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ON THE SECOND LEAST DISTANCE EIGENVALUE OF A GRAPH*

XUEYI HUANG†, QIONGXIANG HUANG†, AND LU LU†

Abstract. Let G be a connected graph on n vertices, and let $D(G)$ be the distance matrix of G. Let $\partial_1(G) \geq \partial_2(G) \geq \cdots \geq \partial_n(G)$ denote the eigenvalues of $D(G)$. In this paper, the connected graphs with $\partial_{n-1}(G)$ at least the smallest root of $x^3 - 3x^2 - 11x - 6 = 0$ are determined. Additionally, some non-isomorphic distance cospectral graphs are given.

Key words. Distance matrix, Second least distance eigenvalue, Distance cospectral graph.

AMS subject classifications. 05C50.

1. Introduction. Let G be a connected simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Denoted by $d(v_i, v_j)$ the length of the shortest path connecting v_i and v_j in G. Let H be a connected subgraph of G and $v \in V(G)$. The *distance* between v and H is defined to be $d(v, H) = \min\{d(v, w) \mid w \in V(H)\}$. Also, the *diameter* and *distance matrix* of G are defined as $d(G) = \max\{d(v_i, v_j) \mid v_i, v_j \in V(G)\}$ and $D(G) = [d(v_i, v_j)]_{n\times n}$, respectively. The characteristic polynomial $\Phi_G(x) = \det(xI - D(G))$ of $D(G)$ is also called the *distance polynomial* of G.

Since $D(G)$ is a real and symmetric, its eigenvalues can be listed as $\partial_1(G) \geq \partial_2(G) \geq \cdots \geq \partial_n(G)$. These eigenvalues are also called the *distance eigenvalues* of G. The *distance spectrum* of G, denoted by $\mathrm{Spec}_D(G)$, is the multiset of distance eigenvalues of G. Two connected graphs are said to be *distance cospectral* if they share the same distance spectrum, and the graph G is called *determined by its distance spectrum* if any connected graph distance cospectral with G must be isomorphic to it.

Let $N_G(v)$ denote the neighborhood of $v \in V(G)$, $G[X]$ the induced subgraph of G on $X \subseteq V(G)$, and $D_G(X)$ the principle submatrix of $D(G)$ corresponding to $G[X]$. Also, we denote by K_n and P_n the complete graph and path on n vertices, respectively.

For a connected graph G whose vertices are labeled as $v_1, v_2, \ldots, v_n$, and a sequence of graphs $H_1, H_2, \ldots, H_n$, the corresponding *generalized lexicographic product* $G[H_1, \ldots, H_n]$ is defined as the graph obtained from G by replacing v_i with the graph H_i for $1 \leq i \leq n$, and connecting all edges between H_i and H_j if v_i is adjacent to v_j for $1 \leq i \neq j \leq n$. For example, Figure 1 illustrates the graph $P_4[K_{a_1}, K_{a_2}, K_{a_3}, K_{a_4}]$, where A_i denotes the vertex subset of $P_4[K_{a_1}, K_{a_2}, K_{a_3}, K_{a_4}]$ corresponding to K_{a_i} for $1 \leq i \leq 4$ and the line segments represent connecting all edges between A_i and A_{i+1} for $1 \leq i \leq 3$.

Connected graphs whose distance eigenvalues satisfy special conditions and the study of whether such graphs are determined by their distance spectra have received some attention recently. Lin et al. [4] (see also Yu [9]) proved that $\partial_n(G) = -2$ if and only if G is a complete multipartite graph, and conjectured that complete multipartite graphs are determined by their distance spectra. Recently, Jin and Zhang [1]

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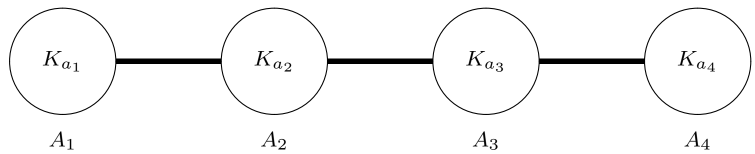

FIGURE 1. *The graph* $P_4[K_{a_1}, K_{a_2}, K_{a_3}, K_{a_4}]$.

confirmed the conjecture. Lin et al. [5, 3] characterized all connected graphs with $\partial_n(G) \geq -1-\sqrt{2}$ and $\partial_{n-1}(G) = -1$, and showed that these graphs are determined by their distance spectra. Li and Meng [2] extended the result to connected graphs with $\partial_n(G) \geq -\frac{1+\sqrt{17}}{2}$. Xing and Zhou [8] determined all connected graphs with $\partial_2(G) < -2+\sqrt{2}$, and Liu et al. [6] generalized the result to $\partial_2(G) \leq \frac{17-\sqrt{329}}{2}$ and proved that all these graphs are determined by their distance spectra. Very recently, Lu et al. [7] characterized all connected graphs with $\partial_n(G) \geq -3$ and $\partial_3(G) \leq -1$, and particularly, they determined all connected graphs with exactly two distance eigenvalues different from -1 and -3, which are also shown to be determined by their distance spectra. It is worth mentioning that most of above graphs are of diameter 2, and that only a few infinite families of non-isomorphic distance cospectral graphs are known up to now.

In this paper, we determine all connected graphs with $\partial_{n-1}(G) > \alpha$ (the diameter of these graphs could be 2 or 3), where $\alpha \approx -1.5709$ is the least root of $x^3 - 3x^2 - 11x - 6 = 0$. This extends a result of Lin et al. [5]. Furthermore, we give some infinite families of non-isomorphic distance cospectral graphs.

