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## INERTIA SETS ALLOWED BY MATRIX PATTERNS\*

A.H. BERLINER<sup>†</sup>, D.D. OLESKY<sup>‡</sup>, AND P. VAN DEN DRIESSCHE<sup>§</sup>

**Abstract.** Motivated by the possible onset of instability in dynamical systems associated with a zero eigenvalue, sets of inertias  $\mathbb{S}_n$  and  $\mathbb{S}_n^*$  for sign and zero-nonzero patterns, respectively, are introduced. For an  $n \times n$  sign pattern  $\mathcal{A}$  that allows inertia  $(0, n-1, 1)$ , a sufficient condition is given for  $\mathcal{A}$  and every superpattern of  $\mathcal{A}$  to allow  $\mathbb{S}_n$ , and a family of such irreducible sign patterns for all  $n \geq 3$  is specified. All zero-nonzero patterns (up to equivalence) that allow  $\mathbb{S}_3^*$  and  $\mathbb{S}_4^*$  are determined, and are described by their associated digraphs.

**Key words.** Sign pattern, Zero-nonzero pattern, Inertia, Digraph.

**AMS subject classifications.** 15B35, 05C20, 15A18, 15C50.

**1. Introduction.** The *inertia* of a real matrix  $A_n$  of order  $n$  is an ordered triple  $(n_+, n_-, n_0)$  of nonnegative integers summing to  $n$ , where  $n_+, n_-, n_0$  are the number of eigenvalues of  $A_n$  with positive, negative, zero real parts, respectively. In a dynamical system, the presence of a zero eigenvalue of the Jacobian matrix at an equilibrium may herald the onset of instability. With the variation of a parameter, the eigenvalues may move from all having negative real parts (linear stability) to having a simple zero eigenvalue, that then moves to have a positive real part (instability), while the remaining eigenvalues continue to have negative real parts. This corresponds to the inertia going from  $(0, n, 0)$  to  $(0, n-1, 1)$  to  $(1, n-1, 0)$ . An example from a dynamical system in ecology is given in Section 6.

With this motivation, we define the inertia set  $\mathbb{S}_n$  with  $n \geq 2$  as

$$\mathbb{S}_n = \{(0, n, 0), (0, n-1, 1), (1, n-1, 0)\}.$$

We are interested in the inertias of irreducible patterns, both sign patterns having entries in  $\{+, -, 0\}$ , and zero-nonzero patterns with entries in  $\{0, *\}$ . If  $\mathcal{A}_n$  denotes a sign pattern of order  $n$ , then  $\mathcal{A}_n$  has inertia  $i(\mathcal{A}_n) = \{i(A_n) : A_n \in Q(\mathcal{A}_n)\}$ , where  $Q(\mathcal{A}_n)$  denotes the set of all real matrices having sign pattern  $\mathcal{A}_n$ , i.e., matrix  $A_n$  is a *realization* of  $\mathcal{A}_n$ . The inertia of a zero-nonzero pattern is defined analogously. We identify some irreducible sign patterns that *allow*  $\mathbb{S}_n$ , i.e.,  $i(\mathcal{A}_n) \supseteq \mathbb{S}_n$ , or *require*  $\mathbb{S}_n$ , i.e.,  $i(\mathcal{A}_n) = \mathbb{S}_n$ .

If  $\mathcal{A}_n$  is a zero-nonzero pattern, then  $(n_+, n_-, n_0) \in i(\mathcal{A}_n)$  if and only if its *reversal*  $(n_-, n_+, n_0) \in i(\mathcal{A}_n)$ . Thus, we define the inertia set

$$\mathbb{S}_n^* = \{(0, n, 0), (0, n-1, 1), (1, n-1, 0), (n, 0, 0), (n-1, 0, 1), (n-1, 1, 0)\},$$

whose elements are the elements of  $\mathbb{S}_n$  and their reversals. Using analogous definitions, we identify some irreducible zero-nonzero patterns that allow  $\mathbb{S}_n^*$ . If sign pattern  $\mathcal{A}_n$  allows  $\mathbb{S}_n$ , then obviously the associated

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zero-nonzero pattern allows  $\mathbb{S}_n^*$ . However, the corresponding result with “allows” replaced by “requires” may not be true (see, e.g., Example 1.1). For both sign and zero-nonzero patterns, we identify them up to *equivalence*, i.e., up to any combination of transposition and permutation similarity, and signature similarity for sign patterns, since these leave the spectrum unchanged.

