM$. Therefore, $E_2AD_1(D_2M)=E_2$, and according to [17, Theorem 2], there exists $Y\in\mathbb{C}_{r,n}$ such that $D_2M=(E_2AD_1)^\dagger E_2+\left[I_r-(E_2AD_1)^\dagger(E_2AD_1)\right]Y$. Thus, $B=DM=D_1D_2M$ implies

$$B=D_1\left[(E_2AD_1)^\dagger E_2+\left(I_r-(E_2AD_1)^\dagger E_2AD_1\right)Y\right].$$

Now suppose that $B\in\mathbb{C}_{m,n}$ has this form. Observe that $B=D_1Z$ for some matrix Z. Thus, $B=D_1D_2D_2^\dagger Z=DD_2^\dagger Z$ implies that $\mathcal{R}(B)\subseteq\mathcal{R}(D)$. In addition,

$$EAB=E_1E_2AD_1\left[(E_2AD_1)^\dagger E_2+\left(I_r-(E_2AD_1)^\dagger E_2AD_1\right)Y\right]=E_1E_2AD_1(E_2AD_1)^\dagger E_2.$$

Since $\mathrm{rk}(E_2AD_1)=\mathrm{rk}(E_2)$ and $\mathcal{R}(E_2AD_1)\subseteq\mathcal{R}(E_2)$, $\mathcal{R}(E_2AD_1)=\mathcal{R}(E_2)$. Therefore, $E_2AD_1(E_2AD_1)^\dagger=P_{\mathcal{R}(E_2AD_1)}=P_{\mathcal{R}(E_2)}$, which implies $E_2AD_1(E_2AD_1)^\dagger E_2=E_2$. Thus, $EAB=E_1E_2=E$. In particular, $B\in\mathcal{I}(A)^{\|D,E}_{right}$.

To prove statement (ii), apply Remark 2.8 (i) and what has been proved. ∎

Next two particular cases will be derived from Theorem 3.22.

Corollary 3.23. *Let $D,E\in\mathbb{C}_{m,n}$ and consider $D=D_1D_2$ and $E=E_1E_2$ two full rank factorizations of D and E, respectively. The matrix $A\in\mathbb{C}_{n,m}$ is left and right (D,E)-invertible if and only if* $\mathrm{rk}(E_2)=\mathrm{rk}(D_1)=\mathrm{rk}(E_2AD_1)$*. Moreover, in this case,*

$$\mathcal{I}(A)^{\|D,E}_{left}=\mathcal{I}(A)^{\|D,E}_{right}=\{D_1(E_2AD_1)^{-1}E_2\}.$$

Proof. The first statement can be derived from Theorem 3.22.

Note that $\mathrm{rk}(D)=\mathrm{rk}(D_1)$, $\mathrm{rk}(E)=\mathrm{rk}(E_2)$ and $\mathrm{rk}(EAD)=\mathrm{rk}(E_2AD_1)$ (see the proof of Theorem 3.22). Then, according to Theorem 3.20, $\mathcal{I}(A)^{\|D,E}_{left}=\mathcal{I}(A)^{\|D,E}_{right}$ is a singleton. In addition, Theorem 3.22 implies $D_1(E_2AD_1)^\dagger E_2\in\mathcal{I}(A)^{\|D,E}_{left}\cap\mathcal{I}(A)^{\|D,E}_{right}$. Thus, $\mathcal{I}(A)^{\|D,E}_{left}=\mathcal{I}(A)^{\|D,E}_{right}=\{D_1(E_2AD_1)^\dagger E_2\}$. However, since $\mathrm{rk}(E_2)=r=\mathrm{rk}(D_1)$ and $(E_2AD_1)^\dagger\in\mathbb{C}_r$ is such that $\mathrm{rk}(E_2AD_1)=r$, $(E_2AD_1)^\dagger=(E_2AD_1)^{-1}$. ∎

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To end this section, the case of left and right inverses along a matrix will be presented.

COROLLARY 3.24. *Let $D \in \mathbb{C}_{m,n}$ and consider $D = D_1D_2$ a full rank factorizations of D. The matrix $A \in \mathbb{C}_{n,m}$ is left or right invertible along D if and only if* $\mathrm{rk}(D_1) = \mathrm{rk}(D_2AD_1)$. *Moreover, in this case, the unique left inverse of A along D and the unique right inverse of A along D coincide with $D_1(D_2AD_1)^{-1}D_2$.*

Proof. Apply Theorem 3.7 and Corollary 3.23. ∎

4. The (D, E)-inverse of arbitrary matrices. First of all the (b, c)-inverse will be extended to rectangular matrices. Compare with Definition 2.1 and recall the observation before Definition 2.7.

DEFINITION 4.1. Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. The matrix A is said to be (D, E)- *invertible*, if there exists a matrix $X \in \mathbb{C}_{m,n}$ such that $XAD = D$, $EAX = E$, $\mathcal{R}(X) \subseteq \mathcal{R}(D)$, and $\mathcal{N}(E) \subseteq \mathcal{N}(X)$.

Under the same conditions as in Definition 4.1, note that $\mathcal{R}(X) \subseteq \mathcal{R}(D)$ (respectively $\mathcal{N}(E) \subseteq \mathcal{N}(X)$) is equivalent to $\mathcal{R}(X) = \mathcal{R}(D)$ (respectively $\mathcal{N}(E) = \mathcal{N}(X)$). In the following theorem, it will be proved that the (D, E)-inverse of a matrix A is unique, if it exists.

THEOREM 4.2. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *The (D, E)-inverse of the matrix A exists.*
(ii) *The matrix A is both left and right (D, E)-invertible.*

Furthermore, in this case, the (D, E)-inverse of the matrix A is unique.

Proof. It is enough to prove that statement (ii) implies statement (i). To this end, apply Proposition 3.5. Proposition 3.5 also proves that there is only one (D, E)-inverse of A, when it exists. ∎

According to Theorem 4.2, if the matrix $A \in \mathbb{C}_{n,m}$ has a (D, E)-inverse ($D, E \in \mathbb{C}_{m,n}$), then it will be denoted by $A^{\|(D,E)}$. In addition, note that according to Definition 2.5 and [14, Corollary 3.7], when the matrices A, D and E are square, Definition 4.1 reduces to the (b, c)-inverse ([9, Definition 1.3], i.e., Definition 2.1). In the following remark some basic results on this inverse that can be derived from what has been proved in Sections 2 and 3 will be collected.

REMARK 4.3. Let $D, E \in \mathbb{C}_{m,n}$ and consider $A \in \mathbb{C}_{n,m}$.

(i) The matrix A is (D, E)-invertible if and only if $A^* \in \mathbb{C}_{m,n}$ is (E^*, D^*)-invertible (E^*, $D^* \in \mathbb{C}_{n,m}$). Moreover, in this case, $(A^*)^{\|(E^*,D^*)} = (A^{\|(D,E)})^*$. Apply Theorem 4.2 and Remark 2.8 (i)-(ii).
(ii) Let $D', E' \in \mathbb{C}_{m,n}$ be such that $\mathcal{R}(D) = \mathcal{R}(D')$ and $\mathcal{N}(E) = \mathcal{N}(E')$. Necessary and sufficient for $A^{\|(D,E)}$ to exist is that $A^{\|(D',E')}$ exists. Furthermore, in this case, $A^{\|(D',E')} = A^{\|(D,E)}$. Apply Theorem 4.2 and Remark 2.8 (iii)-(iv).
(iii) Theorem 3.4 and Theorem 3.20 characterize a matrix A such that $A^{\|(D,E)}$ exists.
(iv) When A is (D, E)-invertible, $A^{\|(D,E)}$ can be represented as in Proposition 3.5, Corollary 3.16 and Remark 3.17.

Although some results have been presented in connection to left and right (D, E)-inverses ($D, E \in \mathbb{C}_{m,n}$), they deserve to be considered again for the (D, E)-inverse. Recall that according to [9, Remark 2.4] (see also [9, Theorem 2.2]), when the matrices A, D, E are square, A is (D, E)-invertible if and only if $\mathrm{rk}(D) = \mathrm{rk}(EAD) = \mathrm{rk}(E)$. In the following theorem, this result will be extended to arbitrary matrices.

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THEOREM 4.4. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *The (D, E)-inverse of A exists.*
(ii) $\mathrm{rk}(D) = \mathrm{rk}(E) = \mathrm{rk}(EAD)$.

Furthermore, in this case, $A^{\|(D,E)} = D(EAD)^{\dagger}E$.

Proof. Apply Theorem 4.2, Proposition 3.5 and Theorem 3.20. ∎

Now a corollary will be derived from Theorem 4.4.

COROLLARY 4.5. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ be such that $A^{\|(D,E)}$ exists. Then* $\mathrm{rk}(AD) = \mathrm{rk}(EA) = \mathrm{rk}(E) = \mathrm{rk}(D)$.