2. Main results. First of all, we present a result on $\partial_n(G)$, which is useful in the following.

LEMMA 2.1 ([3]). *Let G be a connected graph on n vertices. Then $\partial_n(G) \leq -d(G)$ where $d(G)$ is the diameter of G and the equality holds if and only if G is a complete multipartite graph.*

A *Hermitian matrix* is a square matrix with complex entries that is equal to its own conjugate transpose. Note that all the eigenvalues of a Hermitian matrix are real, and any real symmetric matrix is always a Hermitian matrix. The following result is well known.

LEMMA 2.2 (Cauchy Interlacing Theorem). *Let A be a Hermitian matrix of order n, and B a principle submatrix of A of order m. If $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of A and $\mu_1(B) \geq \mu_2(B) \geq \cdots \geq \mu_m(B)$ the eigenvalues of B, then $\lambda_i(A) \geq \mu_i(B) \geq \lambda_{n-m+i}(A)$ for $i = 1, \ldots, m$.*

Let G be a connected graph on n vertices, and let $S = \{v_1, \ldots, v_p\} \subseteq V(G)$ ($p \geq 2$) be a clique of G such that $N_G(v_i) \setminus S = N_G(v_j) \setminus S$ for $1 \leq i, j \leq p$. Take $\mathbf{x}_\ell \in \mathbb{R}^n$ ($2 \leq \ell \leq p$) as the vector defined on $V(G)$ with $\mathbf{x}_\ell(v_1) = 1$, $\mathbf{x}_\ell(v_\ell) = -1$ and $\mathbf{x}_\ell(v) = 0$ for $v \notin \{v_1, v_\ell\}$, then one can easily verify that $D(G)\mathbf{x}_\ell = -\mathbf{x}_\ell$. This implies that -1 is a distance eigenvalue of G with multiplicity at least $p-1$ (cf. [7]). If there are r disjoint subsets $S_1, \ldots, S_r$ ($|S_i| = p_i \geq 2$) of $V(G)$ sharing the same property as S, then we may conclude that -1 is a distance eigenvalue of G with multiplicity at least $\sum_{i=1}^{r} p_i - r$. Thus, we have the following result.

LEMMA 2.3. *Let G be a connected graph. If $S_1, \ldots, S_r$ ($|S_i| = p_i \geq 2$) are disjoint cliques of G such that, for each $1 \leq i \leq r$, $N_G(u) \setminus S_i = N_G(v) \setminus S_i$ for any $u, v \in S_i$, then -1 is a distance eigenvalue of G with multiplicity at least $\sum_{i=1}^{r} p_i - r$.*

For a connected graph G, the vertex partition $\Pi : V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ is called a *distance equitable partition* if, for any $v \in V_i$, $\sum_{u \in V_j} d(v, u) = b_{ij}$ is a constant only dependent on i, j ($1 \leq i, j \leq k$). The matrix $B_\Pi = (b_{ij})_{k \times k}$ is called the *distance divisor matrix* of G with respect to Π. The following lemma

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states that each eigenvalue of B_Π is also the distance eigenvalue of G.

LEMMA 2.4 ([7]). *Let G be a connected graph with distance matrix $D(G)$, and let $\Pi : V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ be a distance equitable partition of G with distance divisor matrix B_Π. Then $\det(xI - B_\Pi) | \det(xI - D(G))$, and the largest eigenvalue of B_Π is $\partial_1(G)$.*

The following two lemmas give the distance polynomials of $P_4[K_{a_1}, K_{a_2}, K_{a_3}, K_{a_4}]$ and $P_3[K_{b_1}, K_{b_2}, K_{b_3}]$, which are the graphs we need to consider in our main result.

LEMMA 2.5. *Let $G = P_4[K_{a_1}, K_{a_2}, K_{a_3}, K_{a_4}]$ with $a_1, a_2, a_3, a_4 \geq 1$. Then the distance polynomial of G is given by*

$$\Phi_G(x) = (x+1)^{a_1+a_2+a_3+a_4-4}\Phi_1(x), \tag{2.1}$$

where $\Phi_1(x) = x^4 - (a_1+a_2+a_3+a_4-4)x^3 - [3a_1a_3 + 8a_1a_4 + 3a_2a_4 + 3(a_1+a_2+a_3+a_4) - 6]x^2 + [a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 - (6a_1a_3 + 16a_1a_4 + 6a_2a_4) - 3(a_1+a_2+a_3+a_4) + 4]x + a_1a_2a_3a_4 + a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4 - (3a_1a_3 + 8a_1a_4 + 3a_2a_4) - (a_1+a_2+a_3+a_4) + 1$. *In particular, -1 is not a zero of $\Phi_1(x)$.*

Proof. As shown in Figure 1, let A_i denote the vertex subset of G corresponding to K_{a_i} for $1 \leq i \leq 4$. Then it is easy to see that $\Pi : V(G) = A_1 \cup A_2 \cup A_3 \cup A_4$ is a distance equitable partition of G, and the corresponding distance divisor matrix is

$$B_\Pi = \begin{bmatrix} a_1 - 1 & a_2 & 2a_3 & 3a_3 \\ a_1 & a_2 - 1 & a_3 & 2a_4 \\ 2a_1 & a_2 & a_3 - 1 & a_4 \\ 3a_1 & 2a_2 & a_3 & a_4 - 1 \end{bmatrix}.$$

By Lemma 2.4, we have $\Phi_1(x) = \det(xI - B_\Pi) \mid \Phi_G(x)$, where $\Phi_1(x)$ is given in Eq. (2.1). Furthermore, from Lemma 2.3 we know that -1 is a distance eigenvalue of G with multiplicity at least $a_1+a_2+a_3+a_4-4$. Thus, our result follows because -1 is not a zero of $\Phi_1(x)$ due to $\Phi_1(-1) = a_1a_2a_3a_4 > 0$. ☐

Using the same method as in Lemma 2.5, one can also obtain the distance polynomial of $P_3[K_{b_1}, K_{b_2}, K_{b_3}]$.

