Electronic Journal of Linear Algebra

Volume 32 *Volume 32 (2017)* Article 27

2017

On the Location of Eigenvalues of Real Matrices

Rachid Marsli
King Fahd University of Petroleum and Minerals, rmarsliz@kfupm.edu.sa

Frank Hall
Georgia State University, fhall@gsu.edu

Recommended Citation
Marsli, Rachid and Hall, Frank. (2017), "On the Location of Eigenvalues of Real Matrices", *Electronic Journal of Linear Algebra*, Volume 32, pp. 357-364.
DOI: https://doi.org/10.13001/1081-3810.3544

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 357-364, September 2017.
http://repository.uwyo.edu/ela

ON THE LOCATION OF EIGENVALUES OF REAL MATRICES*

RACHID MARSLI† AND FRANK J. HALL‡

Abstract. The research in this paper is motivated by a recent work of I. Barany and J. Solymosi [I. Barany and J. Solymosi. Gershgorin disks for multiple eigenvalues of non-negative matrices. Preprint arXiv no. 1609.07439, 2016.] about the location of eigenvalues of nonnegative matrices with geometric multiplicity higher than one. In particular, an answer to a question posed by Barany and Solymosi, about how the location of the eigenvalues can be improved in terms of their geometric multiplicities is obtained. New inclusion sets for the eigenvalues of a real square matrix, called Geršgorin discs of the second type, are introduced. It is proved that under some conditions, an eigenvalue of a real matrix is in a Geršgorin disc of the second type. Some relationships between the geometric multiplicities of eigenvalues and these new inclusion sets are established. Some other related results, consequences, and examples are presented. The results presented here apply not only to nonnegative matrices, but extend to all real matrices, and some of them do not depend on the geometric multiplicity.

Key words. Real matrix, Eigenvalue, Geršgorin disc, Radius.

AMS subject classifications. 15A18, 15A42.

1. Introduction. In 1931, S. Geršgorin published his famous theorem about the location of eigenvalues, [3]. The first part of this theorem states that every eigenvalue of a given $n \times n$ matrix is located in the union of its Geršgorin discs. There is an abundant literature about Geršgorin's theorem and follow-up results. Some recommended references are Chapter 6 from the book [4] by R. Horn and C. Johnson, as well as the book [9] by R. Varga.

The work we have recently done in [2], [5], [6], [7] and [8] shows a strong connection between Geršgorin theory and the geometric multiplicities of eigenvalues. The main theorem in [5] states:

THEOREM 1.1. *Let A be an $n \times n$ matrix and let λ be an eigenvalue of A with geometric multiplicity k. Then λ is in at least k Geršgorin discs of A.*

In the proof of this theorem, the following key preliminary result was employed.

THEOREM 1.2. *Let S be a k-dimensional subspace of $\mathbb{C}^n$. There is a basis $\{v_1, v_2, \ldots, v_k\}$ of S with the following property: For each $i = 1, 2, \ldots, k$, there is a distinct integer p_i, with $1 \leq p_i \leq n$ and $p_i \neq p_j$ for $i \neq j$, such that a largest modulus entry of each v_i is in position p_i.*

More recently, in [7], Theorem 1.1 has been improved in the following way.

THEOREM 1.3. *Let A be an $n \times n$ matrix and let λ be an eigenvalue of A with geometric multiplicity k. Construct the $n \times n$ matrix $C_k(A)$ in the following way: In every row of A, replace the smallest k-1 off-diagonal entries in absolute value by zeros. Then λ is in at least k Geršgorin discs of $C_k(A)$.*

Some consequences and applications of the above theorems to the general case of square matrices, as

*Received by the editors on May 10, 2016. Accepted for publication on August 16, 2017. Handling Editor: Raphael Loewy.
†Preparatory Math Program, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia (rmarsliz@kfupm.edu.sa). Supported by grant no. SR161014 of King Fahd University of Petroleum and Minerals.
‡Department of Mathematics and Statistics, Georgia State University, Atlanta, Georgia, USA (fhall@gsu.edu).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 357-364, September 2017.
http://repository.uwyo.edu/ela

well to some large classes of matrices, have been discussed in the five articles mentioned above. To cite just few of them, in [6], it has been proven that:

THEOREM 1.4. *let A be an $n \times n$ matrix with s distinct eigenvalues $\lambda_1, \dots, \lambda_s$, which have geometric multiplicities $k_1, \dots, k_s$ and algebraic multiplicities $n_1, \dots, n_s$, respectively. Then for any integers r_i such that $k_i \le r_i \le n_i$, there is a matrix B similar to A such that λ_i is in precisely r_i Geršgorin discs of B, $i = 1, \dots, s$.*

In [8], we have the following result about the ranks of matrices:

THEOREM 1.5. *Let A be an $n \times n$ matrix, and let k be an integer with $1 \le k \le n$. Define the matrix C_k as in Theorem 1.3, and let $R_i'(C_k)$ be the sum of the absolute values of all off-diagonal elements from the i^{th} row of C_k. If $|a_{ii}| > R'(C_k)$ for more than $n-k$ values of i, then the rank of A is strictly greater than $n-k$; equivalently, the geometric multiplicity of 0, as a potential eigenvalue of A, cannot exceed $k-1$.*

Notice that $C_1(A)$ is the matrix A itself. Therefore, the case $k = 1$ in this theorem represents the classical theorem about the non-singularity of strictly diagonally dominant matrices.

In this paper, we continue our work about the location of eigenvalues, but this time we are inspired by a recent work of I. Barany and J. Solymosi [1] about the location of eigenvalues of nonnegative matrices with geometric multiplicity higher than one. In particular, we will answer a question posed in [1] about how the location of the eigenvalues can be improved in terms of their geometric multiplicities. However, the results that we present here apply not only to nonnegative matrices, but extend to all real matrices, and some of them, as we will see, do not depend on the geometric multiplicity at all.

2. Geršgorin discs of the second type. There are many nonnegative, even irreducible, $n \times n$ matrices, with an eigenvalue of high geometric multiplicity. As for examples, the (symmetric) adjacency matrix of the star graph on n vertices has 0 as an eigenvalue with multiplicity $n-2$, while the (symmetric) adjacency matrix of the complete graph on n vertices has -1 as an eigenvalue with multiplicity $n-1$. In [1], I. Barany and J. Solymosi came up with the following nice result about nonnegative matrices.

THEOREM 2.1. *Let $A = [a_{ij}]$ be an $n \times n$ nonnegative matrix and let λ be an eigenvalue of A with geometric multiplicity at least two. Then λ is in a half Geršgorin disk of A.*

Here, the expression "half Geršgorin disc" means a disc centered at a diagonal element a_{ii} of A, and with its radius equal to the partial sum of the $\lfloor \frac{n}{2} \rfloor$ largest non-diagonal elements, in absolute value, chosen from row i of A. The proof of Theorem 2.1 is based on the following two results that can be found in [1], about vector arrangements and geometric estimates.

THEOREM 2.2. *Let $V = \{v_1, v_2, \dots, v_n\} \subset \mathbb{R}^d$ with $\sum_{i=1}^{n} v_i = 0$. Further, let $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n \ge 0$ be nonnegative real numbers and set $\beta = \alpha_{\lfloor \frac{n}{2} \rfloor + 1}$. Then, for every permutation σ of $\{1, 2, \dots, n\}$, we have*

$$\left\| \sum_{i=1}^{n} \alpha_i v_{\sigma(i)} \right\| \le \max_{1 \le i \le n} \|v_i\| \sum_{i=1}^{n} |\alpha_i - \beta|.$$

COROLLARY 2.3. *Under the above conditions,*

$$\left\| \sum_{i=1}^{n} \alpha_i v_{\sigma(i)} \right\| \le \max_{1 \le i \le n} \|v_i\| \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \alpha_i.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 357-364, September 2017.
http://repository.uwyo.edu/ela

Barany and Solymosi have mentioned in their paper that the sum $\sum_{i=1}^{n} |\alpha_i - x|$ takes its minimum at $x = \alpha_{\frac{n+1}{2}}$, when n is odd, and for every real number $x \in [\alpha_{\frac{n}{2}}, \alpha_{\frac{n}{2}+1}]$, when n is even. Because of the importance of this idea for the analysis used in our work, we present it in the form of a lemma, which can be proved using Calculus.