Proof. Apply Theorem 4.4 and Theorem 3.4. ∎

REMARK 4.6. Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. Note that the condition in Corollary 4.5 does not imply that $A^{\|(D,E)}$ exists. In fact, consider

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad D = E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $EAD = 0$ and $\mathrm{rk}(D) = \mathrm{rk}(E) = 1$, according to Theorem 4.4, $A^{\|(D,E)}$ does not exist. However, $\mathrm{rk}(AD) = 1$ and $\mathrm{rk}(EA) = 1$.

In the following theorem the (D, E)-inverse will be characterized using full rank factorizations.

THEOREM 4.7. *Let $D, E \in \mathbb{C}_{m,n}$ and consider $D = D_1D_2$ and $E = E_1E_2$ two full rank factorizations of D and E, respectively. The matrix $A \in \mathbb{C}_{n,m}$ is (D, E)-invertible if and only if* $\mathrm{rk}(E_2) = \mathrm{rk}(D_1) = \mathrm{rk}(E_2AD_1)$. *Moreover, in this case, $A^{\|(D,E)} = D_1(E_2AD_1)^{-1}E_2$.*

Proof. Apply Theorem 4.2 and Corollary 3.23. ∎

In the following corollaries two particular cases will be derived from Theorem 4.7.

COROLLARY 4.8. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{n,m}$ be such that $A^{\|(D,E)}$ exists. The following statements hold.*

(i) *If the columns of D are linearly independent, then $A^{\|(D,E)} = D(AD)^{-1}$.*
(ii) *If the rows of E are linearly independent, then $A^{\|(D,E)} = (EA)^{-1}E$.*

Proof. If D has full column rank, then $D = DI_n$ is a full rank factorization of D. Since $A^{\|(D,E)}$ exists, according to Theorem 4.4, $\mathrm{rk}(E) = \mathrm{rk}(D)$. Consequently, E has full column rank and $E = EI_n$ is a full rank factorization of E. Then, according to Theorem 4.7, $A^{\|(D,E)} = D(AD)^{-1}$.

Apply a similar argument to prove statement (ii), using in particular the full rank factorizations $E = I_mE$ and $D = I_mD$ and Theorem 4.7. ∎

COROLLARY 4.9. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ be such that* $\mathrm{rk}(D) = \mathrm{rk}(E) = 1$. *If $D = \mathbf{d}_1\mathbf{d}_2^*$ and $E = \mathbf{e}_1\mathbf{e}_2^*$ ($\mathbf{d}_1, \mathbf{e}_1 \in \mathbb{C}_{m,1}$, $\mathbf{d}_2, \mathbf{e}_2 \in \mathbb{C}_{n,1}$) are full rank factorizations of D and E, respectively, then $A^{\|(D,E)}$ exists if and only if $\mathbf{e}_2^*A\mathbf{d}_1 \neq 0$. Moreover, in this case,*

$$A^{\|(D,E)} = \frac{1}{\mathbf{e}_2^*A\mathbf{d}_1}\mathbf{d}_1\mathbf{e}_2^*.$$

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Proof. According to Theorem 4.7, the matrix A is (D, E)-invertible if and only if $\mathrm{rk}(\mathbf{d}_1) = \mathrm{rk}(\mathbf{e}_2^*) = \mathrm{rk}(\mathbf{e}_2^* A\mathbf{d}_1)$. Therefore, $A^{\|(D,E)}$ exists if and only if $\mathrm{rk}(\mathbf{e}_2^* A\mathbf{d}_1) = 1$. But observe that $\mathbf{e}_2^* A\mathbf{d}_1$ is an scalar. Thus, the first part of the theorem has been proved. The expression of $A^{\|(D,E)}$ also follows from Theorem 4.7. ∎

Next an application of Theorem 4.7 will lead to representations of several generalized inverses in terms of a full rank representation. The explicit expression for $A^\dagger$ is attributed to C.C. MacDufee by Ben-Israel and Greville in [2]. Ben-Israel and Greville report that around 1959, MacDufee was the first to point out that a full-rank factorization of A leads to the mentioned formula.

COROLLARY 4.10. *Let $A \in \mathbb{C}_n$ and consider a full rank factorization $A = FG$. Then, the following statements hold.*

(i) $A^\dagger = G^*(F^*AG^*)^{-1}F^*$.
(ii) *A is group invertible if and only if GF is nonsingular; in this case, $A^\# = F(GF)^{-2}G$ (see [7]).*
(iii) *A is core invertible if and only if GF is nonsingular; in this case, $A^{\oplus} = F(GF)^{-1}F^\dagger$. The same equivalent condition holds for the dual core invertibility of A; in this case, $A_{\oplus} = G^\dagger(GF)^{-1}G$.*
(iv) *$A^\dagger_{M,N} = N^{-1}G^*(F^*MAN^{-1}G^*)^{-1}F^*M$, where M and N are nonsingular and positive.*

Proof. Observe that $A^* = G^*F^*$ is a full rank factorization of A^*.

Recall that according to [9, p. 1912], A is Moore-Penrose invertible if and only if A is (A^*, A^*)-invertible, and in this case, $A^\dagger = A^{\|(A^*,A^*)}$. Now apply Theorem 4.7 with $D = E = A^* = G^*F^*$.

According to [9, p. 1910], A is group invertible if and only if A is (A, A)-invertible, and in this case, $A^\# = A^{\|(A,A)}$. According to Theorem 4.7, A is group invertible if and only if $\mathrm{rk}(G) = \mathrm{rk}(F) = \mathrm{rk}(GFGF)$. Thus, if $r = \mathrm{rk}(A)$, then $GF \in \mathbb{C}_r$ is invertible. To prove the formula representing $A^\#$, apply Theorem 4.7 with $D = E = A = FG$.

Recall that according to [1, p. 684], A is core invertible if and only if it is group invertible. Thus, according to what has been proved, the first part of statement (iii) holds. In addition, according to [19, Theorem 4.4] (i), necessary and sufficient for A to be core invertible is that A is (A, A^*)-invertible, and in this case, $A^{\oplus} = A^{\|(A,A^*)}$. Now, apply Theorem 4.7 with $D = A = FG$ and $E = A^* = G^*F^*$. Observe however first that if $r = \mathrm{rk}(A)$, then $F^*F \in \mathbb{C}_r$ is such that $\mathrm{rk}(F^*F) = \mathrm{rk}(F) = r$, and recall that $(F^*F)^{-1}F^* = F^\dagger$ ([18, Theorem 1]). Then,

$$A^{\oplus} = A^{\|(A,A^*)} = F(F^*FGF)^{-1}F^* = F(GF)^{-1}(F^*F)^{-1}F^* = F(GF)^{-1}F^\dagger.$$

Note that A is dual core invertible if and only if A^* is core invertible and $A_{\oplus} = [(A^*)^{\oplus}]^*$. Thus, the second part of statement (iii) can be derived from what has been proved and

$$A_{\oplus} = [(G^*(F^*G^*)^{-1}(G^*)^\dagger)]^* = G^\dagger(GF)^{-1}G.$$

According to [4, Theorem 3.2], A is weighted Moore-Penrose invertible with weights M and N if and only if A is invertible along $N^{-1}A^*M$. In addition, according to [9, Definition 6.1], this is equivalent to the fact that A is $(N^{-1}A^*M, N^{-1}A^*M)$-invertible. Now apply Theorem 4.7 with $D = E = N^{-1}A^*M$ and consider the full rank factorization $N^{-1}A^*M = (N^{-1}G^*)(F^*M)$. ∎

To end this section, the set of all matrices $A \in \mathbb{C}_{n,m}$ such that they are (D, E)-invertible will be studied (D, $E \in \mathbb{C}_{m,n}$). First some notation needs to be introduced.

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Given D, $E \in \mathbb{C}_{m,n}$, let $\mathbb{C}_{n,m}^{\|D,E}$ denote the set of all matrices $A \in \mathbb{C}_{n,m}$ such that the (D,E)-inverse of A exists, i.e.,

$$\mathbb{C}_{n,m}^{\|D,E} = \left\{ A \in \mathbb{C}_{n,m} : A^{\|(D,E)} \text{ exists} \right\}.$$

THEOREM 4.11. *Let $D, E \in \mathbb{C}_{m,n}$. Necessary and sufficient for $\mathbb{C}_{n,m}^{\|D,E} \neq \emptyset$ is that $\mathrm{rk}(E) = \mathrm{rk}(D)$. Moreover, in this case, $\mathbb{C}_{n,m}^{\|D,E} = (\mathbb{C}_{n,m})_{left}^{\|D,E} = (\mathbb{C}_{n,m})_{right}^{\|D,E}$.*

Proof. Apply Theorem 4.2 and Corollary 3.12. ☐

Observe that Corollary 3.16 gives an explicit representation of $\mathbb{C}_{n,m}^{\|D,E}$.