LEMMA 2.6. *Let $G = P_3[K_{b_1}, K_{b_2}, K_{b_3}]$ with $b_1, b_2, b_3 \geq 1$. Then the distance polynomial of G is given by*

$$\Phi_G(x) = (x+1)^{b_1+b_2+b_3-3}\Phi_2(x), \tag{2.2}$$

where $\Phi_2(x) = x^3 - (b_1+b_2+b_3-3)x^2 - [2(b_1+b_2+b_3) + 3b_1b_3 - 3]x + b_1b_2b_3 - 3b_1b_3 - (b_1+b_2+b_3) + 1$. *In particular, -1 is not a zero of $\Phi_2(x)$.*

The following lemma is crucial for the proof of our main result.

LEMMA 2.7. *If G is a connected graph on n ($n \geq 4$) vertices with $\partial_{n-1}(G) > \alpha$, where $\alpha \approx -1.5709$ is the least root of $x^3 - 3x^2 - 11x - 6 = 0$, then each matrix listed below cannot be the principle submatrix of*

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$D(G)$:

$$A_1 \begin{bmatrix} 0&1&2&3&1\\ 1&0&1&2&2\\ 2&1&0&1&2\\ 3&2&1&0&2\\ 1&2&2&2&0 \end{bmatrix},\ A_2 \begin{bmatrix} 0&1&2&3&1\\ 1&0&1&2&2\\ 2&1&0&1&2\\ 3&2&1&0&3\\ 1&2&2&3&0 \end{bmatrix},\ A_3 \begin{bmatrix} 0&1&2&3&1\\ 1&0&1&2&2\\ 2&1&0&1&3\\ 3&2&1&0&2\\ 1&2&3&2&0 \end{bmatrix},\ A_4 \begin{bmatrix} 0&1&2&3&1\\ 1&0&1&2&2\\ 2&1&0&1&3\\ 3&2&1&0&3\\ 1&2&3&3&0 \end{bmatrix},\ A_5 \begin{bmatrix} 0&1&2&3&2\\ 1&0&1&2&1\\ 2&1&0&1&2\\ 3&2&1&0&2\\ 2&1&2&2&0 \end{bmatrix},$$

$$A_6 \begin{bmatrix} 0&1&2&3&2\\ 1&0&1&2&1\\ 2&1&0&1&2\\ 3&2&1&0&3\\ 2&1&2&3&0 \end{bmatrix},\ A_7 \begin{bmatrix} 0&1&2&3&2\\ 1&0&1&2&1\\ 2&1&0&1&1\\ 3&2&1&0&2\\ 2&1&1&2&0 \end{bmatrix},\ A_8 \begin{bmatrix} 0&1&2&3&1\\ 1&0&1&2&2\\ 2&1&0&1&1\\ 3&2&1&0&2\\ 1&2&1&2&0 \end{bmatrix},\ A_9 \begin{bmatrix} 0&1&2&3&1&1\\ 1&0&1&2&1&1\\ 2&1&0&1&2&2\\ 3&2&1&0&2&2\\ 1&1&2&2&0&2\\ 1&1&2&2&2&0 \end{bmatrix},\ A_{10} \begin{bmatrix} 0&1&2&3&1&1\\ 1&0&1&2&1&1\\ 2&1&0&1&2&2\\ 3&2&1&0&2&3\\ 1&1&2&2&0&2\\ 1&1&2&3&2&0 \end{bmatrix},$$

$$A_{11} \begin{bmatrix} 0&1&2&3&1&1\\ 1&0&1&2&1&1\\ 2&1&0&1&2&2\\ 3&2&1&0&3&2\\ 1&1&2&3&0&2\\ 1&1&2&2&2&0 \end{bmatrix},\ A_{12} \begin{bmatrix} 0&1&2&3&1&1\\ 1&0&1&2&1&1\\ 2&1&0&1&2&2\\ 3&2&1&0&3&3\\ 1&1&2&3&0&2\\ 1&1&2&3&2&0 \end{bmatrix},\ A_{13} \begin{bmatrix} 0&1&2&3&1&1\\ 1&0&1&2&1&1\\ 2&1&0&1&1&1\\ 3&2&1&0&2&2\\ 1&1&1&2&0&2\\ 1&1&1&2&2&0 \end{bmatrix},\ A_{14} \begin{bmatrix} 0&1&2&3&1&1\\ 1&0&1&2&1&1\\ 2&1&0&1&2&1\\ 3&2&1&0&2&2\\ 1&1&2&2&0&2\\ 1&1&1&2&2&0 \end{bmatrix},$$