Lemma 2.4. *Consider the real function of the real variable $f(x) = \sum_{i=1}^{n} |x - \beta_i|$, with $\beta_1 \geq \cdots \geq \beta_n$ not necessarily distinct n real numbers.*

1. *If n is odd, then $\min_{x\in\mathbb{R}} f(x) = \Big(\beta_1 + \cdots + \beta_{\frac{n-1}{2}}\Big) - \Big(\beta_{\frac{n+3}{2}} + \cdots + \beta_n\Big)$. This minimum is reached when $x = \beta_{\frac{n+1}{2}}$.*
2. *If n is even, then $\min_{x\in\mathbb{R}} f(x) = \big(\beta_1 + \cdots + \beta_{\frac{n}{2}}\big) - \big(\beta_{\frac{n}{2}+1} + \cdots + \beta_n\big)$. This takes place for every $x \in [\beta_{\frac{n}{2}}, \beta_{\frac{n}{2}+1}]$ if $\beta_{\frac{n}{2}} \neq \beta_{\frac{n}{2}+1}$, and only for $x = \beta_{\frac{n}{2}}$ if $\beta_{\frac{n}{2}} = \beta_{\frac{n}{2}+1}$.*

Definition 2.5. Let $A = [a_{ij}]$ be an $n \times n$ matrix, and let $x_{i1} \geq \cdots \geq x_{in}$ be a rearrangement in non-increasing order of $a_{i1}, \ldots, a_{ii-1}, 0, a_{ii+1}, \ldots, a_{in}$ for $i = 1, \ldots, n$. We call a Geršgorin disc of A of the first type a usual Geršgorin disc centered at a diagonal element of the matrix A and with its radius equal to the sum of the absolute values of the off-diagonal elements belonging to the same row as the center of the disc. A Geršgorin disc of the second type of A, denoted $\hat{D}(a_{ii}, \hat{r}_i)$, satisfies the following conditions:

1. Its center a_{ii} is the diagonal element from the i^{th} row of A.
2. Its radius is
   (a) $\hat{r}_i = \sum_{j=1}^{\frac{n-1}{2}} x_{ij} - \sum_{j=\frac{n+3}{2}}^{n} x_{ij}$, if n is odd.
   (b) $\hat{r}_i = \sum_{j=1}^{\frac{n}{2}} x_{ij} - \sum_{j=\frac{n}{2}+1}^{n} x_{ij}$, if n is even.

The following theorem states a sufficient condition for an eigenvalue to be in a Geršgorin disc of the second type.

Theorem 2.6. *Let $A = [a_{ij}]$ be an $n \times n$ real matrix, and suppose that λ is an eigenvalue of A associated with an eigenvector orthogonal to the all $1'$s vector. Then λ is in a Geršgorin disc of the second type of A.*

Proof. Let $e = (1, \ldots, 1)^T$ be the all 1's vector in $\mathbb{R}^n$. Suppose that λ is associated with the eigenvector $v = (v_1, \ldots, v_n)^T$ and $v \cdot e = 0$; that is, $\sum v_i = 0$. Without loss of generality, suppose that v_1 is the largest, in absolute value, among the elements of v. Then $Av = \lambda v$ implies

$$(\lambda - a_{11})v_1 = \sum_{j=2}^{n} a_{1j} v_j = (0 - x)v_1 + \sum_{j=2}^{n} (a_{1j} - x)v_j,$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 357-364, September 2017.
http://repository.uwyo.edu/ela

since $\sum_{j=1}^{n} xv_j = 0$. Hence,

$$\begin{aligned}|\lambda - a_{11}|\,|v_1| &= \left|(0-x)v_1 + \sum_{j=2}^{n}(a_{1j}-x)v_j\right| \\ &\le |0-x|\,|v_1| + \sum_{j=2}^{n}|(a_{1j}-x)|\,|v_j| \\ &\le \left(|0-x| + \sum_{j=2}^{n}|(a_{1j}-x)|\right)|v_1|;\end{aligned}$$

that is,

$$|\lambda - a_{11}| \le \min_{x\in\mathbb{R}}\left(|x| + \sum_{j=2}^{n}|(a_{1j}-x)|\right).$$

Therefore, the theorem follows by recalling Lemma 2.4 and using Definition 2.5. ☐

Remark 2.7. In contrast to Theorem 1.1, Theorem 1.3 and Theorem 2.1, the geometric multiplicity does not play any role in Theorem 2.6.