5. Invertible matrices along a fixed matrix. Now the case $D = E$ will be considered. Compare with Definition 2.2 and recall the observation before Definition 2.7.

DEFINITION 5.1. Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$. The matrix A is said to be *invertible along the matrix* D, if there exists a matrix $X \in \mathbb{C}_{m,n}$ such that $XAD = D = DAX$, $\mathcal{R}(X) \subseteq \mathcal{R}(D)$, and $\mathcal{N}(D) \subseteq \mathcal{N}(X)$.

According to Theorem 4.2, if the inverse of the matrix $A \in \mathbb{C}_{n,m}$ along the matrix $D \in \mathbb{C}_{m,n}$ exists, then it is unique and it will be denoted by $A^{\|D}$. In addition, when A and D are square matrices, since according to [9, Proposition 6.1], the inverse of A along D coincides with the (D,D)-inverse of A, Definition 5.1 reduces to the notion of the inverse along an element in a ring of square matrices ([15, Definition 4], i.e., Definition 2.2). Moreover, under the same conditions as in Definition 5.1, note that $\mathcal{R}(X) \subseteq \mathcal{R}(D)$ (respectively $\mathcal{N}(D) \subseteq \mathcal{N}(X)$) is equivalent to $\mathcal{R}(X) = \mathcal{R}(D)$ (respectively $\mathcal{N}(D) = \mathcal{N}(X)$). Now characterizations of the inverse along a matrix will be given.

THEOREM 5.2. *Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *The matrix A is left invertible along D.*
(ii) *The matrix A is right invertible along D.*
(iii) *The matrix A is invertible along D.*

Proof. Apply Theorem 4.2 and Theorem 3.7. ☐

In the following remark, several results on this inverse that can be derived from what has been proved in Sections 2, 3 and 4 will be collected.

REMARK 5.3. Let $D \in \mathbb{C}_{m,n}$ and consider $A \in \mathbb{C}_{n,m}$.

(i) A is invertible along D if and only if A^* is invertible along D^*. Moreover, in this case, $(A^*)^{\|D^*} = (A^{\|D})^*$. Apply Remark 4.3 (i) to the case $E = D$.
(ii) Let $D' \in \mathbb{C}_{m,n}$ be such that $\mathcal{R}(D) = \mathcal{R}(D')$ and $\mathcal{N}(D) = \mathcal{N}(D')$. Necessary and sufficient for $A^{\|D}$ to exist is that $A^{\|D'}$ exists. Furthermore, in this case, $A^{\|D'} = A^{\|D}$. Apply Remark 4.3 (ii) to the case $E = D$.
(iii) Theorem 3.6 and Theorem 3.7 characterize matrices A such that $A^{\|D}$ exists. Compare Theorem 3.7 (v)-(vi) with [15, Theorem 7] and [16, Theorem 2.1].
(iv) When A is invertible along D, $A^{\|D}$ can be represented as in Corollary 3.8 and Corollary 3.18.

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(v) According to Remark 3.17 applied to the case $D = E$, another representation of $A^{\|D}$, when it exists, is the following. In the decomposition $\mathbb{C}^n = \mathcal{N}(D) \oplus \mathcal{X}$, it holds that

$$\mathcal{N}(A^{\|D}) = \mathcal{N}(D) \quad \text{and} \quad A^{\|D}\mathbf{y} = \phi^{-1}(\mathbf{y}),\ \mathbf{y} \in \mathcal{X},$$

where $\phi : \mathcal{R}(D) \to \mathcal{X}$ is the map of Theorem 3.3.

(vi) According to Corollary 4.5 applied to the case $D = E$, if $A^{\|D}$ exists, then $\text{rk}(AD) = \text{rk}(DA) = \text{rk}(D)$.

Some results, however, deserve to be presented separately.

Corollary 5.4. *Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *A is invertible along D.*
(ii) $\text{rk}(D) = \text{rk}(DAD)$.

Furthermore, in this case, $A^{\|D} = D(DAD)^{\dagger}D$.

Proof. Apply Theorem 4.4 for the case $D = E$. ∎

Corollary 5.5. *Let $D \in \mathbb{C}_{m,n}$ and consider $D = D_1D_2$ a full rank factorization of D. The matrix $A \in \mathbb{C}_{n,m}$ is invertible along D if and only if $\text{rk}(D_1) = \text{rk}(D_2AD_1)$. Moreover, in this case, $A^{\|D} = D_1(D_2AD_1)^{-1}D_2$.*

Proof. Apply Theorem 4.7 for the case $D = E$. ∎

Corollary 5.6. *Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$.*

(i) *Suppose that the columns of D are linearly independent. A is invertible along D if and only if $AD \in \mathbb{C}_n$ is nonsingular. Moreover, in this case, $A^{\|D} = D(AD)^{-1}$.*
(ii) *Suppose that the rows of D are linearly independent. A is invertible along D if and only if $DA \in \mathbb{C}_m$ is invertible. Moreover, in this case, $A^{\|D} = (DA)^{-1}D$.*

Proof. Suppose that the columns of D are linearly independent. According to Theorem 5.2 and Theorem 3.7 (v), the characterization of the inverse of A along D holds. To prove the formula that represents $A^{\|D}$, apply Corollary 4.8 (i).

To prove statement (ii), apply a similar argument to the one used to prove statement (i), using in particular Theorem 3.7 (vi) and Corollary 4.8 (ii). ∎

Corollary 5.7. *Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$. If $\text{rk}(D) = 1$, then $A^{\|D}$ exists if and only if $\text{tr}(AD) \neq 0$. Moreover, in this case,*

$$A^{\|D} = \frac{1}{\text{tr}(AD)}D.$$

Proof. Let $D = \mathbf{d}_1\mathbf{d}_2^*$ ($\mathbf{d}_1 \in \mathbb{C}_{m,1}$ and $\mathbf{d}_2 \in \mathbb{C}_{n,1}$) be a full rank factorization of D, According to Corollary 4.9 applied to the case $D = E$, $\mathbf{d}_2^*A\mathbf{d}_1 = \text{tr}(\mathbf{d}_2^*A\mathbf{d}_1) = \text{tr}(A\mathbf{d}_1\mathbf{d}_2^*) = \text{tr}(AD)$. ∎

Let $\mathbb{C}_{n,m}^{\|D}$ stand for the set of all matrices $A \in \mathbb{C}_{n,m}$ such that A is invertible along D, i.e., $\mathbb{C}_{n,m}^{\|D} = \{A \in \mathbb{C}_{n,m} : A^{\|D} \text{ exists}\}$. The following corollary proves that the set under consideration is nonempty.

Corollary 5.8. *Let $D \in \mathbb{C}_{m,n}$. Then, $\mathbb{C}_{n,m}^{\|D} \neq \emptyset$.*

Proof. Apply Theorem 5.2, Corollary 3.11 and Corollary 3.18. ∎

Observe that Corollary 3.18 gives an explicit representation of $\mathbb{C}_{n,m}^{\|D}$.

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6. Outer and inner inverses. In the following theorem it will be proved that the notion introduced in Definition 4.1 is an outer inverse.

Theorem 6.1. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. If A is (D,E)-invertible, then $A^{\|(D,E)}$ is an outer inverse of A.*

Proof. According to Definition 4.1, since $\mathcal{R}(A^{\|(D,E)}) \subseteq \mathcal{R}(D)$, there exists $M \in \mathbb{C}_n$ such that $A^{\|(D,E)} = DM$. Thus, $A^{\|(D,E)}AA^{\|(D,E)} = A^{\|(D,E)}ADM = DM = A^{\|(D,E)}$. □

Corollary 6.2. *Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$. If A is invertible along D, then $A^{\|D}$ is an outer inverse of A.*

Given $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ such that $A^{\|(D,E)}$ exists, according to Corollary 3.18 and Theorem 4.4,

$$\mathrm{rk}(A^{\|(D,E)}) = \mathrm{rk}(D) = \mathrm{rk}(E) \leq \mathrm{rk}(A).$$

In particular, when A, D and E are square matrices, $A^{\|(D,E)}$ is nonsingular if and only if D or E are nonsingular. In addition, if $A^{\|(D,E)}$ is nonsingular, then A is nonsingular. However, if A is nonsingular and D, E are such that $A^{\|(D,E)}$ exists, then it may be happen that D, E, or $A^{\|(D,E)}$ are singular. For example, take

$$A = I_2 \quad \text{and} \quad D = E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

According to Theorem 4.4, $A^{\|(D,E)}$ exists ($A^{\|(D,E)} = D$). However, it is possible to characterize when $\mathrm{rk}(A) = \mathrm{rk}(D)$. This characterization is linked with the following observation: $A^{\|(D,E)}$ is always an outer inverse of A, but it is not necessarily an inner inverse of A. To prove this characterization some preparation is needed first.