$$A_{15} \begin{bmatrix} 0&1&2&3&1&1\\ 1&0&1&2&1&1\\ 2&1&0&1&2&1\\ 3&2&1&0&3&2\\ 1&1&2&3&0&2\\ 1&1&1&2&2&0 \end{bmatrix},\ A_{16} \begin{bmatrix} 0&1&2&3&1&2\\ 1&0&1&2&1&1\\ 2&1&0&1&2&1\\ 3&2&1&0&2&1\\ 1&1&2&2&0&1\\ 2&1&1&1&1&0 \end{bmatrix},\ A_{17} \begin{bmatrix} 0&1&2&3&1&2\\ 1&0&1&2&1&2\\ 2&1&0&1&2&1\\ 3&2&1&0&2&1\\ 1&1&2&2&0&1\\ 2&2&1&1&1&0 \end{bmatrix},\ A_{18} \begin{bmatrix} 0&1&2&3&1&2\\ 1&0&1&2&1&1\\ 2&1&0&1&1&1\\ 3&2&1&0&2&1\\ 1&1&1&2&0&2\\ 2&1&1&1&2&0 \end{bmatrix},$$

$$A_{19} \begin{bmatrix} 0&1&2&2\\ 1&0&1&2\\ 2&1&0&2\\ 2&2&2&0 \end{bmatrix},\ A_{20} \begin{bmatrix} 0&1&2&1&2\\ 1&0&1&2&2\\ 2&1&0&2&1\\ 1&2&2&0&1\\ 2&2&1&1&0 \end{bmatrix},\ A_{21} \begin{bmatrix} 0&1&2&1&2\\ 1&0&1&2&1\\ 2&1&0&2&1\\ 1&2&2&0&1\\ 2&1&1&1&0 \end{bmatrix},\ A_{22} \begin{bmatrix} 0&1&2&1&1\\ 1&0&1&2&1\\ 2&1&0&2&1\\ 1&2&2&0&1\\ 1&1&1&1&0 \end{bmatrix},\ A_{23} \begin{bmatrix} 0&1&2&2\\ 1&0&1&1\\ 2&1&0&2\\ 2&1&2&0 \end{bmatrix},$$

$$A_{24} \begin{bmatrix} 0&1&2&1\\ 1&0&1&2\\ 2&1&0&1\\ 1&2&1&0 \end{bmatrix},\ A_{25} \begin{bmatrix} 0&1&2&1&1\\ 1&0&1&1&1\\ 2&1&0&2&2\\ 1&1&2&0&2\\ 1&1&2&2&0 \end{bmatrix},\ A_{26} \begin{bmatrix} 0&1&2&1&1\\ 1&0&1&1&1\\ 2&1&0&1&1\\ 1&1&1&0&2\\ 1&1&1&2&0 \end{bmatrix},\ A_{27} \begin{bmatrix} 0&1&2&1&1\\ 1&0&1&1&1\\ 2&1&0&2&1\\ 1&1&2&0&2\\ 1&1&1&2&0 \end{bmatrix},\ A_{28} \begin{bmatrix} 0&1&2&1&2\\ 1&0&1&1&1\\ 2&1&0&2&1\\ 1&1&2&0&1\\ 2&1&1&1&0 \end{bmatrix}.$$

Proof. Assume that there exists some i $(1 \le i \le 28)$ such that A_i $(|A_i| = m)$ is the principle submatrix of $D(G)$. Then the second least eigenvalue of A_i satisfies $\partial_{m-1}(A_i) \ge \partial_{n-1}(G) > \alpha$ by Lemma 2.2, which is a contradiction because $\partial_{m-1}(A_i) \le \alpha$ according to Table 1. ◻

TABLE 1
The second least eigenvalue ∂_{m-1} of A_i $(1 \le i \le 28)$.

A_i	∂_{m-1}	A_i	∂_{m-1}	A_i	∂_{m-1}	A_i	∂_{m-1}	A_i	∂_{m-1}
A_1	-2.2442	A_2	-1.8864	A_3	-2.6300	A_4	-2.1466	A_5	-2
A_6	-2	A_7	-1.8010	A_8	-2	A_9	-2	A_{10}	-2.0671
A_{11}	-2.0671	A_{12}	-2	A_{13}	-2	A_{14}	-2.2598	A_{15}	-1.8894
A_{16}	-1.6527	A_{17}	-2	A_{18}	-1.6527	A_{19}	-2	A_{20}	-2.6180
A_{21}	-2	A_{22}	α	A_{23}	-2	A_{24}	-2	A_{25}	-2
A_{26}	-2	A_{27}	α	A_{28}	α	—	—	—	—

The *join* of two graphs G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H.

Now we are in a position to prove the main result of this paper.

THEOREM 2.8. *Let G be a connected graph on n $(n \ge 3)$ vertices. If $\partial_{n-1}(G) > \alpha$, where $\alpha \approx -1.5709$ is the least root of $x^3 - 3x^2 - 11x - 6 = 0$, then one of the following occurs:*

(1) $\alpha < \partial_{n-1}(G) < -1$ and $G = P_4[K_{a_1}, K_{a_2}, K_{a_3}, K_{a_4}]$ $(a_1 + a_2 + a_3 + a_4 = n)$ with $a_1, a_2, a_3, a_4 \ge 1$ satisfying $\Phi_1(\alpha) < 0$, where $\Phi_1(x)$ is given in Eq. (2.1);
(2) $\partial_{n-1}(G) = -1$ and $G = P_3[K_{b_1}, K_{b_2}, K_{b_3}] = K_{b_2} \vee (K_{b_1} \cup K_{b_3})$ $(b_1 + b_2 + b_3 = n \ge 4)$ with $b_1, b_2, b_3 \ge 1$ or $G = K_n$;
(3) $\partial_{n-1}(G) = \sqrt{3} - 1$ and $G = P_3$.