Next, we give an answer to the question posed in [1], about how to improve the location of eigenvalues with geometric multiplicities higher than 2. To do this, we need the following result, which can be easily verified.

Lemma 2.8. *Let S be a subspace of $\mathbb{C}^n$ of dimension $k \ge 2$, and let $w \ne 0$ be a vector in $\mathbb{C}^n$. Then there are at least $(k-1)$ linearly independent unit vectors in S that are orthogonal to w.*

The next corollary follows directly from the above lemma.

Corollary 2.9. *Let S_1 and S_2 be two subspaces of $\mathbb{C}^n$ of dimensions k_1 and k_2, respectively, with $k_2 > k_1 \ge 1$. Then there are at least $(k_2 - k_1)$ linearly independent vectors in S_2 that are orthogonal to S_1.*

Proof. Use Lemma 2.8 and induction. ☐

Theorem 2.10. *Let A be an $n\times n$ real matrix, and let λ be an eigenvalue of A with geometric multiplicity $k \ge 2$. Then λ is in the intersection of at least one Geršgorin disc of the first type of $C_k(A)$, and at least $(k-1)$ Geršgorin discs of the second type of A, each one of the discs being constructed from a different row of A. The matrix $C_k(A)$ is defined as in Theorem 1.3.*

Proof. The eigenspace X of λ has dimension k. According to Lemma 2.8, there are $(k-1)$ linearly independent eigenvectors $v_1, v_2, \ldots, v_{k-1}$ from X that are orthogonal to the all one's vector $e = (1, 1, \ldots, 1)^T$. These vectors span a subspace of X in which all the vectors are orthogonal to e, and by Theorem 1.2, we can construct another basis $\{u_1, u_2, \ldots, u_{k-1}\}$ of this subspace, such that each vector in this basis is orthogonal to the all one's vector, and at the same time, has an element of largest absolute value in a different position compared to the others. With a reasoning similar to the one used in the proof of Theorem 2.6, each one of these $(k-1)$ vectors allows us to construct from a distinct row of A, a Geršgorin disc, but this time, of the second type. Now we use Theorem 1.3 to conclude that λ is in the intersection of these $(k-1)$ discs with at least one Geršgorin disc of the first type of $C_k(A)$, constructed from a different row. ☐

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 357-364, September 2017.
http://repository.uwyo.edu/ela

Remark 2.11. Theorem 2.1 becomes now a consequence of Theorem 2.10 for if A is nonnegative, then each one of its discs of the second type is contained in one of the so called, in Theorem 2.1, "half Geršgorin discs of A" constructed from the same row.

Example 2.12. Let

$$A = \begin{bmatrix} 8 & 2 & 1 & 0 & 1 & 1 \\ 1 & 9 & 1 & 3 & 2 & 1 \\ 8 & 10 & 3 & 3 & 3 & 2 \\ 15 & 12 & 3 & 4 & 4 & 3 \\ 16 & 20 & 4 & 6 & 7 & 4 \\ 9 & 18 & 3 & 6 & 5 & 4 \end{bmatrix}.$$

The Geršgorin discs of the first type of A are

$$D_1(8,5),\ D_2(9,8),\ D_3(3,26),\ D_4(4,37),\ D_5(7,50) \text{ and } D_6(4,41),$$

the Geršgorin discs of the first type of $C_3(A)$ are

$$D_{1C}(8,3),\ D_{2C}(9,5),\ D_{3C}(3,18),\ D_{4C}(4,27),\ D_{5C}(7,36) \text{ and } D_{6C}(4,27),$$

while the Geršgorin discs of the second type of A are

$$\hat{D}_1(8,3),\ \hat{D}_2(9,4),\ \hat{D}_3(3,16),\ \hat{D}_4(4,25),\ \hat{D}_5(7,34) \text{ and } \hat{D}_6(4,25).$$

Now, $\lambda = 1$ is an eigenvalue of A with geometric multiplicity 4 and is included in the intersection of the 3^{rd} Geršgorin disc of the first type of $C_3(A)$ and the last 3 Geršgorin discs of the second type of A. Therefore, Theorem 2.10 is satisfied.