Theorem 6.3. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ be such that $A^{\|(D,E)}$ exists. Then, the following statements hold.*

(i) $\mathcal{R}(D) \oplus \mathcal{N}(A) = \mathcal{N}(AA^{\|(D,E)}A - A)$.
(ii) $\mathrm{rk}(A) = \mathrm{rk}(D) + \mathrm{rk}(AA^{\|(D,E)}A - A)$.
(iii) $\mathcal{R}(A) + \mathcal{N}(E) = \mathbb{C}^n$ *and* $\mathcal{R}(A) \cap \mathcal{N}(E) = \mathcal{R}(AA^{\|(D,E)}A - A)$.

Proof. The inclusion $\mathcal{N}(A) \subseteq \mathcal{N}(AA^{\|(D,E)}A - A)$ is evident. The equality $A^{\|(D,E)}AD = D$ leads to $\mathcal{R}(D) \subseteq \mathcal{N}(AA^{\|(D,E)}A - A)$. If $\mathbf{x} \in \mathcal{R}(D) \cap \mathcal{N}(A)$, then there exists $\mathbf{u} \in \mathbb{C}^n$ such that $\mathbf{x} = D\mathbf{u}$, and therefore, $\mathbf{x} = D\mathbf{u} = A^{\|(D,E)}AD\mathbf{u} = A^{\|(D,E)}A\mathbf{x} = \mathbf{0}$. Thus, $\mathcal{R}(D) \oplus \mathcal{N}(A) \subseteq \mathcal{N}(AA^{\|(D,E)}A - A)$.

In order to prove the opposite inclusion, take $\mathbf{y} \in \mathcal{N}(AA^{\|(D,E)}A - A)$. Now, $A(A^{\|(D,E)}A\mathbf{y} - \mathbf{y}) = \mathbf{0}$ and the decomposition $\mathbf{y} = (\mathbf{y} - A^{\|(D,E)}A\mathbf{y}) + A^{\|(D,E)}A\mathbf{y}$ proves that $\mathcal{N}(AA^{\|(D,E)}A - A) \subseteq \mathcal{R}(D) \oplus \mathcal{N}(A)$ (because $\mathcal{R}(A^{\|(D,E)}) = \mathcal{R}(D)$).

Statement (ii) follows from statement (i).

According to Remark 4.3 (i), $(A^*)^{\|(E^*,D^*)}$ exists, so that, according to statement (i) applied to A^*, D^* and E^*, $\mathcal{N}(A^*) \cap \mathcal{R}(E^*) = 0$. Then,

$$[\mathcal{R}(A) + \mathcal{N}(E)]^\perp = \mathcal{R}(A)^\perp \cap \mathcal{N}(E)^\perp = \mathcal{N}(A^*) \cap \mathcal{R}(E^*) = 0.$$

Therefore, $\mathcal{R}(A) + \mathcal{N}(E) = \mathbb{C}^n$. The inclusion $\mathcal{R}(AA^{\|(D,E)}A - A) \subseteq \mathcal{R}(A) \cap \mathcal{N}(E)$ follows from $EAA^{\|(D,E)} = E$.

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According to statement (ii), Theorem 4.4 and

$$\begin{aligned}\dim(\mathcal{R}(A) \cap \mathcal{N}(E)) &= \dim \mathcal{R}(A) + \dim \mathcal{N}(E) - \dim(\mathcal{R}(A) + \mathcal{N}(E)) \\ &= \mathrm{rk}(A) + n - \mathrm{rk}(D) - n = \mathrm{rk}(AA^{\|(D,E)}A - A),\end{aligned}$$

$\mathcal{R}(AA^{\|(D,E)}A - A)$ and $\mathcal{R}(A) \cap \mathcal{N}(E)$ have the same dimension. Therefore, both subspaces are equal. □

The following corollary characterizes when $A^{\|(D,E)}$ is an inner inverse of A.

Corollary 6.4. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ be such that $A^{\|(D,E)}$ exists. Then, the following statements are equivalent.*

(i) $AA^{\|(D,E)}A = A$.
(ii) $\mathrm{rk}(A) = \mathrm{rk}(D)$.
(iii) $\mathcal{R}(D) \oplus \mathcal{N}(A) = \mathbb{C}^m$.
(iv) $\mathcal{R}(A) \oplus \mathcal{N}(E) = \mathbb{C}^n$.

Proof. Apply Theorem 6.3. □

If Corollary 6.4 is applied for $D = E$, then it is possible to characterize when the inverse along an element is an inner inverse.

Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. Since $A^{\|(D,E)}$ is an outer inverse, if it exists, $A^{\|(D,E)}A$ and $AA^{\|(D,E)}$ are idempotents. Now some properties of these idempotents will be studied.

Theorem 6.5. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ be such that $A^{\|(D,E)}$ exists.*

(i) $A^{\|(D,E)}A$ *and* $AA^{\|(D,E)}$ *are idempotents,* $\mathcal{R}(A^{\|(D,E)}A) = \mathcal{R}(D)$, $\mathcal{R}(AA^{\|(D,E)}) = \mathcal{R}(AD)$ *and* $\mathrm{rk}(D) = \mathrm{rk}(AA^{\|(D,E)}) = \mathrm{rk}(A^{\|(D,E)}A) = \mathrm{rk}(E)$.
(ii) $\mathcal{N}(AA^{\|(D,E)}) = \mathcal{N}(E)$ *and* $\mathcal{N}(A^{\|(D,E)}A) = \mathcal{N}(EA)$.
(iii) $\mathcal{N}(A^{\|(D,E)}A) = \mathcal{N}(A)$ *if and only if* $A^{\|(D,E)}$ *is an inner inverse of* A.
(iv) $\mathcal{R}(AA^{\|(D,E)}) = \mathcal{R}(A)$ *if and only if* $A^{\|(D,E)}$ *is an inner inverse of* A.
(v) $AA^{\|(D,E)}$ *is an orthogonal projector if and only if* $\mathcal{R}(AD) = \mathcal{R}(E^*)$.
(vi) $A^{\|(D,E)}A$ *is an orthogonal projector if and only if* $\mathcal{R}((EA)^*) = \mathcal{R}(D)$.

Proof. Since $A^{\|(D,E)}$ is an outer inverse (Theorem 6.1), $A^{\|(D,E)}A$ and $AA^{\|(D,E)}$ are idempotents and $\mathcal{R}(A^{\|(D,E)}A) = \mathcal{R}(A^{\|(D,E)})$. Moreover, since $\mathcal{R}(A^{\|(D,E)}) = \mathcal{R}(D)$, according to Theorem 4.4,

$$\mathrm{rk}(E) = \mathrm{rk}(D) = \mathrm{rk}(A^{\|(D,E)}) = \mathrm{rk}(A^{\|(D,E)}A).$$

In addition, according to Remark 4.3 (i) and what has been proved, $\mathrm{rk}(E) = \mathrm{rk}(E^*) = \mathrm{rk}((A^{\|(D,E)})^*A^*) = \mathrm{rk}(AA^{\|(D,E)})$. Moreover,

$$\mathcal{R}(AD) = \mathcal{R}(AA^{\|(D,E)}AD) \subseteq \mathcal{R}(AA^{\|(D,E)}).$$

However, since according to Corollary 4.5, $\mathrm{rk}(AD) = \mathrm{rk}(D) = \mathrm{rk}(AA^{\|(D,E)})$, and $\mathcal{R}(AA^{\|(D,E)}) = \mathcal{R}(AD)$.

Since $A^{\|(D,E)}$ is an outer inverse, $\mathcal{N}(AA^{\|(D,E)}) = \mathcal{N}(A^{\|(D,E)}) = \mathcal{N}(E)$. Note that since $EA = EAA^{\|(D,E)}A$, $\mathcal{N}(A^{\|(D,E)}A) \subseteq \mathcal{N}(EA)$. However, since according to Corollary 4.5, $\mathrm{rk}(EA) = \mathrm{rk}(D) = \mathrm{rk}(A^{\|(D,E)}A)$, $\dim \mathcal{N}(A^{\|(D,E)}A) = \dim \mathcal{N}(EA)$. Therefore, $\mathcal{N}(A^{\|(D,E)}A) = \mathcal{N}(EA)$.

Naturally, $\mathcal{N}(A) \subseteq \mathcal{N}(A^{\|(D,E)}A)$. Since $\dim \mathcal{N}(A^{\|(D,E)}A) = m - \mathrm{rk}(A^{\|(D,E)}A) = m - \mathrm{rk}(D)$ and $\dim \mathcal{N}(A) = m - \mathrm{rk}(A)$, $\mathcal{N}(A) = \mathcal{N}(A^{\|(D,E)}A)$ if and only if $\mathrm{rk}(A) = \mathrm{rk}(D)$. Now apply Corollary 6.4.

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Note that $A^{\|(D,E)}$ is an inner inverse of A if and only if $(A^{\|(D,E)})^*$ is an inner inverse of A^*. Now, according to statement (iii), this is equivalent to $\mathcal{N}((A^{\|(D,E)})^*A^*) = \mathcal{N}(A^*)$, which in turn is equivalent to $\mathcal{R}(AA^{\|(D,E)}) = \mathcal{R}(A)$.

