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SPECTRAL DYNAMICS OF GRAPH SEQUENCES GENERATED BY SUBDIVISION AND TRIANGLE EXTENSION*

HAIYAN CHEN† AND FUJI ZHANG‡

Abstract. For a graph G and a unary graph operation X, there is a graph sequence $\{G_k\}$ generated by $G_0 = G$ and $G_{k+1} = X(G_k)$. Let $Sp(G_k)$ denote the set of normalized Laplacian eigenvalues of G_k. The set of limit points of $\bigcup_{k=0}^{\infty} Sp(G_k)$, $\liminf_{k\to\infty} Sp(G_k)$ and $\limsup_{k\to\infty} Sp(G_k)$ are considered in this paper for graph sequences generated by two operations: subdivision and triangle extension. It is obtained that the spectral dynamic of graph sequence generated by subdivision is determined by a quadratic function, which is closely related to the the well-known logistic map; while that generated by triangle extension is determined by a linear function. By using the knowledge of dynamic system, the spectral dynamics of graph sequences generated by these two operations are characterized. For example, it is found that, for any initial non-trivial graph G, chaos takes place in the spectral dynamics of iterated subdivision graphs, and the set of limit points is the entire closed interval $[0, 2]$.

Key words. Dynamical system, Eigenvalue, Subdivision graph, Triangle extension graph.

AMS subject classifications. 05C50, 26A18.

1. Introduction.

Let X be a unary operation of graphs. Starting from any graph G, we may iterate the operation X to obtain a graph sequence

$$X^0(G) = G, \;\; X^1(G) = X(G), \;\; X^2(G) = X(X(G)), \;\; \ldots, \;\; X^k(G) = X(X^{k-1}(G)), \;\ldots$$

Statistical physics motivated recent research on the limit behavior of some parameters related to graphs, such as the number of spanning trees, the number of perfect matchings, Kirchhoff index, energy [27, 32–35]. The spectra of a graph is a fruitful topic in algebraic graph theory. The roots of characteristic polynomial of adjacency, Laplacian and normalized Laplacian are called adjacency, Laplacian and normalized Laplacian spectrum, respectively. Many papers and books have been published on spectra of graphs (see for example [14, 15] and the references cited therein). The adjacency spectral dynamics of graph sequences were first studied in [35] by Chen, Chen and one of the present authors, where the graph sequence in consideration is generated by the graph operation of clique-inserting (or called para-line in [28]). It is showed in [35] that for any initial r-regular graph G with $r > 2$, the set of limit points of the adjacency eigenvalues of all graphs in the sequence is a fractal with the maximum r and the minimum -2, and that the fractal is independent of the structure of the G as long as the degree r of G is fixed. In view of the rich and colorful phenomena in dynamical systems, one naturally wants to investigate spectral dynamics of graph sequences generated by other unary operations. In this paper, we shall study the normalized Laplacian spectral dynamics of graph sequences generated by subdivision and triangle extension, respectively. Now we first give the definitions of subdivision and triangle extension.

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Let $G = (V, E)$ be a graph with vertex set V and edge set E.

The *subdivision operation* for an edge $\{u, v\} \in E$ is the deletion of $\{u, v\}$ from G and the addition of two edges $\{u, w\}$ and $\{w, v\}$ along with the new vertex w (so, the three edges $\{u, v\}$, $\{u, w\}$ and $\{w, v\}$ consist of a triangle). The *subdivision graph* $S(G)$ of G is the graph obtained from G by doing subdivision for every edge of G.

The *triangle extension operation* for an edge $\{u, v\} \in E$ is the addition of two edges $\{u, w\}$ and $\{w, v\}$ along with the new vertex w. The *triangle extension graph* $R(G)$ of G is the graph obtained from G by doing triangle extension for every edge of G.

Note that the only difference between $S(G)$ and $R(G)$ is whether we keep the original edges in G (for triangle extension) or not (for subdivision). But we shall see that the normalized Laplacian spectral dynamics of graph sequences generated by these two operations are very different. The set of limit points is the entire internal $[0, 2]$ for the subdivision, while the set of limit points is $\{0\}$ for the triangle extension.