It is often convenient to refer to a zero-nonzero pattern  $\mathcal{A} = [\alpha_{ij}]$  (or a matrix  $A = [a_{ij}]$ ) of order  $n$  by its associated digraph, a digraph on  $n$  vertices where there is an arc from vertex  $i$  to vertex  $j$  if and only if  $\alpha_{ij} = *$  (or  $a_{ij} \neq 0$ ). In the case that  $\mathcal{A}$  is a sign pattern, the arc from vertex  $i$  to vertex  $j$  is signed as  $\alpha_{ij}$ . Two digraphs are equivalent if and only if their associated patterns are equivalent. In Sections 4 and 5, we use the same naming conventions and notation for digraphs as in [1].

For  $n = 2$ , there are two nonequivalent sign patterns that allow  $\mathbb{S}_2$ , namely  $\begin{bmatrix} - & - \\ - & - \end{bmatrix}$  and  $\begin{bmatrix} - & - \\ + & + \end{bmatrix}$  (see [15]). The first sign pattern requires  $\mathbb{S}_2$ , whereas the second is spectrally arbitrary (i.e., it can attain every spectrum allowed by a real matrix). Thus, the order 2 zero-nonzero pattern with each entry nonzero allows but does not require  $\mathbb{S}_2^*$ .

To illustrate the inertia sets for  $n = 3$ , consider the following sign pattern.

EXAMPLE 1.1. Sign pattern  $\mathcal{B}_3$  and any matrix realization  $B_3$  are given by

$$\mathcal{B}_3 = \begin{bmatrix} - & 0 & + \\ + & 0 & - \\ 0 & + & 0 \end{bmatrix} \quad \text{and} \quad B_3 = \begin{bmatrix} -a & 0 & 1 \\ c & 0 & -b \\ 0 & 1 & 0 \end{bmatrix},$$

with  $a, b, c > 0$ . Note that without loss of generality, two entries on the 3-cycle in the digraph associated with  $\mathcal{B}_3$  can be set equal to 1. The characteristic polynomial of  $B_3$  is

$$p_{B_3}(z) = z^3 + az^2 + bz + ab - c.$$

If  $c < ab$ , then  $i(B) = (0, 3, 0)$ , if  $c = ab$ , then  $i(B_3) = (0, 2, 1)$ , and if  $c > ab$ , then  $i(B_3) = (1, 2, 0)$ . Thus,  $\mathcal{B}_3$  allows  $\mathbb{S}_3$ . There are ten possible inertias for a matrix of order 3, and the remaining seven can be eliminated for  $\mathcal{B}_3$  as follows:

- Since  $\text{tr}(B_3) < 0$ , inertias  $(3, 0, 0)$ ,  $(2, 0, 1)$  are not possible.
- Since  $b \neq 0$ , the only eigenvalue allowed on the imaginary axis by  $B_3$  is a simple 0 eigenvalue, thus eliminating inertias  $(0, 0, 3)$ ,  $(1, 0, 2)$ , and  $(0, 1, 2)$ .
- If one eigenvalue of  $B_3$  is 0, then the other two have negative real parts, thus eliminating inertia  $(1, 1, 1)$ .
- If  $c < ab$  in  $B_3$ , then the Routh-Hurwitz condition (see, for example, [7, p. 194])  $ab > ab - c$  is satisfied, eliminating inertia  $(2, 1, 0)$ .