Observe that $AA^{\|(D,E)}$ is an orthogonal projector if and only if

$$\mathcal{R}(AD) = \mathcal{R}(AA^{\|(D,E)}) = \mathcal{N}(AA^{\|(D,E)})^\perp = \mathcal{N}(E)^\perp = \mathcal{R}(E^*).$$

The proof of statement (vi) follows Remark 4.3 (i) and statement (v). □

By particularizing $D = E$ in Theorem 6.5, the case of the inverses along a fixed matrix can be obtained.

Given $A \in \mathbb{C}_n$ such that A is group invertible, according to [1] and [19], $AA^{\#\!\!\!\!\bigcirc}$ and $A_{\#\!\!\!\!\bigcirc}A$ are orthogonal projectors. In the next corollaries, similar properties for several generalized inverses will be characterized using Theorem 6.5. Note that the following identities hold: $\mathcal{R}(XY) = \mathcal{R}(X)$ and $\mathcal{R}(XX^*) = \mathcal{R}(X)$, where Y is a nonsingular matrix and X is any matrix. In addition, recall that a matrix $A \in \mathbb{C}_n$ is said to be EP, if $AA^\dagger = A^\dagger A$. It is well known that this condition is equivalent to $\mathcal{R}(A) = \mathcal{R}(A^*)$.

Corollary 6.6. *Consider $A \in \mathbb{C}_n$ a group invertible matrix. The following statements are equivalent.*

(i) *$A^{\#\!\!\!\!\bigcirc}A$ is an orthogonal projector.*
(ii) *$AA_{\#\!\!\!\!\bigcirc}$ is an orthogonal projector.*
(iii) *A is EP.*

Proof. According to Theorem 2.4 (i) and Theorem 6.5 (vi), $A^{\#\!\!\!\!\bigcirc}A$ is an orthogonal projector if and only if $\mathcal{R}((AA^*A)^*) = \mathcal{R}(AA^*)$. However, $\mathcal{R}(AA^*) = \mathcal{R}(A)$ and $\mathcal{R}((AA^*A)^*) = \mathcal{R}(A^*AA^*) = \mathcal{R}(A^*A) = \mathcal{R}(A^*)$.

The proof of the equivalence of the statements (ii) and (iii) is similar. □

Corollary 6.7. *Let $A \in \mathbb{C}_n$ and consider M, $N \in \mathbb{C}_n$ nonsingular and positive. The following statements hold.*

(i) *$AA^\dagger_{M,N}$ is an orthogonal projector if and only if $\mathcal{R}(A) = \mathcal{R}(MA)$.*
(ii) *$A^\dagger_{M,N}A$ is an orthogonal projector if and only if $\mathcal{R}(A^*) = \mathcal{R}(N^{-1}A^*)$.*

Proof. According to Theorem 2.4 (ii) and Theorem 6.5 (v), $AA^\dagger_{M,N}$ is an orthogonal projector if and only if $\mathcal{R}(AN^{-1}A^*M) = \mathcal{R}((N^{-1}A^*M)^*)$. This last condition is equivalent to $\mathcal{R}(AN^{-1}A^*) = \mathcal{R}(MA)$. Since N is nonsingular and positive, there exists a Hermitian and nonsingular matrix Q such that $N^{-1} = Q^2$. Define $R = AQ$. Then, $AN^{-1}A^* = RR^*$, and thus,

$$\mathcal{R}(AN^{-1}A^*) = \mathcal{R}(RR^*) = \mathcal{R}(R) = \mathcal{R}(AQ) = \mathcal{R}(A).$$

Similarly, according to Theorem 2.4 (ii) and Theorem 6.5 (vi), $A^\dagger_{M,N}A$ is an orthogonal projector if and only if $\mathcal{R}((N^{-1}A^*MA)^*) = \mathcal{R}(N^{-1}A^*M)$. This equality is equivalent to $\mathcal{R}(A^*MA) = \mathcal{R}(N^{-1}A^*)$. Since M is nonsingular and positive, there exists a Hermitian and nonsingular matrix P such that $M = P^2$. Then $A^*MA = A^*P^*PA = (PA)^*PA$, and thus,

$$\mathcal{R}(A^*MA) = \mathcal{R}((PA)^*PA) = \mathcal{R}((PA)^*) = \mathcal{R}(A^*P^*) = \mathcal{R}(A^*). \quad \square$$

7. The outer inverse with prescribed range and null space. Firstly, the relationship between the outer inverse with prescribed range and null space and the (D, E)-inverse will be considered ($D, E \in \mathbb{C}_{m,n}$).

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THEOREM 7.1. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *The matrix A is (D, E)-invertible.*
(ii) *The outer inverse $A^{(2)}_{\mathcal{R}(D),\mathcal{N}(E)}$ exists.*

Furthermore, in this case, $A^{\|(D,E)} = A^{(2)}_{\mathcal{R}(D),\mathcal{N}(E)}$.

Proof. Suppose that statement (i) holds and let $H = A^{\|(D,E)}$. Then, according to Theorem 6.1, $H = HAH$. In addition, according to Definition 4.1, $\mathcal{N}(H) = \mathcal{N}(E)$ and $\mathcal{R}(H) = \mathcal{R}(D)$. In particular, $A^{(2)}_{\mathcal{R}(D),\mathcal{N}(E)}$ exists and $A^{(2)}_{\mathcal{R}(D),\mathcal{N}(E)} = H$.

Now suppose that statement (ii) holds and let $L = A^{(2)}_{\mathcal{R}(D),\mathcal{N}(E)}$. In particular, $\mathcal{R}(L) \subseteq \mathcal{R}(D)$ and $\mathcal{N}(E) \subseteq \mathcal{N}(L)$. Since L is an outer inverse of A and $\mathcal{R}(L) = \mathcal{R}(D)$, it is not difficult to prove that $LAD = D$. In addition, since $AL\mathbf{x} - \mathbf{x} \in \mathcal{N}(L) = \mathcal{N}(E)$, for all $\mathbf{x} \in \mathbb{C}^n$, $EAL = E$. Therefore, $L = A^{\|(D,E)}$. □

COROLLARY 7.2. *Let $A \in \mathbb{C}_{n,m}$ and $D \in \mathbb{C}_{m,n}$. The following statements are equivalent.*

(i) *The matrix A is invertible along D.*
(ii) *The outer inverse $A^{(2)}_{\mathcal{R}(D),\mathcal{N}(D)}$ exists.*

Furthermore, in this case, $A^{\|D} = A^{(2)}_{\mathcal{R}(D),\mathcal{N}(D)}$.

Proof. Apply Theorem 7.1 to the case $D = E$. □

Due to Theorem 7.1, the properties of the outer inverse with prescribed range and null space can be easily proved for the (D, E)-inverse $(D, E \in \mathbb{C}_{m,n})$. Here only some of the most well known result are considered. Other results and the case of the inverse along a fixed matrix, i.e., when $D = E$, are left to the reader.

COROLLARY 7.3. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. $A^{\|(D,E)}$ is the unique matrix X that satisfies the following equations:*

$$XAX = X, \qquad AX = P_{\mathcal{R}(AD),\mathcal{N}(E)}, \qquad XA = P_{\mathcal{R}(D),\mathcal{N}(EA)}.$$

Proof. Apply Theorem 7.1 and [24, Theorem 1]. Note that, if $\mathcal{T} = \mathcal{R}(D)$ and $\mathcal{S} = \mathcal{N}(E)$, then $A(\mathcal{T}) = \mathcal{R}(AD)$ and $(A^*(\mathcal{S}^\perp))^\perp = \mathcal{N}(EA)$. □

COROLLARY 7.4. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. Suppose that there exists $G \in \mathbb{C}_{m,n}$ such that $\mathcal{R}(G) = \mathcal{R}(D)$ and $\mathcal{N}(G) = \mathcal{N}(E)$. If $A^{\|(D,E)}$ exists, then $AG \in \mathbb{C}_m$ and $GA \in \mathbb{C}_n$ are group invertible and*

$$A^{\|(D,E)} = G(AG)^\# = (GA)^\# G = [GA\,|_{\mathcal{R}(G)}]^{-1} G.$$

Proof. Apply Theorem 7.1, [21, Theorem 2.1] and [21, Theorem 2.3]. □

COROLLARY 7.5. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. Suppose that there exists $G \in \mathbb{C}_{m,n}$ such that $\mathcal{R}(G) = \mathcal{R}(D)$ and $\mathcal{N}(G) = \mathcal{N}(E)$. If $A^{\|(D,E)}$ exists, then*

$$A^{\|(D,E)} = \lim_{\varepsilon \to 0} (GA - \varepsilon I_m)^{-1} G = \lim_{\varepsilon \to 0} G(AG - \varepsilon I_n)^{-1}.$$