The normalized Laplacian matrix is closely related to random walks on graphs and discrete geometric analysis [4, 9, 24, 29–31]. Now many mathematical results have been obtained (see [5–7, 10–13, 19, 21, 22, 26], for example). By the way normalized Laplacian spectrum provided a powerful weapon in applications such as machine learning, ratio cut partitioning and clustering, see for example [1, 17, 20, 25].

In this paper, all graphs are assumed to be simple and connected. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. Its *adjacency matrix* is defined to be the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if v_i is adjacent to v_j; and $a_{ij} = 0$, otherwise. Its *incidence matrix* is defined to be the $n \times m$ matrix $B(G) = (b_{ij})$, where $b_{ij} = 1$ if v_i is incident with e_j; and $b_{ij} = 0$, otherwise. Let d_i denote the degree of vertex v_i, $D(G) - A(G)$ is called the (combinatorial) *Laplacian matrix* of G, where $D(G) = diag(d_1, d_2, \dots, d_n)$ is the degree diagonal matrix of G. The *normalized Laplacian matrix* of G is defined as [14]:

$\mathcal{L}(G) = (l_{ij}) = D(G)^{-1/2}(D(G) - A(G))D(G)^{-1/2}$, that is

$$l_{ij} = \begin{cases} 1, & \text{if } i = j; \\ -1/\sqrt{d_i d_j}, & \text{if } v_i \text{ is adjacent to } v_j; \\ 0, & \text{otherwise.} \end{cases}$$

In the following, for simplicity, when we say eigenvalues and the characteristic polynomial of G, we always mean eigenvalues and the characteristic polynomial of $\mathcal{L}(G)$. The following theorem gives some basic properties of the spectrum of $\mathcal{L}(G)$. More related results can be seen in [2, 8, 14].

THEOREM 1.1. *For a connected graph G, we have:*

(i) all eigenvalues of $\mathcal{L}(G)$ lie in the interval $[0, 2]$;

(ii) 0 *is always an eigenvalue of* $\mathcal{L}(G)$;

(iii) 2 *is an eigenvalue of $\mathcal{L}(G)$ if and only if G is bipartite.*

Given a graph G and a unary operation X, let $Sp(X^k(G))$ denote the set of eigenvalues of $X^k(G)$, $k = 0, 1, 2, \dots$ Let also

$$Sp^X(G) = \bigcup_{k=0}^{\infty} Sp(X^k(G))$$

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denote the union of the eigenvalue sets of all graphs in the sequence $\{X^k(G)\}_{k\geq 0}$. Then in this paper, on the one hand, we concern the set of limit points of the set $Sp^X(G)$, which is denoted by $\Lambda^X(G)$; on the other hand, we concern the supremun and infimum limits of the sequence of sets $Sp(X^k(G))$, that is,

$$\limsup_{k\to\infty} Sp(X^k(G)) = \bigcap_{k=1}^{\infty}\bigcup_{l=k}^{\infty} Sp(X^l(G));$$

$$\liminf_{k\to\infty} Sp(X^k(G)) = \bigcup_{k=1}^{\infty}\bigcap_{l=k}^{\infty} Sp(X^l(G)).$$

These three sets $\Lambda^X(G), \limsup_{k\to\infty} Sp(X^k(G))$ and $\liminf_{k\to\infty} Sp(X^k(G))$ can be very different. Note that $x \in \Lambda^X(G)$ if and only if there exists a point sequence $\{x_k\}, x_k \in Sp^X(G)$ such that $x_k \to x$. Also note that $x \in \limsup_{k\to\infty} Sp(X^k(G))$ if and only if there exists a subsequence $\{Sp(X^{k_i}(G))\}$ of $\{Sp(X^k(G))\}$ such that $x \in Sp(X^{k_i}(G))$ for all i; and $x \in \liminf_{k\to\infty} Sp(X^k(G))$ if and only if there exists some $h > 0$ such that $x \in Sp(X^l(G))$ for all $l > h$. So, $x \in \Lambda^X(G)$ may not be an eigenvalue of any graph in the graph sequence $\{X^k(G)\}$, while if $x \in \limsup_{k\to\infty} Sp(X^k(G))$ or $x \in \limsup_{k\to\infty} Sp(X^k(G))$, then x must be an eigenvalue of infinite many graphs in this sequence.