Thus, in fact, sign pattern  $\mathcal{B}_3$  requires  $\mathbb{S}_3$ . The associated zero-nonzero pattern  $\mathcal{B}_3^*$  has a realization

$$F_3 = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

with  $i(F_3) = (1, 1, 1)$ . Thus, the zero-nonzero pattern  $\mathcal{B}_3^*$  allows but does not require  $\mathbb{S}_3^*$ . Note that  $F_3$  is not equivalent to a realization of the sign pattern  $\mathcal{B}_3$ . Up to equivalence,  $\mathcal{B}_3$  is the only signing of  $\mathcal{B}_3^*$  that

requires or allows  $\mathbb{S}_3$ , since it is the only such sign pattern that allows inertia  $(0, 3, 0)$ , i.e., is potentially stable [14].

A related set of *refined inertias*, in which  $n_0$  is split into zero eigenvalues denoted by  $n_z$  and nonzero pure imaginary eigenvalues denoted by  $2n_p$ , giving the 4-tuple  $(n_+, n_-, n_0, 2n_p)$ , was introduced for sign patterns in [2]. The set  $\mathbb{H}_n$  is defined as

$$\mathbb{H}_n = \{(0, n, 0, 0), (0, n - 2, 0, 2), (2, n - 2, 0, 0)\},$$

and can herald the onset of instability by a pair of nonzero pure imaginary eigenvalues. For further results on  $\mathbb{H}_n$ , see [8, 9, 10, 11], and for the related set  $\mathbb{H}_n^*$  pertaining to zero-nonzero patterns, see [1]. For some of our results we use techniques that are found in these cited papers.

We begin by considering sign patterns, giving sufficient conditions for a sign pattern and its superpatterns to allow  $\mathbb{S}_n$ . Recall that  $\mathcal{B}_n$  is a *superpattern* of a sign pattern  $\mathcal{A}_n$  if  $\mathcal{B}_n$  is obtained from  $\mathcal{A}_n$  by replacing some (or possibly none) of the zero entries of  $\mathcal{A}_n$  with  $-$  or  $+$ . In Section 3, we use this result and a bordering technique to construct a family of irreducible sign patterns that allows  $\mathbb{S}_n$ . In Sections 4 and 5, we identify all zero-nonzero patterns (up to equivalence) that allow  $\mathbb{S}_n^*$  for  $n = 3$  and  $n = 4$ , respectively. In Section 6, we give an ecological example as an application of our sign pattern results.

**2. Preliminary results on sign patterns allowing  $\mathbb{S}_n$ .** In this section, we begin with some observations, give an example of a family of irreducible sign patterns that allows  $\mathbb{S}_n$  and requires  $\mathbb{S}_n$  for small orders, and conclude with a theorem on superpatterns that is used to obtain results in the following sections.

OBSERVATION 2.1. *If sign pattern  $\mathcal{A}_n$  allows  $\mathbb{S}_n$ , then any superpattern  $\hat{\mathcal{A}}_n$  allows  $\mathbb{S}_n$  if and only if  $\hat{\mathcal{A}}_n$  allows inertia  $(0, n - 1, 1)$ .*

A sign pattern  $\mathcal{A}_n$  is *combinatorially singular* if  $\det(A_n) = 0$  for all  $A_n \in Q(\mathcal{A}_n)$ , and it is *sign nonsingular* if  $\det(A_n) \neq 0$  for all  $A_n \in Q(\mathcal{A}_n)$ . A zero-nonzero pattern is *combinatorially nonsingular* if the determinant is nonzero for every matrix realization of the pattern.

OBSERVATION 2.2. *If sign pattern  $\mathcal{A}_n$  allows  $\mathbb{S}_n$ , then  $\mathcal{A}_n$  is not combinatorially singular and not sign nonsingular.*

For sign patterns this observation means that there are at least two terms in the determinant of  $A_n \in Q(\mathcal{A}_n)$  and they are of opposite signs. The next observation follows as in Lemma 3.1 of [2].