Proof. Apply Theorem 7.1 and [21, Theorem 2.4]. □

COROLLARY 7.6. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. Suppose that there exists $G \in \mathbb{C}_{m,n}$ such that $\mathcal{R}(G) = \mathcal{R}(D)$ and $\mathcal{N}(G) = \mathcal{N}(E)$. If $A^{\|(D,E)}$ exists, then*

$$A^{\|(D,E)} = \int_0^\infty \exp[-G(GAG)^* GAt] G(GAG)^* G \, \mathrm{d}t.$$

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Proof. Apply Theorem 7.1 and [23, Theorem 2.2]. ∎

COROLLARY 7.7. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$. Suppose that there exists $G \in \mathbb{C}_{m,n}$ such that $\mathcal{R}(G) = \mathcal{R}(D)$ and $\mathcal{N}(G) = \mathcal{N}(E)$. If $A^{\|(D,E)}$ exists and the nonzero spectrum of GA lies in the open left half plane, then*

$$A^{\|(D,E)} = -\int_0^\infty \exp(GAt)G\,\mathrm{d}t.$$

Proof. Apply Theorem 7.1 and [22]. ∎

8. Continuity and differentiability. First of all note that if $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ are such that $A^{\|(D,E)}$ exists and $D', E' \in \mathbb{C}_{n,m}$ are such that $D = DD'D$ and $E = EE'E$, then according to Definition 4.1, $DD'A^{\|(D,E)} = A^{\|(D,E)}$ (because $\mathcal{R}(A^{\|(D,E)}) \subseteq \mathcal{R}(D)$) and $A^{\|(D,E)} = A^{\|(D,E)}E'E$. The last identity can be easily derived from the fact that there exists a matrix $N \in \mathbb{C}_m$ such that $A^{\|(D,E)} = NE$ (because $\mathcal{N}(E) \subseteq \mathcal{N}(A^{\|(D,E)})$).

In order to characterize the continuity of the (D,E)-inverse, a technical lemma is needed.

LEMMA 8.1. *Let $A, B \in \mathbb{C}_{n,m}$ and $D, E, F, G \in \mathbb{C}_{m,n}$ be such that $A^{\|(D,E)}$ and $B^{\|(F,G)}$ exist. Let D', E', F' and $G' \in \mathbb{C}_{n,m}$ be such that $D = DD'D$, $E = EE'E$, $F = FF'F$ and $G = GG'G$. Then*

$$\begin{aligned} B^{\|(F,G)} - A^{\|(D,E)} &= B^{\|(F,G)}(G'G - E'E)(I_n - AA^{\|(D,E)}) + B^{\|(F,G)}(A - B)A^{\|(D,E)} \\ &\quad + (I_m - B^{\|(F,G)}B)(FF' - DD')A^{\|(D,E)}. \end{aligned}$$

Proof. Since $DD'A^{\|(D,E)} = A^{\|(D,E)}$ and $B^{\|(F,G)}BF = F$, then

$$\begin{aligned} B^{\|(F,G)}BA^{\|(D,E)} - A^{\|(D,E)} &= -(I_m - B^{\|(F,G)}B)DD'A^{\|(D,E)} \\ &= [(I_m - B^{\|(F,G)}B)FF' - (I_m - B^{\|(F,G)}B)DD']A^{\|(D,E)} \\ &= (I_m - B^{\|(F,G)}B)(FF' - DD')A^{\|(D,E)}. \end{aligned}$$

In addition, since $B^{\|(F,G)} = B^{\|(F,G)}G'G$ and $EAA^{\|(D,E)} = E$,

$$\begin{aligned} B^{\|(F,G)} - B^{\|(F,G)}AA^{\|(D,E)} &= B^{\|(F,G)}G'G(I_n - AA^{\|(D,E)}) \\ &= B^{\|(F,G)}[G'G(I_n - AA^{\|(D,E)}) - E'E(I_n - AA^{\|(D,E)})] \\ &= B^{\|(F,G)}(G'G - E'E)(I_n - AA^{\|(D,E)}). \end{aligned}$$

Thus,

$$\begin{aligned} B^{\|(F,G)} - A^{\|(D,E)} &= B^{\|(F,G)}(G'G - E'E)(I_n - AA^{\|(D,E)}) + B^{\|(F,G)}AA^{\|(D,E)} \\ &\quad + (I_m - B^{\|(F,G)}B)(FF' - DD')A^{\|(D,E)} - B^{\|(F,G)}BA^{\|(D,E)} \\ &= B^{\|(F,G)}(G'G - E'E)(I_n - AA^{\|(D,E)}) + B^{\|(F,G)}(A - B)A^{\|(D,E)} \\ &\quad + (I_m - B^{\|(F,G)}B)(FF' - DD')A^{\|(D,E)}. \quad \square \end{aligned}$$

Next a result regarding the continuity of the (D,E)-inverse will be presented.

THEOREM 8.2. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ be such that $A^{\|(D,E)}$ exists and consider $(A_k)_{k\in\mathbb{N}} \subset \mathbb{C}_{n,m}$ and $(D_k)_{k\in\mathbb{N}}$, $(E_k)_{k\in\mathbb{N}} \subset \mathbb{C}_{m,n}$ such that $A_k^{\|(D_k,E_k)}$ exists for each $k \in \mathbb{N}$. Let $D', E' \in \mathbb{C}_{n,m}$ and $(D'_k)_{k\in\mathbb{N}}$,*

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$(E'_k)_{k\in\mathbb{N}} \subset \mathbb{C}_{n,m}$ be such that $D = DD'D$, $E = EE'E$, $D_k = D_kD'_kD_k$ and $E_k = E_kE'_kE_k$, for each $k \in \mathbb{N}$. Suppose that $(A_k)_{k\in\mathbb{N}}$, $(D_kD'_k)_{k\in\mathbb{N}}$ and $(E'_kE_k)_{k\in\mathbb{N}}$ converge to A, DD' and $E'E$, respectively. Then, the following statements are equivalent.

(i) $(A_k^{\|(D_k,E_k)})_{k\in\mathbb{N}}$ *converges to* $A^{\|(D,E)}$.
(ii) *The sequence* $(A_k^{\|(D_k,E_k)})_{k\in\mathbb{N}}$ *is bounded.*

Proof. Apply Lemma 8.1. □

If the Moore-Penrose inverse is used, then a more general result can be presented.

Theorem 8.3. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ be such that $A^{\|(D,E)}$ exists and consider $(A_k)_{k\in\mathbb{N}} \subset \mathbb{C}_{n,m}$ and $(D_k)_{k\in\mathbb{N}}$, $(E_k)_{k\in\mathbb{N}} \subset \mathbb{C}_{m,n}$ such that $A_k^{\|(D_k,E_k)}$ exists for each $k \in \mathbb{N}$. Suppose that $(A_k)_{k\in\mathbb{N}}$ converges to A, $(D_k)_{k\in\mathbb{N}}$ converges to D, and $(E_k)_{k\in\mathbb{N}}$ converges to E. Then, the following statements are equivalent.*

(i) $(A_k^{\|(D_k,E_k)})_{k\in\mathbb{N}}$ *converges to* $A^{\|(D,E)}$.
(ii) *The sequence $(A_k^{\|(D_k,E_k)})_{k\in\mathbb{N}} \subset \mathbb{C}_{m,n}$ is bounded and the sequences $(D_k^\dagger)_{k\in\mathbb{N}}$ and $(E_k^\dagger)_{k\in\mathbb{N}}$ converge to $D^\dagger$ and $E^\dagger$, respectively.*

Proof. Suppose that $(A_k^{\|(D_k,E_k)})_{k\in\mathbb{N}}$ converges to $A^{\|(D,E)}$. Then, the sequence $(A_k^{\|(D_k,E_k)}A_k)_{k\in\mathbb{N}}$ converges to $A^{\|(D,E)}A$. Consequently,

$$\lim_{k\to\infty} \operatorname{tr}(A_k^{\|(D_k,E_k)}A_k) = \operatorname{tr}(A^{\|(D,E)}A).$$

Since $A^{\|(D,E)}$ is an outer inverse (Theorem 6.1), $A^{\|(D,E)}A$ is an idempotent. Thus,

$$\operatorname{tr}(A^{\|(D,E)}A) = \operatorname{rk}(A^{\|(D,E)}A) = \operatorname{rk}(A^{\|(D,E)}).$$

Similarly, $\operatorname{tr}(A_k^{\|(D,E)}A) = \operatorname{rk}(A_k^{\|(D,E)})$, for $k \in \mathbb{N}$. As a result, for sufficiently large $k \in \mathbb{N}$, $\operatorname{rk}(A_k^{\|(D,E)}) = \operatorname{rk}(A^{\|(D,E)})$. However, according to Theorem 4.4

$$\operatorname{rk}(D_k) = \operatorname{rk}(E_k) = \operatorname{rk}(A_k^{\|(D,E)}) = \operatorname{rk}(A^{\|(D,E)}) = \operatorname{rk}(D) = \operatorname{rk}(E).$$

Therefore, according to [20], $(D_k^\dagger)_{k\in\mathbb{N}}$ converges to $D^\dagger$ and $(E_k^\dagger)_{k\in\mathbb{N}}$ to $E^\dagger$. The remaining part of statement (ii) is evident.