The rest of the paper is organized as follows. In Section 2, we focus on the spectral dynamics of the graph sequences $\{S^k(G)\}_{k\geq 0}$ generated by the subdivision S. We first use algebraic method to establish an explicit relation between the characteristic polynomial of $S(G)$ and that of G. Then by connecting this relation with the well-known logistic map, we not only show that $\Lambda^S(G) = [0, 2]$ for any initial non-trivial graph G, but also give an explicit characteristic of $\limsup_{k\to\infty} Sp(S^k(G))$ and $\liminf_{k\to\infty} Sp(S^k(G))$ in terms of period points of the logistic map. In Section 3, we give explicit results for the iterated-triangle-extension graph sequences $\{R^k(G)\}_{k\geq 0}$ for any initial graph G. In Section 4, as a conclusion, we first summarize the results that we have obtained, then point out problems which need further study.

In this paper, we follow standard notation and terminology. The reader may refer to [3, 15] for graph theory, and [16] for dynamical systems.

2. Spectral dynamics of iterated subdivision graphs. Let G be a graph with n vertices and m edges, the *characteristic polynomial* of G will be denoted by $\Phi(G, x)$, that is $\Phi(G, x) = det(xI - \mathcal{L}(G))$. In this section, we first give an explicit expression for the characteristic polynomials of $S(G)$ in terms of that of G. Then based on the expression, we study the spectral dynamics of graph sequence $\{S^k(G)\}_{k\geq 0}$.

For simplicity, $\mathcal{L}(G)$ is often written $\mathcal{L}$ when the graph G is implied. This abbreviation applies to $A(G), B(G)$ and $D(G)$ as well. We also write $|M|$ for $det(M)$. First note that

$$\begin{aligned}\Phi(G; x) &= \left|xI - D^{-1/2}(D - A)D^{-1/2}\right| = \left|(x-1)I + D^{-1/2}AD^{-1/2}\right| \\ &= \left|D^{1/2}\right|\left|(x-1)I + D^{-1}A\right|\left|D^{-1/2}\right| = \left|(x-1)I + D^{-1}A\right|,\end{aligned} \tag{2.1}$$

and

$$BB^T = A + D. \tag{2.2}$$

we also will use the following known result.

LEMMA 2.1. [18] *Let N be a non-singular square matrix. Then*

$$\begin{vmatrix} P & Q \\ M & N \end{vmatrix} = |N|\left|P - QN^{-1}M\right|.$$

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Now we are ready to obtain the relation between $\Phi(S(G);x)$ and $\Phi(G;x)$.

THEOREM 2.2. *Let G be a graph with n vertices and m edges. Then*

$$\Phi(S(G);x) = \frac{(-1)^n(x-1)^{m-n}}{2^n}\Phi(G;2x(2-x)).$$

Proof. By the definition of $S(G)$, we have

$$A(S(G)) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \quad \text{and} \quad D(S(G)) = \begin{pmatrix} D & 0 \\ 0 & 2I_m \end{pmatrix}.$$

Thus,

$$D(S(G))^{-1}A(S(G)) = \begin{pmatrix} 0 & D^{-1}B \\ \frac{1}{2}B^T & 0 \end{pmatrix}.$$