OBSERVATION 2.3. *If sign pattern  $\mathcal{A}_n$  has no zero entry on its main diagonal, then it allows  $\mathbb{S}_n$  if and only if it allows inertia  $(0, n - 1, 1)$ .*

Note that Observations 2.1, 2.2, and 2.3 with sign nonsingular replaced with combinatorially nonsingular hold for zero-nonzero patterns and  $\mathbb{S}_n^*$ .

We next consider a sign pattern that has all of its diagonal entries negative, and the other nonzero entries correspond to a positive  $n$ -cycle in the digraph associated with the sign pattern.

THEOREM 2.1. *Let  $\mathcal{C}_n$  be the sign pattern of a positive  $n$ -cycle, i.e.,  $\mathcal{C}_n = [\gamma_{ij}]$  has  $\gamma_{12} = \gamma_{23} = \dots = \gamma_{n-1,n} = \gamma_{n1} = +$  and all other entries zero, and let  $\mathcal{I}_n$  be the sign pattern with each diagonal entry  $+$  and all other entries zero. Sign pattern  $\mathcal{C}_n - \mathcal{I}_n$  allows  $\mathbb{S}_n$  for all  $n \geq 3$ , but requires  $\mathbb{S}_n$  only for  $n = 3$  and 4.*

*Proof.* The eigenvalues of matrix  $C_n \in Q(\mathcal{C}_n)$  are the  $n$ th roots of a positive scalar multiple of 1, thus  $C_n$  has a simple positive eigenvalue  $a$ . With  $I_n$  denoting the identity matrix of order  $n$ , the matrix  $C_n - aI_n$  has inertia  $(0, n - 1, 1)$ , and Observation 2.3 gives the allow result. The other seven (twelve) inertias for  $n = 3$  ( $n = 4$ ) can be eliminated by using the facts that any matrix realization of  $C_n - \mathcal{I}_n$  has negative trace, is essentially nonnegative, and has a simple zero as the only eigenvalue on the imaginary axis, together with continuity and the Routh-Hurwitz conditions. It follows that  $\mathcal{C}_3 - \mathcal{I}_3$  ( $\mathcal{C}_4 - \mathcal{I}_4$ ) requires  $\mathbb{S}_3$  ( $\mathbb{S}_4$ ). For  $n \geq 5$ , matrix  $C_n - I_n$  may have more than one eigenvalue in the right half plane, and thus,  $C_n - \mathcal{I}_n$  does not require  $\mathbb{S}_n$ .  $\square$

Now we give sufficient conditions for a sign pattern and its superpatterns to allow  $\mathbb{S}_n$ . Let  $A = [a_{ij}]$  be a real matrix of order  $n$  having  $m \geq n$  nonzero entries and  $i(A) = (0, n - 1, 1)$ . Let matrix  $X$  be the same as  $A$ , except that these  $m$  nonzero entries of  $A$ , namely  $a_{i_1, j_1}, \dots, a_{i_m, j_m}$ , are replaced by variables  $x_1, \dots, x_m$ , respectively. The characteristic polynomial of  $X$  is

$$c_X(z) = z^n + p_1 z^{n-1} + \dots + p_{n-1} z + p_n,$$

with coefficients  $p_1, \dots, p_n$  depending on  $x_1, \dots, x_m$ . The  $n \times m$  Jacobian matrix  $J$  of  $A$  has  $(i, j)$ -entry equal to  $\frac{\partial p_i(x_1, \dots, x_m)}{\partial x_j}$  evaluated at  $(x_1, \dots, x_m) = (a_{i_1, j_1}, \dots, a_{i_m, j_m})$ . If  $J$  has rank  $n$ , then  $A$  allows a Jacobian matrix of full rank. This definition, which uses a rectangular Jacobian matrix as in [12], is equivalent to the determinantal property that  $A$  “allows a nonzero Jacobian” as defined in [4].