Remark 2.13. A disc of the second type is a subset of the disc of the first type constructed from the same row and having the same center. From the proof of the previous theorem, it can be understood that the real matrix A does not have to be nonnegative. However, from Definition 2.5, it can be clearly seen that this theorem is more efficient for nonnegative and non-positive matrices.

3. Some other inclusion sets. Here, we present a theorem that gives better location of the eigenvalues depending on their geometric multiplicities. According to this theorem, the higher is the geometric multiplicity, the better is the location. The construction of the discs described in the following definition is a little more complicated than the construction of those in Definition 2.5, and the use will depend on the geometric multiplicity.

Definition 3.1. Let $A = [a_{ij}]$ be an $n \times n$ matrix, and let $x_{i1} \geq \cdots \geq x_{in}$ be a rearrangement in nonincreasing order of $a_{i1}, \ldots, a_{ii-1}, 0, a_{ii+1}, \ldots, a_{in}$. Let k be an integer such that $1 \leq k \leq n$. For $i = 1, \ldots, n$, define the disc $D_i^k(A)$ in the following way:

1. Its center a_{ii} is the diagonal element from the i^{th} row of A.
2. Its radius is
   (a) $r_{ik} = \displaystyle\sum_{j=1}^{\frac{n-k+1}{2}} x_{ij} - \sum_{j=\frac{n+k+1}{2}}^{n} x_{ij}$, if $n-k$ is odd.
   (b) $r_{ik} = \displaystyle\sum_{j=1}^{\frac{n-k+2}{2}} x_{ij} - \sum_{j=\frac{n+k}{2}}^{n} x_{ij}$, if $n-k$ is even.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 357-364, September 2017.
http://repository.uwyo.edu/ela

Note that when $k = 2$, these discs are the same as the discs of the second type of A. A reason for phrasing this definition in this manner is the following result, which is then used in the proof of Theorem 3.4.

Lemma 3.2. *Let A be an $n \times n$ real matrix and let A_s be a principal submatrix of A of size $(n-k+2) \times (n-k+2)$. Let $\hat{D}_i(A_s)$ be the disc of the second type of A_s constructed from its i^{th} row. There exists an index $j \in \{1, \dots, n\}$ such that $\hat{D}_i(A_s) \subseteq D_j^k(A)$, where $D_j^k(A)$ is as in Definition 3.1. In fact, the index j is such that the i^{th} row of A_s is a part of the j^{th} row of A.*

Proof. Suppose that the i^{th} row of $A_s = [b_{ij}]$ is a part of the j^{th} row of $A = [a_{ij}]$. Since every diagonal element of A_s is a diagonal element of A, it follows that $b_{ii} = a_{jj}$. Let $x_{j1} \geq \cdots \geq x_{jn}$ be a rearrangement in non-increasing order of $a_{j1}, \dots, a_{j,j-1}, 0, a_{j,j+1}, \dots, a_{jn}$ and let $y_{i1} \geq \cdots \geq y_{i,n-k+2}$ be a rearrangement in non-increasing order of $b_{i1}, \dots, b_{i,i-1}, 0, b_{i,i+1}, \dots, b_{i,n-k+2}$. We have $\{y_{i1}, \dots, y_{i,n-k+2}\} \subset \{x_{j1}, \dots, x_{jn}\}$. Therefore, by going back to Definitions 2.5 and 3.1, we can see that the radius of $\hat{D}_i(A_s)$ is less than or equal to the radius of $D_j^k(A)$, and since these two discs have the same center, it follows that $\hat{D}_i(A_s) \subseteq D_j^k(A)$. ∎

Lemma 3.3. *Let A be an $n \times n$ matrix and let λ be an eigenvalue of A with geometric multiplicity k. Let $\hat{A}$ be an $m \times m$ principal submatrix of A with $m > n-k$. Then λ is an eigenvalue of $\hat{A}$ with geometric multiplicity at least $m+k-n$.*

A version of this lemma can be found in [4]. The newer version that we present here can be found in [7].