So, by Lemma 2.1, (2.1) and (2.2), we have

$$\begin{aligned}
\Phi(S(G);x) = &\begin{vmatrix} (x-1)I_n & D^{-1}B \\ \frac{1}{2}B^T & (x-1)I_m \end{vmatrix} = (x-1)^m \left|(x-1)I_n - \frac{D^{-1}BB^T}{2(x-1)}\right| \\
=&(x-1)^m \left|(x-1)I_n - \frac{D^{-1}(A+D)}{2(x-1)}\right| \\
=&\frac{(-1)^n(x-1)^{m-n}}{2^n}\left|(1-2(x-1)^2)I_n + D^{-1}A\right| \\
=&\frac{(-1)^n(x-1)^{m-n}}{2^n}\Phi\left(G;2(1-(x-1)^2)\right) \\
=&\frac{(-1)^n(x-1)^{m-n}}{2^n}\Phi(G;2x(2-x)). \qquad \square
\end{aligned}$$

Let $f(x) = 2x(2-x)$ and let $f^{-1}(x) = \left\{1 \pm \sqrt{\frac{2-x}{2}}\right\}$ represent the pre-image of x under f, i.e., $f(f^{-1}(x)) = x$. Then from the above theorem, we can immediately derive the following result.

COROLLARY 2.3. *Let G be a connected graph with n vertices and $m(m > 0)$ edges. If $Sp(G) = \{\lambda_1, \lambda_2, \dots, \lambda_s\}$ with $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_s$, then*

$$Sp(S(G)) = \begin{cases} \left\{1; 1 \pm \sqrt{\frac{2-\lambda_i}{2}}, i = 1, 2, \dots, s\right\}, & \textit{if } m > n \textit{ and } G \textit{ is non-bipartite}; \\ \left\{1 \pm \sqrt{\frac{2-\lambda_i}{2}}, i = 1, 2, \dots, s\right\}, & \textit{otherwise}. \end{cases}$$

Proof. First note that $m \geq n-1$ since G is connected and $\lambda_s = 2$ if G is bipartite by Theorem 1.1 (iii). Now if $m = n-1$, then G is bipartite. In this case, the result can be checked directly from Theorem 2.2. If $m \geq n$, from Theorem 2.2, we see that μ is an eigenvalue of $S(G)$ if and only if $\mu \in f^{-1}(\lambda_i)$ for some $i \in \{1, 2, \dots, s\}$ or $\mu = 1$. Since $1 \in f^{-1}(\lambda_s)$ if G is bipartite, we have the result. $\square$

Note that for any graph G, $S(G)$ must be bipartite. So, to study the asymptotic properties of the sequence $\{S^k(G)\}_{k\geq 0}$, without loss of generality, we may suppose that G itself is bipartite. Thus, by Corollary 2.3, to obtain the set $Sp(S^k(G))$ for general k, we only need to consider the backwards and forwards iterations of the quadratic map

$$f : x \to 2x(2-x).$$

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From Corollary 2.3, μ is an eigenvalue of $S^k(G)$ if and only if $\mu \in f^{-k}(\lambda_i)$ for some $i \in \{1, 2, \ldots, s\}$ where $f^{-k}(x) = f^{-1}(f^{-k+1}(x))$. Now we define the following affine transformation:

$$h : x \to \frac{x}{2}$$

and consider the well studied logistic map with parameter $b = 4$:

$$g : x \to 4x(1 - x).$$

Obviously, h is a homeomorphism. It is easy to check the fact that

$$h \circ f(x) = g \circ h(x).$$

This fact indicates that g and f are topologically conjugate to each other via the homeomorphism h. Note that for the logistic map $g(x) = 4x(1 - x)$, the k-th iterated function $g^k(x)$ can be expressed explicitly as follows [23]:

$$g^k(x) = \sin^2(2^k \arcsin \sqrt{x}).$$

So, we have

$$f^k(x) = (h^{-1} \circ g \circ h)^k(x) = h^{-1} \circ g^k \circ h(x) = 2\sin^2(2^k \arcsin \sqrt{x/2}).$$

Hence, for any $x \in [0, 2]$, we have

$$f^{-k}(x) = \left\{ 2\sin^2\left(\frac{\arccos(1 - x) + 2l\pi}{2^{k+1}}\right),\ l = 0, 1, \ldots, 2^k - 1 \right\}. \tag{2.3}$$

Now we are ready to give the set $Sp(S^k(G))$ explicitly for any k.