**THEOREM 2.2.** *Let  $\mathcal{A}$  be an  $n \times n$  sign pattern that allows inertia  $(0, n - 1, 1)$  and let  $A \in \mathcal{Q}(\mathcal{A})$  with  $i(A) = (0, n - 1, 1)$ . If  $A$  allows a Jacobian matrix of full rank, then every superpattern  $\hat{\mathcal{A}}$  of  $\mathcal{A}$  (including  $\mathcal{A}$ ) allows  $\mathbb{S}_n$ .*

*Proof.* If  $A$  allows a nonzero Jacobian and  $i(A) = (0, n - 1, 1)$ , then it follows that  $\{(1, n - 1, 0), (0, n, 0)\} \subseteq i(\mathcal{A})$  by Lemma 3.4(i) of [3]. Therefore,  $\mathcal{A}$  allows  $\mathbb{S}_n$ . By Theorem 3.2 of [3], it follows that there exists  $\hat{A} \in \mathcal{Q}(\hat{\mathcal{A}})$  for which  $i(\hat{A}) = (0, n - 1, 1)$ . Thus, by Observation 2.1,  $\hat{\mathcal{A}}$  allows  $\mathbb{S}_n$ .  $\square$

**3. A family of sign patterns that allows  $\mathbb{S}_n$ .** We employ the following bordering technique [13] to construct a family of irreducible sign patterns that allows  $\mathbb{S}_n$ .

For  $n \geq 3$ , consider the family of sign patterns  $\mathcal{A}_n$  associated with the family of matrices  $A_n$  that are obtained from successive borderings of the form

$$A_{n+1} = \begin{bmatrix} I_n & 0 \\ e_n^T & 1 \end{bmatrix} \begin{bmatrix} A_n & e_n \\ 0^T & -1 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -e_n^T & 1 \end{bmatrix} = \begin{bmatrix} A_n - e_n e_n^T & e_n \\ e_n^T A_n & 0 \end{bmatrix},$$

where  $e_n = (0, 0, \dots, 0, 1)^T$  and  $e_n^T A_n$  is the  $n$ th row of  $A_n$ . Note that if  $i(A_n) = (n_+, n_-, n_0)$ , then  $i(A_{n+1}) = (n_+, n_- + 1, n_0)$ .

**EXAMPLE 3.1.** To illustrate the above bordering, let  $n = 3$  and  $A_3$  be the matrix  $B_3$  in Example 1.1 with  $a = b = c = 1$ . Then  $i(B_3) = (0, 2, 1)$  and the bordering gives

$$B_4 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

It can be shown (see the proof of Theorem 3.1) that  $B_4$  allows a Jacobian matrix of rank 4. So, by Theorem 2.2,  $\text{sgn}(B_4)$  allows  $\mathbb{S}_4$ . However,  $\text{sgn}(B_4)$  does not require  $\mathbb{S}_4$  since it allows  $\mathbb{H}_4$  (see G9/23 in Table 3 of

[1]). A second iteration of bordering gives

$$B_5 = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and by similar arguments,  $\text{sgn}(B_5)$  allows  $\mathbb{S}_5$ .

Continued iterations of bordering, as in Example 3.1, lead to irreducible matrices  $B_n = [b_{ij}]$  of order  $n \geq 3$  defined as follows:

- $b_{13} = b_{21} = b_{n2} = 1$  and  $b_{11} = b_{23} = -1$ .
- For  $3 \leq i \leq n - 1$ ,  $b_{ii} = -1$ ,  $b_{i,i+1} = 1$ , and  $b_{i2} = 1$ .
- $b_{ij} = 0$  otherwise.