Proof. If $n = 3$, the result follows by simple computation. Now suppose $n \ge 4$. Let $d(G)$ be the diameter of G. If $d(G) \ge 4$, then $D(P_5)$ is a principle submatrix of $D(G)$, and so we have $-1.5709 \approx \alpha < \partial_{n-1}(G) \le$

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$\partial_4(P_5) \approx -1.7304$ by Lemma 2.2, a contradiction. If $d(G) = 1$, then $G = K_n$ with $\partial_{n-1}(G) = -1 > \alpha$, as required. Thus, we only need to consider the following two cases.

Case 1. $d(G) = 3$.

Let $H = P_4 = v_1v_2v_3v_4$ be a diameter path of G. Then H is an induced subgraph of G and $D(H) = D_G(\{v_1, v_2, v_3, v_4\})$ is a principle submatrix of $D(G)$. Firstly, we have the following claim.

Claim 1.1. $d(v, H) = 1$ for any $v \in V(G) \setminus V(H)$.

If not, we have $2 \le d(v, H) \le 3$ since $d(G) = 3$. Let $d_i = d(v, v_i)$ for $i = 1, 2, 3, 4$. Then $d_i \in \{2, 3\}$ for each i, and the principle submatrix of $D(G)$ corresponding to $\{v_1, v_2, v_3, v_4, v\}$ is of the form

$$D_G(\{v_1, v_2, v_3, v_4, v\}) = \begin{bmatrix} 0 & 1 & 2 & 3 & d_1 \\ 1 & 0 & 1 & 2 & d_2 \\ 2 & 1 & 0 & 1 & d_3 \\ 3 & 2 & 1 & 0 & d_4 \\ d_1 & d_2 & d_3 & d_4 & 0 \end{bmatrix}.$$

In Table 2, we list approximate values of each of the second least eigenvalue of $D_G(\{v_1, v_2, v_3, v_4, v\})$. By Lemma 2.2, we have $-1.5709 \approx \alpha < \partial_{n-1}(G) \le \partial_4(D_G(\{v_1, v_2, v_3, v_4, v\})$, which is impossible according to Table 2. Hence, each vertex in $V(G) \setminus V(H)$ must be adjacent to at least one vertex of H. Thus, we have established Claim 1.1.

Table 2
The second least eigenvalue of $D_G(\{v_1, v_2, v_3, v_4, v\})$.

(d_1, d_2, d_3, d_4)	∂_4	(d_1, d_2, d_3, d_4)	∂_4	(d_1, d_2, d_3, d_4)	∂_4
$(2,2,2,2)$	-2.3956	$(2,2,2,3)$	-2.3810	$(2,2,3,2)$	-3.0586
$(2,2,3,3)$	-2.6028	$(2,3,2,2)$	-3.0586	$(2,3,2,3)$	-3.1163
$(2,3,3,2)$	-3.4142	$(2,3,3,3)$	-3.1014	$(3,2,2,2)$	-2.3810
$(3,2,2,3)$	-3.1436	$(3,2,3,2)$	-3.1163	$(3,2,3,3)$	-3.2798
$(3,3,2,2)$	-2.6028	$(3,3,2,3)$	-3.2798	$(3,3,3,2)$	-3.1014
$(3,3,3,3)$	-3.4142	—	—	—	—

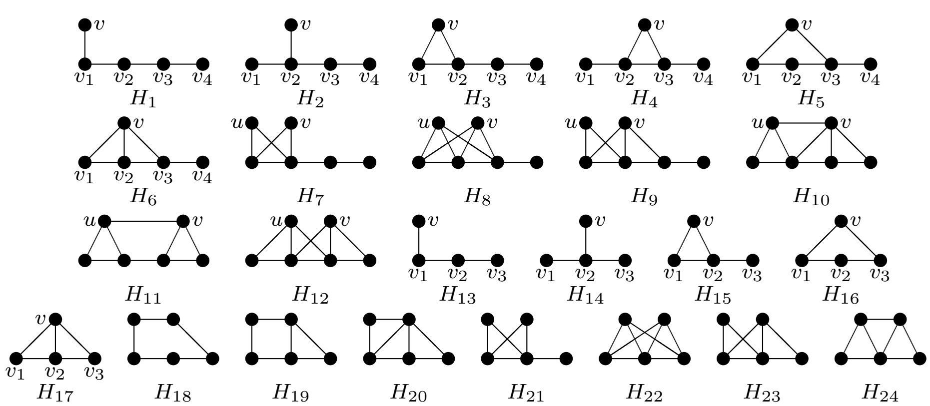

Figure 2. *The graphs H_1–H_{24}.*

Note that $d(v_1, v_4) = 3$. From Claim 1.1 and the symmetry of v_1 and v_4 (resp., v_2 and v_3), for any $v \in V(G) \setminus V(H)$, we can suppose that $G[\{v_1, v_2, v_3, v_4, v\}] \in \{H_1, H_2, H_3, H_4, H_5, H_6\}$ (see Figure 2). If $G[\{v_1, v_2, v_3, v_4, v\}] = H_1$, then $d(v, v_1) = 1$, $d(v, v_2) = 2$, and $d(v, v_3), d(v, v_4) \in \{2, 3\}$. Thus,

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$D_G(\{v_1,v_2,v_3,v_4,v\}) \in \{A_1,A_2,\ A_3,A_4\}$ is a principle submatrix of $D(G)$, contrary to Lemma 2.7. Similarly, if $G[\{v_1,v_2,\ v_3,v_4,v\}] \in \{H_2,H_4,H_5\}$, then we have $D_G(\{v_1,v_2,v_3,v_4,v\}) \in \{A_5,A_6,A_7,A_8\}$, which is impossible. Hence, we conclude that $G[\{v_1,v_2,v_3,v_4,v\}] = H_3$ or H_6 for any $v \in V(G) \setminus V(H)$. Again by considering the symmetry of v_1 and v_4 (resp., v_2 and v_3), we have the following claim.