THEOREM 2.4. *Let G be a bipartite graph with at least one edge. If $Sp(G) = \{\lambda_1 = 0, \lambda_2, \ldots \lambda_s\}$, then*

$$Sp(S^k(G)) = \bigcup_{i=1}^{s} \left\{ 2\sin^2\left(\frac{\arccos(1 - \lambda_i) + 2l\pi}{2^{k+1}}\right),\ l = 0, 1, \ldots, 2^k - 1 \right\}.$$

Proof. First by Corollary 2.3, μ is an eigenvalue of $S^k(G)$ if and only if $\mu \in f^{-k}(\lambda_i)$ for some $i \in \{1, 2, \ldots, s\}$. So, by (2.3), the result follows immediately. ∎

Now recalling that, for a function ψ, a point x is called a *period point* of ψ with period k if $\psi^k(x) = x$. It is known that the logistic map $g(x) = 4x(1 - x)$ has exactly 2^k period points with period k listed below:

$$\left\{ \sin^2 \frac{l\pi}{2^k - 1},\ \sin^2 \frac{(l + 1)\pi}{2^k + 1},\ l = 0, 1, \ldots, 2^{k-1} - 1. \right\}$$

Since f and g are topologically conjugate to each other via the homeomorphism h, we deduce that $f(x) = 2x(2 - x)$ has exactly 2^k period points with period k, and they are

$$\left\{ 2\sin^2 \frac{l\pi}{2^k - 1},\ 2\sin^2 \frac{(l + 1)\pi}{2^k + 1},\ l = 0, 1, \ldots, 2^{k-1} - 1. \right\}$$

Let $P(f)$ denote the set of all period points of f, and let $orb_f(x)$ denote the orbit of x under f. Then we have the following results about supremum limit and infimum limit of $\{Sp(S^k(G))\}_{k \geq 0}$.

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THEOREM 2.5. *Let G be a bipartite graph with at least one edge, and let $Y = Sp(G) \bigcap P(f)$ and $Z = \{\lambda | orb_f(\lambda) \subseteq Sp(G)\}$. Then*

(i) $\limsup_{k\to\infty} Sp(S^k(G)) = \bigcup_{\lambda\in Y} \bigcup_{k=0}^{\infty} f^{-k}(\lambda)$;

(ii) $\liminf_{k\to\infty} Sp(S^k(G)) = \bigcup_{\lambda\in Z} \bigcup_{k=0}^{\infty} f^{-k}(\lambda)$.

Proof. First for (i), given any $\lambda \in Y$ and any k, we may suppose that $f^l(\lambda) = \lambda$ for some positive integer l. Then $f^{tl}(\lambda) = \lambda, t = 1, 2, \ldots$ That is, $\lambda \in f^{-tl}(\lambda), t = 1, 2, \ldots$ This implies that $f^{-k}(\lambda) \subseteq f^{-tl-k}(\lambda) \subseteq Sp(S^{tl+k}(G))$ for all $t = 1, 2, \ldots$, so we have $f^{-k}(\lambda) \subseteq \limsup_{k\to\infty} Sp(S^k(G))$. Hence,

$$\bigcup_{\lambda\in Y} \bigcup_{k=0}^{\infty} f^{-k}(\lambda) \subseteq \limsup_{k\to\infty} Sp(S^k(G)).$$