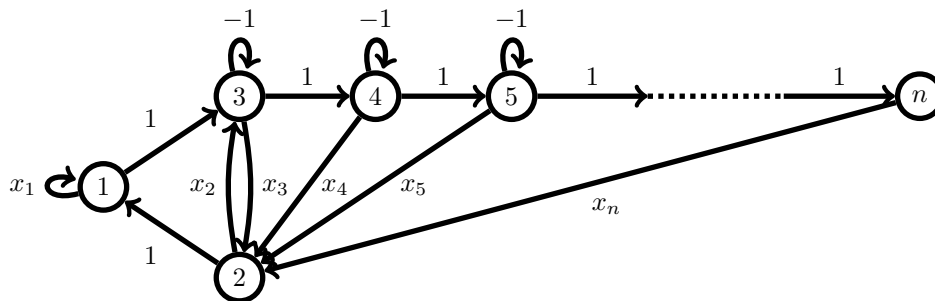
**THEOREM 3.1.** *With matrix  $B_n$  defined as above for  $n \geq 3$ ,  $\mathcal{B}_n = \text{sgn}(B_n)$  and every superpattern of  $\text{sgn}(B_n)$  allow  $\mathbb{S}_n$ .*

*Proof.* Since  $i(B_3) = (0, 2, 1)$ , it follows that  $i(B_4) = (0, 3, 1)$  and by induction,  $i(B_n) = (0, n - 1, 1)$ .

We define  $X_{B_n} = [w_{ij}]$  to be equal to  $B_n$  except that  $n$  nonzero entries of  $B_n$  have been replaced by variables  $x_1, \dots, x_n$  in the following way:

- $w_{13} = w_{21} = 1$ .
- For  $3 \leq i \leq n - 1$ ,  $w_{ii} = -1$  and  $w_{i,i+1} = 1$ .
- $w_{11} = x_1$ ,  $w_{23} = x_2$ , and for  $3 \leq i \leq n$ ,  $w_{i2} = x_i$ .
- $w_{ij} = 0$  otherwise.

The labeled signed digraph of  $X_{B_n}$  is as follows:



It is well-known (see, e.g., [16, page 29-6, Fact 4]) that if

$$p_{X_{B_n}} = z^n + p_1 z^{n-1} + p_2 z^{n-2} + \dots + p_{n-1} z + p_n$$

is the characteristic polynomial of  $X_{B_n}$ , then for  $1 \leq k \leq n$ ,  $p_k$  is the sum of all (disjoint) signed generalized cycle products of order  $k$  in the labeled digraph of  $X_{B_n}$ . Note that a *generalized  $k$ -cycle* in a digraph is a set of disjoint cycles that together cover  $k$  vertices, and a *generalized cycle product of order  $k$*  is the product

of the entries on these cycles. The sign of each generalized cycle product is  $(-1)^q$ , where  $q$  is the number of cycles in the product (see [6, Theorem 1.2]). In order to use Theorem 2.2, we need to determine the entries of the Jacobian matrix of  $X_{B_n}$  as functions of  $x_1, \dots, x_n$ . The  $(i, j)$ -entry of the  $n \times n$  Jacobian matrix  $J_{X_{B_n}}$  is equal to  $\frac{\partial p_i}{\partial x_j}$ .

From the digraph of  $X_{B_n}$ , it can be seen that  $p_1 = n - 3 - x_1$ , so that the first row of  $J_{X_{B_n}}$  is

$$[-1, 0, 0, \dots, 0].$$

Again using the digraph of  $X_{B_n}$ , it follows that

$$p_n = \det X_{B_n} = (-1)^2 x_1 x_2 x_n + (-1)^1 x_n$$

since there are only two generalized  $n$ -cycles. Thus, row  $n$  of  $J_{X_{B_n}}$  is equal to

$$[x_2 x_n, x_1 x_n, 0, \dots, 0, x_1 x_2 - 1].$$

Furthermore, for  $2 \leq k \leq n - 1$ , there exists a generalized  $k$ -cycle in the digraph of  $X_{B_n}$  with cycle product  $(-1)^1 x_2 x_{k+1}$ . This is the only generalized  $k$ -cycle involving  $x_{k+1}$ , and  $x_i$  is not on a generalized  $k$ -cycle for any  $i \geq k + 2$ . Thus, for some possibly nonzero values  $*$ ,