Theorem 3.4. *Let $A = [a_{i,j}]$ be an $n \times n$ real matrix, and let λ be an eigenvalue of A with geometric multiplicity $k \geq 2$. Let $S = \{D_i^k(A)\}_{1 \leq i \leq n}$, where $D_i^k(A)$ is as in Definition 3.1. Then λ is in the intersection of at least one disc of $C_k(A)$ and at least $(k-1)$ discs from S, all the discs being constructed from k different rows of A.*

Proof. Denote A by B_n. Let M_n be a principal submatrix of A, of size $(n+2-k) \times (n+2-k)$. According to Lemma 3.3, λ is an eigenvalue of M_n with geometric multiplicity ≥ 2, Hence, according to Theorem 2.10, λ is in at least one Geršgorin disc of the second type of M_n. Denote this disc by D_1. From B_n, construct the $(n-1) \times (n-1)$ principal matrix B_{n-1} by removing the row of A from which the previous Geršgorin disc of second type of M_n has been constructed, and by removing the corresponding column. Let M_{n-1} be a principal submatrix of B_{n-1}, of size $(n+2-k) \times (n+2-k)$. Then λ is in at least one Geršgorin disc of the second type of M_{n-1}. Denote this disc by D_2. Continue this process until the matrix B_{n+2-k} is constructed. Now, $M_{n+2-k} = B_{n-k+2}$, and λ is in at least one Geršgorin disc of the second type, and one Geršgorin disc of the first type of $M_{n+2-k} = B_{n-k+2}$. Denote the one of the second type by D_{k-1}. The process stops here, because by deleting another row and column, we obtain a matrix with size less than $(n+2-k) \times (n+2-k)$, to which Lemma 3.3 does not apply. Now, λ is in the intersection of the discs $D_1, D_2, \dots, D_{k-1}$, which have been constructed from $(k-1)$ distinct rows. According to Lemma 3.2, each one of these $k-1$ discs is a subset of a disc from S, and from the above constructions, we obtain $(k-1)$ discs from S associated with $(k-1)$ distinct rows of A. According to Theorem 1.3, λ is in the intersection of k discs of the first type of $C_k(A)$ constructed from distinct rows. Combining the last two facts, the theorem is obtained. ∎

Example 3.5. Let A be as in Example 2.12, so that $\lambda = 1$ is an eigenvalue of A with geometric multiplicity $k = 4$. We use Definition 3.1 to construct the following discs:

$$D_1^3(8,3),\ \ D_2^3(9,4),\ \ D_3^3(3,16),\ \ D_4^3(4,24),\ \ D_5^3(7,32)\ \text{ and }\ D_6^3(4,24).$$

Theorem 3.4 is satisfied by the fact that $\lambda = 1 \in \cap_{i=3}^6 D_i^3(A)$. Notice that each one of these discs is smaller than or equal in size to the Geršgorin disc of the second type of A constructed from the same row.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 357-364, September 2017.
http://repository.uwyo.edu/ela

4. Another sufficient condition for the location of an eigenvalue within a disc of the second type. The fact that an eigenvalue is associated with an eigenvector with elements summing to 0, is not a necessary condition for the eigenvalue to be contained in a Geršgorin disc of the second type, but is a sufficient condition according to Theorem 2.6. The following theorem gives a better sufficient condition.

THEOREM 4.1. *Let A be an $n \times n$ real matrix, and let λ be an eigenvalue of A associated with an eigenvector $v = (v_1, \ldots, v_n)^T$ such that $|\sum_{i=1}^n v_i| \leq |v_p|$ for some $p \in \{1, \ldots, n\}$. Let A' be the $(n+1)\times(n+1)$ matrix obtained from A by appending a row and column of zeros. Then λ is in a Geršgorin disc of the second type of A', associated with one of its first n rows.*

Proof. Let

$$A_\lambda = \begin{bmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \lambda \end{bmatrix},$$

and let $v' = (v'_1, \ldots, v'_n, v'_{n+1})^T = \left(v_1, \ldots, v_n, -\sum_{i=1}^n v_i\right)^T$. Then $A_\lambda v' = \lambda v'$, and $\sum_{i=1}^{n+1} v'_i = 0$. Since there is an element of v with its absolute value larger than or equal to $\left|-\sum_{i=1}^n v_i\right|$, there is a $j \in \{1, \ldots, n\}$ such that $\max_{1\leq i\leq n+1} |v'_i| = |v_j| = |v'_j|$. Therefore, with a reasoning similar to the one used in the proof of Theorem 2.6, λ must be in a Geršgorin disc of the second type, constructed from the j^{th} row of A_λ. ∎

REMARK 4.2. It may happen that the i^{th} Geršgorin disc of the second type of A is the same as the i^{th} Geršgorin disc of the second type of A_λ for $i = 1, 2, \ldots, n$, but generally this is not always the case even when A is nonnegative.