Conversely, suppose $x \in \limsup_{k\to\infty} Sp(S^k(G))$. Then there exists a subsequence $\{S^{k_i}(G)\}$ such that $x \in Sp(S^{k_i}(G))$ for each i. That is, $x \in f^{-k_i}(\lambda_{k_i})$ for some $\lambda_{k_i} \in Sp(G)$. Since $Sp(G)$ is a finite set, there must be some $k_i \neq k_j$ such that $\lambda_{k_i} = \lambda_{k_j} = \lambda$. Assume that $k_i < k_j$. Then from $f^{k_j}(x) = f^{k_i}(x) = \lambda$, we deduce $f^{k_j - k_i}(\lambda) = f^{k_j-k_i}(f^{k_i}(x)) = f^{k_j}(x) = \lambda$, which means λ is a period point of f and $x \in f^{-k_i}(\lambda)$. Hence,

$$\limsup_{k\to\infty} Sp(S^k(G)) \subseteq \bigcup_{\lambda\in Y} \bigcup_{k=0}^{\infty} f^{-k}(\lambda).$$

Now we prove (i).

For (ii), suppose $\lambda \in Z$ with the least period l. Then

$$orb_f(\lambda) = \{\lambda, f^1(\lambda) = \lambda_1, \ldots, f^{l-1}(\lambda) = \lambda_{l-1}\} \subseteq Sp(G).$$

Hence, λ belongs to each set below:

$$\{\lambda\}, f^{-1}(\lambda_1), \ldots, f^{-l+1}(\lambda_{l-1}), f^{-l}(\lambda), f^{-l-1}(\lambda_1), \ldots, f^{-2l-1}(\lambda_{l-1}), \ldots$$

This implies that, for any k, $f^{-k}(\lambda)$ is contained in each set listed below:

$$f^{-k}(\lambda), f^{-k-1}(\lambda_1), \ldots, f^{-k-l+1}(\lambda_{l-1}), f^{-k-l}(\lambda), f^{-k-l-1}(\lambda_1), \ldots, f^{-k-2l-1}(\lambda_{l-1}), \ldots$$

So, $f^{-k}(\lambda) \subseteq Sp(S^{t+k}(G))$, $t = 0, 1, 2, \ldots$ By the definition of infimum limit, we have

$$\bigcup_{\lambda\in Z} \bigcup_{k=0}^{\infty} f^{-k}(\lambda) \subseteq \liminf_{k\to\infty} Sp(S^k(G)).$$

Conversely, suppose $x \in \liminf_{k\to\infty} Sp(S^k(G))$, there exists an integer t such that $x \in Sp(S^{t+i}(G))$ for all $i \geq 1$. This means that $x \in f^{-t-i}(\lambda_i)$ for some $\lambda_i \in Sp(G)$. Since $Sp(G)$ is a finite set, there must be some $i \neq j$ such that $\lambda_i = \lambda_j$. Without loss of generality, we assume $\lambda_1, \lambda_2, \ldots, \lambda_q$ are all distinct, but $\lambda_{q+1} = \lambda_1$. Since $x \in f^{-t-i}(\lambda_i)$ for all i, that is, $f^{t+i}(x) = \lambda_i$, we have

$$\lambda_2 = f(\lambda_1),\ \lambda_3 = f^2(\lambda_1),\ \ldots,\ \lambda_q = f^{q-1}(\lambda_1),\ \lambda_1 = f^q(\lambda_1).$$

It follows that $orb_f(\lambda_1) \subseteq Sp(G)$ and $x \in f^{-t-1}(\lambda_1)$. Thus, we have

$$\liminf_{k\to\infty} Sp(X^k(G)) \subseteq \bigcup_{\lambda\in Z} \bigcup_{k=0}^{\infty} f^{-k}(\lambda).$$

Now the proof is completed. ∎

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Recalling that $0 \in Sp(G)$ for any graph G, at the same time 0 is a fixed point of f. So, $0 \in Y$ and $0 \in Z$, thus by Theorem 2.5, we have

$$\bigcup_{k=0}^{\infty} f^{-k}(0) \subseteq \liminf_{k\to\infty} Sp(C^k(G)) \subseteq \limsup_{k\to\infty} Sp(C^k(G)).$$

Furthermore, by (2.3), we have

$$f^{-k}(0) = \left\{ 2\sin^2(\frac{l\pi}{2^k}),\ \ l = 0, 1, \ldots, 2^k - 1 \right\}.$$

From this expression, it is easy to see that $\bigcup_{k=0}^{\infty} f^{-k}(0)$ is dense in $[0, 2]$. So, we derive the following results immediately.