Claim 1.2. For any $v \in V(G) \setminus V(H)$, $N_G(v) \cap V(H) = \{v_1,v_2\}$, $\{v_3,v_4\}$, $\{v_1,v_2,v_3\}$ or $\{v_2,v_3,v_4\}$.

Now we denote by V_1, V_2, V_3 and V_4 the sets of $v \in V(G) \setminus V(H)$ such that $N_G(v) \cap V(H) = \{v_1,v_2\}$, $\{v_1,v_2,v_3\}$, $\{v_2,v_3,v_4\}$ and $\{v_3,v_4\}$, respectively. Then $V(G) \setminus V(H) = V_1 \cup V_2 \cup V_3 \cup V_4$.

For any $u,v \in V_1$, if u and v are not adjacent, then $G[\{v_1,v_2,v_3,v_4,u,v\}] = H_7$, and the corresponding principle submatrix $D_G(\{v_1,v_2,v_3,v_4,u,v\})$ belongs to $\{A_9,A_{10},A_{11},A_{12}\}$ because $d(u,v_1) = d(v,v_1) = d(u,v_2) = d(v,v_2) = 1$, $d(u,v_3) = d(v,v_3) = 2$ and $d(u,v_4), d(v,v_4) \in \{2,3\}$. This is a contradiction by Lemma 2.7, which implies that $G[V_1]$ is a complete graph, and so is $G[V_4]$ by the symmetry. Similarly, if $u,v \in V_2$ are not adjacent, then $G[\{v_1,v_2,v_3,v_4,u,v\}] = H_8$ and the corresponding principle submatrix $D_G(\{v_1,v_2,v_3,v_4,u,v\})$ is equal to A_{13}, a contradiction. Thus, $G[V_2]$ is a complete graph, and so is $G[V_3]$ by the symmetry.

For any $u \in V_1$ and $v \in V_2$, if u and v are not adjacent, then $G[\{v_1,v_2,v_3,v_4,u,v\}] = H_9$ and $D_G(\{v_1,v_2,v_3,v_4,u,v\}) \in \{A_{14},A_{15}\}$, which is impossible and so each vertex of V_1 is adjacent to each vertex of V_2. Also, by the symmetry, each vertex of V_3 is adjacent to each vertex of V_4. Similarly, if $u \in V_1$ and $v \in V_3$ are adjacent, then $G[\{v_1,v_2,v_3,v_4,u,v\}] = H_{10}$ and $D_G(\{v_1,v_2,v_3,v_4,u,v\}) = A_{16}$; if $u \in V_1$ and $v \in V_4$ are adjacent, then $G[\{v_1,v_2,v_3,v_4,u,v\}] = H_{11}$ and $D_G(\{v_1,v_2,v_3,v_4,u,v\}) = A_{17}$. Therefore, there are no edges in G connecting V_1 and V_3 (resp., V_2 and V_4 by the symmetry), and V_1 and V_4.

For any $u \in V_2$ and $v \in V_3$, if u and v are not adjacent, then $G[\{v_1,v_2,v_3,v_4,u,v\}] = H_{12}$ and $D_G(\{v_1,v_2,v_3,v_4,u,v\}) = A_{18}$, contrary to Lemma 2.7. Thus, each vertex of V_2 is adjacent to each vertex of V_3.

Now put $V_i' = V_i \cup \{v_i\}$ with $|V_i'| = a_i$ for $i = 1,2,3,4$. Then $V(G) = V_1' \cup V_2' \cup V_3' \cup V_4'$. Note that v_1 (resp., v_4) is adjacent to each vertex of $V_1 \cup V_2'$ (resp., $V_3' \cup V_4$) but none of $V_3' \cup V_4'$ (resp., $V_1' \cup V_2'$), and v_2 (resp., v_3) is adjacent to each vertex of $V_1' \cup V_2 \cup V_3'$ (resp., $V_2' \cup V_3 \cup V_4'$) but none of V_4' (resp., V_1'). Combining this with above arguments, we may conclude that $G[V_i'] = K_{a_i}$ for $i = 1,2,3,4$ and $G = P_4[K_{a_1},K_{a_2},K_{a_3},K_{a_4}]$.

By Lemma 2.5, the distance polynomial of G is $\Phi_G(x) = (x+1)^{a_1+a_2+a_3+a_4-4}\Phi_1(x)$, where $\Phi_1(x)$ is given in Eq. (2.1). Note that $\Phi_1(-1) = a_1a_2a_3a_4 > 0$. Since $\partial_1(G) > 0$ and $\partial_n(G) \leq -3$ (by Lemma 2.1) are zeros of $\Phi_1(x)$, we claim that $\Phi_1(x)$ has two zeros in $(-1,+\infty)$ and one zero in $(-\infty,-3]$. Thus, $\partial_{n-1}(G) \in [\partial_n(G),-1)$, and so $\partial_{n-1}(G) > \alpha$ if and only if $\Phi_1(\alpha) < 0$.

Case 2. $d(G) = 2$.

As above, let $H = P_3 = v_1v_2v_3$ be the diameter path of G. Then H is a induced subgraph of G and $D(H) = D_G(\{v_1,v_2,v_3\})$ is a principle submatrix of G. We also have the following claim.

Claim 2.1. $d(v,H) = 1$ for any $v \in V(G) \setminus V(H)$.