EXAMPLE 4.3. Let

$$A = \begin{bmatrix} -4 & -17 & -11 & -2 \\ 0 & 4 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 1 & -22 & -7 & -1 \end{bmatrix}.$$

Then

$$A' = \begin{bmatrix} -4 & -17 & -11 & -2 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 1 & -22 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The discs of the second type of the matrix A, with respect to the increasing order of the rows, are $\hat{D}_{1A}(-4, 26), \hat{D}_{2A}(4, 0), \hat{D}_{3A}(3, 1)$ and $\hat{D}_{4A}(-1, 30)$, while those of A' are $\hat{D}_{1A'}(-4, 28), \hat{D}_{2A'}(4, 0), \hat{D}_{3A'}(3, 1)$ and $\hat{D}_{4A'}(-1, 29)$. The fifth disc of A' is trivial and useless in this example. The eigenvalues and eigenvectors of A are:

$\lambda_1 = -3,$ associated to $v_1 = [2\ \ 0\ \ 0\ \ -1]^T,$

$\lambda_2 = -2,$ associated to $v_2 = [1\ \ 0\ \ 0\ \ -1]^T,$

$\lambda_3 = 3,$ associated to $v_3 = [1\ \ 0\ \ -1\ \ 2]^T,$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 32, pp. 357-364, September 2017.
http://repository.uwyo.edu/ela

$\lambda_4 = 4$, associated to $v_4 = [\,0 \quad -1 \;\; 1 \;\; 3\,]^T$.

Each one of the above eigenvectors possesses an element with absolute value greater than or equal to the absolute value of the sum of all its elements. As it can be easily seen, each one of the eigenvalues of A is in the intersection of some Geršgorin discs of the second type of A', in accordance to Theorem 4.1.

5. Conclusion. The work done in this paper applies to real matrices since it is based, in large part, on Lemma 2.4. In the case of complex non-real matrices, we may obtain discs of different algebraic structure, since we cannot generally form a positive real radius by summing non-real elements. This is worthy of a separate study. Another point that is worth mentioning is that it is not rare for an eigenvalue to be associated with an eigenvector such that the absolute value of one of its elements is greater than or equal to the absolute value of all its elements. In other words, for every real matrix, there is a chance that one or more of the eigenvalues are located in some Geršgorin discs of the second type of A', where A' is defined as in Theorem 4.1. In connection to that, we state the following open question.

Open Question 5.1. For every $n \times n$ real matrix A, with $n \geq 2$, is it the case that at least one eigenvalue is located in a Geršgorin disc of the second type of A', where A' is defined as in Theorem 4.1?

REFERENCES

[1] I. Barany and J. Solymosi. Gershgorin disks for multiple eigenvalues of non-negative matrices. Preprint arXiv no. 1609.07439, 2016.

[2] M. Fiedler, F.J. Hall, and R. Marsli. Geršgorin discs revisited. *Linear Algebra Appl.*, 438:598–603, 2013.

[3] S. Gerschgorin. die Abgrenzung der Eigenwerte einer Matrix. *Izv. Akad. Nauk.USSR Otd. Fiz.-Mat. Nauk*, 6:749-754, 1931.

[4] R.A. Horn and C.R. Johnson. *Matrix Analysis*, second edition. University Press, Cambridge, 2013.

[5] R. Marsli and F.J. Hall. Geometric multiplicities and Geršgorin discs. *Amer. Math. Monthly*, 120:452–455, 2013.

[6] R. Marsli and F.J. Hall. Further results on Geršgorin discs. *Linear Algebra Appl.*, 439:189–195, 2013.

[7] R. Marsli and F.J. Hall. Some refinements of Geršgorin discs. *Int. J. Algebra*, 7:573–580, 2013.

[8] R. Marsli and F.J. Hall. Some new inequalities on geometric multiplicities and Geršgorin discs. *Int. J. Algebra*, 8:135–147, 2014.

[9] R.S. Varga. *Geršgorin and His Circles.* Springer Series in Computational Mathematics, Vol. 36, Springer Verlag, Berlin, 2004.