Theorem 2.6. *Let G be a connected graph with at leat one edge. Then*

(i) $\overline{\liminf_{k\to\infty} Sp(S^k(G))} = [0, 2]$;

(ii) $\overline{\limsup_{k\to\infty} Sp(S^k(G))} = [0, 2]$;

(iii) $\Lambda^S(G) = [0, 2]$.

3. Spectral dynamics of iterated triangle extension graphs.

Theorem 3.1. *Let G be a graph with n vertices and m edges. Then*

$$\Phi(R(G); x) = (x-1)^{m-n} \left(\frac{2x-3}{4}\right)^n \Phi(G; 2x).$$

Proof. By the definition of $R(G)$, we have

$$A(R(G)) = \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \quad \text{and} \quad D(R(G)) = \begin{pmatrix} 2D & 0 \\ 0 & 2I_m \end{pmatrix}.$$

Thus,

$$D(R(G))^{-1}A(R(G)) = \begin{pmatrix} \frac{1}{2}D^{-1}A & \frac{1}{2}D^{-1}B \\ \frac{1}{2}B^T & 0 \end{pmatrix}.$$

So, by Lemma 2.1, (2.1) and (2.2) again, we have

$$\begin{aligned}
\Phi(R(G); x) =& \begin{vmatrix} (x-1)I_n + \frac{1}{2}D^{-1}A & \frac{1}{2}D^{-1}B \\ \frac{1}{2}B^T & (x-1)I_m \end{vmatrix} \\
=&(x-1)^m \left| (x-1)I_n + \frac{1}{2}D^{-1}A - \frac{D^{-1}BB^T}{4(x-1)} \right| \\
=&(x-1)^m \left| (x-1)I_n + \frac{1}{2}D^{-1}A - \frac{D^{-1}(A+D)}{4(x-1)} \right| \\
=&\frac{(x-1)^{m-n}}{4^n} \left| (4(x-1)^2 - 1)I_n + (2(x-1) - 1)D^{-1}A \right| \\
=&(x-1)^{m-n} \left(\frac{2x-3}{4}\right)^n \left| (2(x-1)+1)I_n + D^{-1}A \right| \\
=&(x-1)^{m-n} \left(\frac{2x-3}{4}\right)^n \Phi(G; 2x). \qquad \square
\end{aligned}$$

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From Theorem 3.1, we immediately have the following results.

THEOREM 3.2. *Let G be a connected graph with n vertices and $m(m>0)$ edges. If $Sp(G)=\{\lambda_1,\lambda_2,\ldots,\lambda_s\}$ with $0=\lambda_1<\lambda_2<\cdots<\lambda_s$, then*

$$Sp(R(G))=\begin{cases}\left\{0,\frac{\lambda_2}{2},\ldots,\frac{\lambda_{s-1}}{2}\right\}\bigcup\left\{\frac{3}{2}\right\}, & \text{if } m=n-1;\\ \left\{0,\frac{\lambda_2}{2},\ldots,\frac{\lambda_{s-1}}{2},\frac{\lambda_s}{2}\right\}\bigcup\left\{\frac{3}{2}\right\}, & \text{if } m=n;\\ \left\{0,\frac{\lambda_2}{2},\ldots,\frac{\lambda_{s-1}}{2},\frac{\lambda_s}{2}\right\}\bigcup\left\{1,\frac{3}{2}\right\}, & \text{if } m>n.\end{cases}$$

COROLLARY 3.3. *Let G be a connected graph with n vertices and m edges. If $m>n$ and $Sp(G)=\{\lambda_1=0,\lambda_2,\ldots,\lambda_s\}$, then*

$$Sp(R^k(G))=\left\{0,\frac{\lambda_2}{2^k},\ldots,\frac{\lambda_s}{2^k}\right\}\bigcup\left\{\frac{1}{2^i},\frac{3}{2^{i+1}},\ i=0,1,\ldots,k-1\right\}.$$