Assume that there exists some $v \in V(G) \setminus V(H)$ such that $d(v,H) > 1$. Then $d(v,v_i) = 2$ for $i = 1,2,3$ because $d(G) = 2$. Thus, $D_G(\{v_1,v_2,v_3,v\}) = A_{19}$ is a principle submatrix of G, which is a contradiction by Lemma 2.7.

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By Claim 2.1, for any $v \in V(G) \setminus V(H)$, the induced subgraph $G[\{v_1, v_2, v_3, v\}]$ must be one of $\{H_{13}, H_{14}, H_{15}, H_{16}, H_{17}\}$. If $G[\{v_1, v_2, v_3, v\}] = H_{13}$, then $d(v, v_2) = d(v, v_3) = 2$ because $d(G) = 2$. Thus, there exists some other $u \in V(G)$ which is adjacent to both v and v_3, and so $G[\{v_1, v_2, v_3, v, u\}] \in \{H_{18}, H_{19}, H_{20}\}$, which is impossible because A_{20}, A_{21} and A_{22} are not principle submatrices of $D(G)$. Similarly, if $G[\{v_1, v_2, v_3, v\}] \in \{H_{14}, H_{16}\}$, one can also deduce a contradiction because $D(G)$ cannot contain A_{23} and A_{24} as its principle submatrices. Thus, $G[\{v_1, v_2, v_3, v\}] = H_{15}$ or H_{17}, and we have the following claim.

Claim 2.2. For any $v \in V(G) \setminus V(H)$, $N_G(v) \cap V(H) = \{v_1, v_2\}$, $\{v_1, v_2, v_3\}$ or $\{v_2, v_3\}$.

Denote by V_1, V_2 and V_3 the sets of $v \in V(G) \setminus V(H)$ such that $N_G(v) \cap V(H) = \{v_1, v_2\}$, $\{v_1, v_2, v_3\}$ and $\{v_2, v_3\}$, respectively. Let $V_i' = V_i \cup \{v_i\}$ with $|V_i'| = b_i$ for $i = 1, 2, 3$. Then $V(G) = V_1' \cup V_2' \cup V_3'$. We claim that H_{21}, H_{22}, H_{23} and H_{24} are not induced subgraphs of G because A_{25}, A_{26}, A_{27} and A_{28} are not principle submatrices of $D(G)$ by Lemma 2.7. Hence, as in Case 1, we may conclude that $G[V_i'] = K_{b_i}$ for $i = 1, 2, 3$, each vertex of V_1' (resp., V_3') is adjacent to each vertex of V_2' but none of V_3' (resp., V_1') and each vertex of V_2' is adjacent to each vertex of $V_1' \cup V_3'$. Therefore, we have $G = P_3[K_{b_1}, K_{b_2}, K_{b_3}]$.

According to Lemma 2.6, the distance polynomial of G is given by $\Phi_G(x) = (x+1)^{b_1+b_2+b_3-3}\Phi_2(x)$, where $\Phi_2(x)$ is shown in Eq. (2.2). Since $\Phi_2(-1) = b_1 b_2 b_3 > 0$, and $\partial_1(G) > 0$, $\partial_n(G) \le -2$ (by Lemma 2.1) are two zeros of $\Phi_2(x)$, we conlude that $\Phi_2(x)$ has two zeros in $(-1, +\infty)$ and one zero in $(-\infty, -2]$, implying that $\partial_{n-1}(G)$ is not a zero of $\Phi_2(x)$. Hence, we have $\partial_{n-1}(G) = -1$ due to $n \ge 4$, as required.

We complete the proof. $\square$

From Theorem 2.8, we obtain that $\partial_{n-1}(G) \in (\alpha, -1)$ if and only if $G = P_4[K_{a_1}, K_{a_2}, K_{a_3}, K_{a_4}]$ with $a_1, a_2, a_3, a_4 \ge 1$ satisfying $\Phi_1(\alpha) < 0$, where $\Phi_1(x)$ is given in Eq. (2.1). Actually, we can determine all the parameters a_1, a_2, a_3, a_4 such that $\partial_{n-1}(P_4[K_{a_1}, K_{a_2}, K_{a_3}, K_{a_4}]) \in (\alpha, -1)$ (or equivalently, $\Phi_1(\alpha) < 0$) by using Lemma 2.2 and solving some inequalities. However, the obtained parameters consist of some infinite families and hundreds of scattered numbers, so we do not list them here, and instead, we just give some examples. For instance, if a_1, a_2 are arbitrary positive integers and $a_3 = a_4 = 1$, then $\Phi_1(\alpha) = -(\alpha^3 + 14\alpha^2 + 24\alpha + 11)a_1 - (\alpha^3 + 6\alpha^2 + 8\alpha + 3)a_2 + (2\alpha + 3)a_1 a_2 + \alpha^4 + 2\alpha^3 - 2\alpha - 1 \approx -3.9700a_1 - 1.3626a_2 - 0.1418a_1 a_2 + 0.4784 < 0$. Thus, $\partial_{n-1}(P_4[K_{a_1}, K_{a_2}, K_1, K_1]) \in (\alpha, -1)$. Similarly, it is easy to check that $\partial_{n-1}(G) \in (\alpha, -1)$ if $G = P_4[K_{a_1}, K_1, K_{a_3}, K_1]$, $P_4[K_{a_1}, K_1, K_1, K_{a_4}]$ or $P_4[K_1, K_{a_2}, K_{a_3}, K_1]$, where a_1, a_2, a_3, a_4 are arbitrary positive integers. Consequently, there are infinitely many graphs satisfying $\partial_{n-1}(G) \in (\alpha, -1)$.