If statement (ii) holds, then apply Theorem 8.2 with $D' = D^\dagger$ and $E' = E^\dagger$, $k \in \mathbb{N}$. □

If Theorem 8.3 is applied for the case $D = E$, then the case of the inverse along a matrix can be obtained.

Now the differentiability will be studied.

Theorem 8.4. *Let $J \subseteq \mathbb{R}$ be an open set and consider $t_0 \in J$. Let functions $\mathcal{A} : J \to \mathbb{C}_{n,m}$ and $\mathcal{D}$, $\mathcal{E} : J \to \mathbb{C}_{m,n}$ be such that $\mathcal{A}(t)$ is $(\mathcal{D}(t), \mathcal{E}(t))$-invertible, for any $t \in J$, and $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{E}$ are differentiable at t_0. Suppose that $f : J \to \mathbb{C}_{m,n}$, $f(t) = \mathcal{A}(t)^{\|(\mathcal{D}(t),\mathcal{E}(t))}$, is a bounded function in J and that the functions $\mathcal{D}$, $\mathcal{E}$ have local constant rank in J. Then, the function f is differentiable at t_0 and*

$$\begin{aligned} f'(t_0) = {} & \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))} \left[\mathcal{G}'(t_0)\mathcal{E}(t_0) + \mathcal{G}(t_0)\mathcal{E}'(t_0)\right] \left[I_n - \mathcal{A}(t_0)\mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}\right] \\ & + \left[I_n - \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}\mathcal{A}(t_0)\right] \left[\mathcal{D}'(t_0)\mathcal{F}(t_0) + \mathcal{D}(t_0)\mathcal{F}'(t_0)\right] \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))} \\ & + \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}\mathcal{A}'(t_0)\mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}, \end{aligned}$$

where $\mathcal{F}$, $\mathcal{G}$: $J \to \mathbb{C}_{n,m}$ are the functions $\mathcal{F}(t) = (\mathcal{D}(t))^\dagger$ and $\mathcal{G}(t) = (\mathcal{E}(t))^\dagger$.

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Proof. Observe that according to Lemma 8.1, for any $t \in J$,

$$\begin{aligned} f(t) - f(t_0) = &\mathcal{A}(t)^{\|(\mathcal{D}(t),\mathcal{E}(t))} \left[\mathcal{E}(t)^\dagger \mathcal{E}(t) - \mathcal{E}(t_0)^\dagger \mathcal{E}(t_0)\right] \left[I_n - \mathcal{A}(t_0)\mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}\right] \\ &+ \left[I_n - \mathcal{A}(t)^{\|(\mathcal{D}(t),\mathcal{E}(t))}\mathcal{A}(t)\right] \left[\mathcal{D}(t)\mathcal{D}(t)^\dagger - \mathcal{D}(t_0)\mathcal{D}(t_0)^\dagger\right] \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))} \\ &+ \mathcal{A}(t)^{\|(\mathcal{D}(t),\mathcal{E}(t))} \left[\mathcal{A}(t_0) - \mathcal{A}(t)\right] \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}. \end{aligned}$$

Now, according to [20], the functions $\mathcal{F}$, $\mathcal{G} : J \to \mathbb{C}_{n,m}$, $\mathcal{F}(t) = (\mathcal{D}(t))^\dagger$ and $\mathcal{G}(t) = (\mathcal{E}(t))^\dagger$ are continuous. Consequently, according to Theorem 8.3,

$$\lim_{t\to t_0} \mathcal{A}(t)^{\|(\mathcal{D}(t),\mathcal{E}(t))} \frac{\left[\mathcal{A}(t_0) - \mathcal{A}(t)\right]}{t - t_0} \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))} = \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))} \mathcal{A}'(t_0) \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}.$$

In addition, according to [12], the functions $\mathcal{F}$, $\mathcal{G} : J \to \mathbb{C}_{n,m}$ are also differentiable. Thus,

$$\begin{aligned} &\lim_{t\to t_0} \mathcal{A}(t)^{\|(\mathcal{D}(t),\mathcal{E}(t))} \frac{\left[\mathcal{E}(t)^\dagger \mathcal{E}(t) - \mathcal{E}(t_0)^\dagger \mathcal{E}(t_0)\right]}{t - t_0} \left[I_n - \mathcal{A}(t_0)\mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}\right] = \\ &\qquad \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))} \left[\mathcal{G}'(t_0)\mathcal{E}(t_0) + \mathcal{G}(t_0)\mathcal{E}'(t_0)\right] \left[I_n - \mathcal{A}(t_0)\mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}\right]. \end{aligned}$$

Similarly,

$$\begin{aligned} &\lim_{t\to t_0} \left[I_n - \mathcal{A}(t)^{\|(\mathcal{D}(t),\mathcal{E}(t))}\mathcal{A}(t)\right] \frac{\left[\mathcal{D}(t)\mathcal{D}(t)^\dagger - \mathcal{D}(t_0)\mathcal{D}(t_0)^\dagger\right]}{t - t_0} \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))} = \\ &\quad \left[I_n - \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}\mathcal{A}(t_0)\right] \left[\mathcal{D}'(t_0)\mathcal{F}(t_0) + \mathcal{D}(t_0)\mathcal{F}'(t_0)\right] \mathcal{A}(t_0)^{\|(\mathcal{D}(t_0),\mathcal{E}(t_0))}. \qquad \square \end{aligned}$$

The differentiability of the inverse along a matrix can be studied if Theorem 8.4 is applied for the case $D = E$.

9. Explicit computations. In this section, some explicit algorithms to compute $A^{\|(D,E)}$ will be given.

THEOREM 9.1. *Let $A \in \mathbb{C}_{n,m}$, $D, E \in \mathbb{C}_{m,n}$, $r = \text{rk}(D)$ and $s = \text{rk}(E)$. If $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is a basis of $\mathcal{R}(D)$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_{n-s}\}$ is a basis of $\mathcal{N}(E)$, then the following statements are equivalent:*

(i) *$A^{\|(D,E)}$ exists.*
(ii) *The matrix $X = [A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_r \ \mathbf{w}_1 \ \cdots \ \mathbf{w}_{n-s}]$ is nonsingular.*

In this case, $A^{\|(D,E)} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_r \ \mathbf{0} \ \cdots \ \mathbf{0}][A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_r \ \mathbf{w}_1 \ \cdots \ \mathbf{w}_{n-s}]^{-1}$.

Proof. If statement (i) holds, then according to Theorem 4.4, $\text{rk}(D) = \text{rk}(E)$. Let $X_1 = [A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_r]$ and $X_2 = [\mathbf{w}_1 \ \cdots \ \mathbf{w}_{n-r}]$. Observe that $n-r = \text{rk}(X_2)$ because $\{\mathbf{w}_i\}_{i=1}^{n-r}$ is a basis. According to Theorem 4.2 and Theorem 3.4, $\text{rk}(X) = \text{rk}(X_1) + \text{rk}(X_2)$ (because $\mathcal{R}(AD) \cap \mathcal{N}(E) = 0$). Since $\{A\mathbf{v}_i\}_{i=1}^r$ span $\mathcal{R}(AD)$ and $r = \text{rk}(D) = \text{rk}(AD) = \dim \mathcal{R}(AD)$, the vectors $\{A\mathbf{v}_i\}_{i=1}^r$ are linearly independent, and thus, $r = \text{rk}(X_1)$. Therefore, $n = \text{rk}(X)$ and by recalling that $X \in \mathbb{C}_n$, the nonsingularity of X is obtained.

Suppose that statement (ii) holds. Since the matrix in statement (ii) must be square, $\text{rk}(D) = r = s = \text{rk}(E)$. In addition, since the matrix in statement (ii) is invertible, $\text{rk}(AD) = \text{rk}(D)$ and $\mathcal{R}(AD) \oplus \mathcal{N}(E) = \mathbb{C}^n$. Consequently, according to Theorem 3.4 and Theorem 4.2, $A^{\|(D,E)}$ exists.