THEOREM 3.4. *Let G be a connected graph with at least one edge. Then*

(i) $\liminf_{k\to\infty}Sp(R^k(G))=\limsup_{k\to\infty}Sp(R^k(G))=\left\{0,\frac{1}{2^i},\frac{3}{2^{i+1}},\ i=0,1,\ldots\right\}$;

(ii) $\Lambda^R(G)=\{0\}$.

Proof. By the definition of the triangle extension operation, for any $k\geq 3$, the number of edges of $R^k(G)$ is greater than the number of its vertices, so the results follow directly from Corollary 3.3. ∎

4. Concluding remarks. From the above results, we see that the spectral dynamics of graph sequences generated by the subdivision and the triangle extension are very different. For the triangle extension, the dynamic properties are determined by linear function $f(x)=\frac{x}{2}$. While for the subdivision, the dynamic properties are determined by quadratic function $f(x)=2x(2-x)$, which is topologically conjugate to the logistic map $g(x)=4x(1-x)$ via the homeomorphism $h(x)=\frac{x}{2}$. Since $g(x)=4x(1-x)$ is chaotic on the interval $[0,1]$, $f(x)=2x(2-x)$ is chaotic on the interval $[0,2]$. So, although $\Lambda^S(G)=[0,2]$ is independent of the initial graph G; $\limsup_{k\to\infty}Sp(S^k(G))$ and $\liminf_{k\to\infty}Sp(S^k(G))$ are indeed depend on the initial graph G. Thus, the first problem pops up:

PROBLEM 1. Characterize $\liminf_{k\to\infty}Sp(S^k(G))$ and $\limsup_{k\to\infty}Sp(S^k(G))$ for some special graphs G, such as the complete graph, the complete bipartite graph, etc.

Furthermore, by the relation between the spectra of the adjacency matrix and the normalized Laplacian matrix of a regular graph, the result obtained in [35] for the adjacency matrix can be translated in terms of the normalized Laplacian matrix as follows:

Let G be an r-regular graph with $r>2$, and let C denote the clique-inserting. Then the dynamic properties of $\{C^k(G)\}$ are determined by quadratic function

$$f'(x)=(r+2)x-rx^2,$$

which is topologically conjugate to the logistic map $g'(x)=(r+2)x(1-x)$ via the homeomorphism $h'(x)=\frac{r}{r+2}x$. Since $g'(x)$ is chaotic on a Cantor set, $f'(x)$ is chaotic on a Cantor set. More clearly, we have the following results:

(i) the set of the limit points of the normalized Laplacian eigenvalues of all graphs in the sequence generated by clique-inserting is a fractal independent of the structure of G as long as the degree of G is fixed. Moreover, the minimum of the limit points is 0, while the maximum is $\frac{r+2}{r}$.

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(ii)

$$\limsup_{k\to\infty} Sp(C^k(G)) = \bigcup_{\lambda\in Y}\bigcup_{k=0}^{\infty} f'^{-k}(\lambda)\bigcup_{k=0}^{\infty} f'^{-k}(1)$$

and

$$\liminf_{k\to\infty} Sp(C^k(G)) = \bigcup_{\lambda\in Z}\bigcup_{k=0}^{\infty} f'^{-k}(\lambda)\bigcup_{k=0}^{\infty} f'^{-k}(1),$$

where $Y = Sp(G)\bigcap P(f')$ *and* $Z = \{\lambda | orb_{f'}(\lambda) \subseteq Sp(G)\}$.

Note that $g(x)$ and $g'(x)$ are logistic maps with parameter $b = 4$ and $b > 4$, respectively. Now we have the second problem.

PROBLEM 2. If there exists some graph sequence such that its spectral dynamic is determined by the logistic map with parameter $b < 4$.

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