Now we consider whether the graphs with $\partial_{n-1}(G) \in (\alpha, -1)$ are determined by their distance spectra. Let $G = P_4[K_{a_1}, K_{a_2}, K_{a_3}, K_{a_4}]$ with $a_1, a_2, a_3, a_4 \ge 1$ satisfying $\Phi_1(\alpha) < 0$, and G' a graph distance cospectral with G. Then G' must be of the form $G' = P_4[K_{a_1'}, K_{a_2'}, K_{a_3'}, K_{a_4'}]$ by Theorem 2.8, and from Lemma 2.5 we deduce that

$$(2.3)\qquad \begin{cases} a_1 + a_2 + a_3 + a_4 = a_1' + a_2' + a_3' + a_4' \\ 3a_1 a_3 + 8a_1 a_4 + 3a_2 a_4 = 3a_1' a_3' + 8a_1' a_4' + 3a_2' a_4' \\ a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4 = a_1' a_2' a_3' + a_1' a_2' a_4' + a_1' a_3' a_4' + a_2' a_3' a_4' \\ a_1 a_2 a_3 a_4 = a_1' a_2' a_3' a_4' \end{cases}$$

by comparing the coefficients of distance polynomials of G and G'. If all the possible solutions of Eq. (2.3) are $(a_1, a_2, a_3, a_4) = (a_1', a_2', a_3', a_4')$ or $(a_1, a_2, a_3, a_4) = (a_4', a_3', a_2', a_1')$, then $G \cong G'$ and so G is determined

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by its distance spectrum. However, it is not the case. Now we give some examples.

EXAMPLE 2.9. Take $(a_1, a_2, a_3, a_4) = (9, 3, 1, 1)$ and $(a'_1, a'_2, a'_3, a'_4) = (3, 1, 9, 1)$, it is easy to check that these parameters satisfy Eq. (2.3). Thus, $\mathrm{Spec}_D(P_4[K_9, K_3, K_1, K_1]) = \mathrm{Spec}_D(P_4[K_3, K_1, K_9, K_1])$, but in fact $P_4[K_9, K_3, K_1, K_1] \ncong P_4[K_3, K_1, K_9, K_1]$ because $P_4[K_9, K_3, K_1, K_1]$ contains a vertex of degree 1 while $P_4[K_3, K_1, K_9, K_1]$ does not. Also note that the second least distance eigenvalue of $P_4[K_9, K_3, K_1, K_1]$ belongs to $(\alpha, -1)$ by above arguments. Thus, there exists some graph with $\partial_{n-1}(G) \in (\alpha, -1)$ that is not determined by its distance spectra.

EXAMPLE 2.10. Note that if (a_1, a_2, a_3, a_4) and (a'_1, a'_2, a'_3, a'_4) satisfy Eq. (2.3), then so is (ma_1, ma_2, ma_3, ma_4) and $(ma'_1, ma'_2, ma'_3, ma'_4)$ for any positive integer m. Hence, for any $m \geq 1$, $P_4[K_{9m}, K_{3m}, K_m, K_m]$ and $P_4[K_{3m}, K_m, K_{9m}, K_m])$ are a pair of distance cospectral graphs according to Example 2.9. Also, they are not isomorphic because they have different minimum degrees, i.e., $2m-1$ and $4m-1$. Using this method, one can obtain many other infinite families of non-isomorphic distance cospectral graphs with the help of computer search, such as $P_4[K_m, K_{2m}, K_{2m}, K_{4m}]$ and $P_4[K_{2m}, K_m, K_{4m}, K_{2m}])$, $P_4[K_m, K_{2m}, K_{3m}, K_{6m}]$ and $P_4[K_{2m}, K_m, K_{6m}, K_{3m}]$, $P_4[K_m, K_{2m}, K_{3m}, K_{9m}]$ and $P_4[K_{2m}, K_{9m}, K_m, K_{3m}])$, $P_4[K_m, K_{2m}, K_{4m}, K_{8m}]$ and $P_4[K_{2m}, K_m, K_{8m}, K_{4m}])$, $P_4[K_m, K_{2m}, K_{5m}, K_{10m}]$ and $P_4[K_{2m}, K_m, K_{10m}, K_{5m}])$, and so on.

By Theorem 2.8, we also obtain that $\partial_{n-1}(G) = -1$ $(n \geq 4)$ if and only if $G = P_3[K_{b_1}, K_{b_2}, K_{b_3}]$ or K_n. Also note that K_n is determined by its distance spectrum. Thus, any graph distance cospectral with $P_3[K_{b_1}, K_{b_2}, K_{b_3}]$ must be of the form $P_3[K_{b'_1}, K_{b'_2}, K_{b'_3}]$ for some b'_1, b'_2, b'_3, and from Lemma 2.6 one can easily deduce that $(b_1, b_2, b_3) = (b'_1, b'_2, b'_3)$ or $(b_1, b_2, b_3) = (b'_3, b'_2, b'_1)$. Thus, $P_3[K_{b_1}, K_{b_2}, K_{b_3}]$ is determined by its distance spectrum, which was mentioned by Lin et al. [5]. Moreover, Theorem 2.8 also implies the following result due to Lin et al. [5].

COROLLARY 2.11 ([5]). *Let G be a connected graph on n vertices. If $n \geq 4$, then $\partial_{n-1}(G) \leq -1$ and the equality holds if and only if $G = K_r \vee (K_s \cup K_t)$ with $r \geq 1$.*

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