Now let $\mathbf{v} \in \mathcal{R}(D)$ and let $\mathbf{x} \in \mathbb{C}^n$ be such that $\mathbf{v} = D\mathbf{x}$. According to Definition 4.1, $A^{\|(D,E)}A\mathbf{v} = A^{\|(D,E)}AD\mathbf{x} = D\mathbf{x} = \mathbf{v}$. In addition, $A^{\|(D,E)}\mathbf{w} = \mathbf{0}$ for any $\mathbf{w} \in \mathcal{N}(E)$. Therefore,

$$A^{\|(D,E)}[A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_r \ \mathbf{w}_1 \ \cdots \ \mathbf{w}_{n-r}] = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_r \ \mathbf{0} \ \cdots \ \mathbf{0}]. \qquad \square$$

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The following `m-file` that can be executed in Matlab or in Octave shows how Theorem 9.1 can be used to compute $A^{\|(D,E)}$.

```
function J = pseudo(A,D,E)
[n m] = size(A);
r = rank(D); s = rank(E);
E1 = null(E); % An orthonormal basis of N(E)
D1 = orth(D); % An orthonormal basis of R(D)
aux = [A*D1 E1];
if not(r==s)
   disp('There does not exist the pseudoinverse')
   disp('because rank(D) is not equal to rank(E)')
else
  if det(aux)==0
   disp('There does not exist the pseudoinverse')
   disp('because the matrix of Th. 9.1 is singular')
   else
     J=[D1 zeros(n,n-r)]*inv(aux);
   end
end
```

Theorem 9.2. *Let $A \in \mathbb{C}_{n,m}$ and $D, E \in \mathbb{C}_{m,n}$ be such that $A^{\|(D,E)}$ exists. Let $r = \mathrm{rk}(D) = \mathrm{rk}(E) = \mathrm{rk}(EAD)$. Let $P \in \mathbb{C}_m$ and $Q \in \mathbb{C}_n$ be two nonsingular matrices such that $PEADQ = \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then*

$$PE = \begin{bmatrix} X \\ 0 \end{bmatrix} \qquad \text{and} \qquad DQ = \begin{bmatrix} Y & 0 \end{bmatrix}, \tag{9.7}$$

where $X \in \mathbb{C}_{r,n}$, $Y \in \mathbb{C}_{m,r}$. Furthermore, $A^{\|(D,E)} = YX$.

Proof. Write P and Q as $P = \left[\begin{smallmatrix} P_1 \\ P_2 \end{smallmatrix}\right]$ and $Q = [Q_1\ Q_2]$, where $P_1 \in \mathbb{C}_{r,m}$, $P_2 \in \mathbb{C}_{m-r,m}$, $Q_1 \in \mathbb{C}_{n,r}$ and $Q_2 \in \mathbb{C}_{n,n-r}$. Now

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PEADQ = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} EAD \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} P_1EADQ_1 & P_1EADQ_2 \\ P_2EADQ_1 & P_2EADQ_2 \end{bmatrix},$$

which implies $P_1EADQ_2 = 0$, $P_2EADQ_1 = 0$ and $P_2EADQ_2 = 0$. Therefore,

$$P_2EADQ = P_2EAD[Q_1\ Q_2] = [P_2EADQ_1\ P_2EADQ_2] = 0.$$

The nonsingularity of Q leads to $P_2EAD = 0$. In a similar way, $EADQ_2 = 0$.

Since $\mathrm{rk}(D) = \mathrm{rk}(E) = \mathrm{rk}(EAD)$, the equalities $\mathcal{R}(E) = \mathcal{R}(EAD)$ and $\mathcal{N}(D) = \mathcal{N}(EAD)$ are obtained. In addition, since $EADQ_2 = 0$ and $\mathcal{N}(EAD) = \mathcal{N}(D)$, it can be deduced that $DQ_2 = 0$. Since $(EAD)^*P_2^* = (P_2EAD)^* = 0$, any column of P_2^* belongs to $\mathcal{N}((EAD)^*) = \mathcal{R}(EAD)^\perp = \mathcal{R}(E)^\perp = \mathcal{N}(E^*)$, and therefore, $E^*P_2^* = 0$, i.e., $P_2E = 0$. If $Y = DQ_1$ and $X = P_1E$, then (9.7) holds.

Now it will be proved that YX satisfies Definition 4.1. First, observe that $XAY = P_1EADQ_1 = I_r$. Now, by (9.7)

$$YXADQ = YXA[Y\ 0] = [YXAY\ 0] = [Y\ 0] = DQ,$$

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and the nonsingularity of Q leads to $YXAD = D$. Similarly,

$$PEAYX = \begin{bmatrix} X \\ 0 \end{bmatrix} AYX = \begin{bmatrix} XAYX \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix} = PE,$$

and thus, $EAYX = E$. Since $YX = DQ_1X$, the inclusion $\mathcal{R}(YX) \subseteq \mathcal{R}(D)$ can be obtained. In addition, since $YX = YP_1E$, it can be deduced $\mathcal{N}(E) \subseteq \mathcal{N}(YX)$. ∎

Remark 9.3. Observe that it is possible to use either the Gaussian elimination method or the singular value decomposition of EAD to determine P and Q. Let $r = \operatorname{rk}(EAD)$.

(i) By using the Gauss-Jordan elimination, there exist an elementary row operation matrix $P \in \mathbb{C}_{m,m}$ and an elementary column operation matrix $Q \in \mathbb{C}_{n,n}$, such that $PEADQ = \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

(ii) Let $EAD = USV^*$ be the singular value decomposition of EAD, where $S = \Sigma \oplus 0$, $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$. Hence, $U^*EADV = \Sigma \oplus 0$, which implies

$$(\Sigma^{-1/2} \oplus I_{m-r})U^*EADV(\Sigma^{-1/2} \oplus I_{n-r}) = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $P = (\Sigma^{-1/2} \oplus I_{m-r})U^*$ and $Q = V(\Sigma^{-1/2} \oplus I_{n-r})$. It is easy to see that P and Q are nonsingular.

Theorem 9.2 and Remark 9.3 (i) yield an elimination method to compute $A^{\|(D,E)}$, which is presented now.

Algorithm 9.4. Compute $A^{\|(D,E)}$.

Input: $A \in \mathbb{C}_{n,m}$, $D, E \in \mathbb{C}_{m,n}$ with $\operatorname{rk}(D) = \operatorname{rk}(E) = \operatorname{rk}(EAD) = r$.

Output: $A^{\|(D,E)}$.

1. Execute elementary row operations on the first m rows of the block matrix

$$G = \begin{bmatrix} EAD & E \\ D & 0 \end{bmatrix}$$

to get

$$G_1 = \begin{bmatrix} \begin{bmatrix} W \\ 0 \end{bmatrix} & \begin{bmatrix} X \\ 0 \end{bmatrix} \\ D & 0 \end{bmatrix}.$$

2. Execute elementary column operations on the first m columns of the block matrix G_1 to get

$$G_2 = \begin{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} X \\ 0 \end{bmatrix} \\ \begin{bmatrix} Y & 0 \end{bmatrix} & 0 \end{bmatrix}.$$

3. $A^{\|(D,E)} = YX$.

Theorem 9.2 and Remark 9.3 (ii) yield a more stable numerical method based on the SVD to compute $A^{\|(D,E)}$, which is shown now.

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ALGORITHM 9.5. Compute $A^{\|(D,E)}$

Input: $A \in \mathbb{C}_{n,m}$, $D, E \in \mathbb{C}_{m,n}$ with $\mathrm{rk}(D) = \mathrm{rk}(E) = \mathrm{rk}(EAD) = r$.

Output: $A^{\|(D,E)}$.

1. Compute the SVD of EAD, i.e., $EAD = USV^*$.
2. $T = S(1:r, 1:r)$, $M = T^{-1/2} \oplus I_{m-r}$, $N = T^{-1/2} \oplus I_{n-r}$.
3. $P = MU^*$, $Q = VN$.
4. $X = PE$, $Y = DQ$.
5. $A^{\|(D,E)} = Y(1:m, 1:r) \cdot X(1:r, 1:n)$.

The following `m-file` shows how Theorem 9.2 and the SVD can be used to compute $A^{\|(D,E)}$.

```
function J = pseudo(A,D,E)
[n m] = size(A); r = rank(D); s = rank(E); t = rank(E*A*D);
if not(r==s)
   disp('There does not exist the pseudoinverse')
   disp('because rank(D) is not equal to rank(E)')
else
  if not(s==t)
   disp('There does not exist the pseudoinverse')
   disp('because rank(D)=rank(E) but not equal to rank(EAD)')
   else
     [U S V] = svd(E*A*D); T = S(1:r,1:r)
     M = [T^(-1/2) zeros(r,m-r); zeros(m-r,r) eye(m-r)];
     N = [T^(-1/2) zeros(r,n-r); zeros(n-r,r) eye(n-r)];
     P = M*U'; Q = V*N; X = P*E; Y = D*Q;
     J = Y(1:m,1:r)*X(1:r,1:n);
   end
end
```

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Added at the end of the revision. Throughout the elaboration of the present article, the source of the definitions of the notions of left and right (b, c)-invertible elements in rings was the work [14]. However, at the end of the revision the authors discovered the article [11] in which the same definitions were given in the context of semigroups.

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