$$J_{X_{B_n}} = \begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & * & -x_2 & 0 & \cdots & 0 & 0 \\ * & * & * & -x_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \cdots & -x_2 & 0 \\ * & * & * & * & \cdots & * & -x_2 \\ x_2 x_n & x_1 x_n & 0 & 0 & \cdots & 0 & x_1 x_2 - 1 \end{bmatrix}.$$

Since  $x_1 = x_2 = -1$  and  $x_n = 1$  achieves inertia  $(0, n - 1, 1)$ , it follows that for these values, the  $(n, n)$ -entry of  $J_{X_{B_n}}$  is 0 and so  $\det J_{X_{B_n}} = (-1)^n$ . Thus,  $\mathcal{B}_n$  allows a Jacobian matrix of full rank, and by Theorem 2.2,  $\mathcal{B}_n$  and every superpattern of  $\mathcal{B}_n$  allow  $\mathbb{S}_n$ .  $\square$

Note that  $\mathcal{B}_n$  does not require  $\mathbb{S}_n$  for  $n \geq 4$  since it allows refined inertia  $(0, n - 2, 0, 2)$ , and is not spectrally arbitrary since any  $B_n \in \mathcal{Q}(\mathcal{B}_n)$  has negative trace.

**4. Zero-nonzero patterns that allow  $\mathbb{S}_3^*$ .** For  $n \geq 3$ , the sign pattern  $\mathcal{B}_n$  in Theorem 3.1 gives an associated zero-nonzero pattern than allows  $\mathbb{S}_n^*$ . In this section, we describe all irreducible nonequivalent zero-nonzero patterns of order 3 that allow  $\mathbb{S}_3^*$ . With digraphs D1–D3 in Appendix A, Table 1 gives realizations  $A = [a_{ij}]$  of three different sign (hence, zero-nonzero) patterns that allow  $\mathbb{S}_3$  (hence,  $\mathbb{S}_3^*$ ). Entries not mentioned are taken to be 0. Each of these realizations has inertia  $(0, 2, 1)$  and allows a Jacobian matrix of full rank (we will illustrate the use of the Jacobian in the next section, which discusses patterns of order 4). By Theorem 2.2, these three zero-nonzero patterns and any superpattern of them allow  $\mathbb{S}_3^*$ . In fact, any irreducible zero-nonzero pattern of order 3 that is not equivalent to a superpattern of one of these three does not allow  $\mathbb{S}_3^*$ . The zero-nonzero pattern associated with digraph D2 with a loop at vertex 2 is combinatorially singular, and all others are combinatorially nonsingular, so do not allow  $\mathbb{S}_3^*$  (see Observation 2.2).

Digraph	Loops at	Nonzero Entries in Realization with Inertia (0, 2, 1)
D1	123	$a_{11} = a_{22} = -1, a_{12} = a_{23} = 1, a_{31} = 2, a_{33} = -2$
D2	13	$a_{11} = -1, a_{12} = a_{23} = a_{21} = 1, a_{32} = a_{33} = -2$
D3	3	$a_{33} = -1, a_{12} = a_{23} = 1, a_{21} = -1/4, a_{31} = 1/4$

TABLE 1  
 Order 3 realizations with inertia (0, 2, 1).

Digraph	Loops at	Nonzero Entries in Realization with Inertia (0, 3, 1)
G1	1234	$a_{11} = a_{22} = -1, a_{12} = a_{23} = a_{34} = 1, a_{33} = -2, a_{44} = -3, a_{41} = 6$
G2	134	$a_{11} = a_{21} = a_{33} = a_{44} = -1, a_{12} = a_{23} = a_{34} = a_{42} = 1$
G3	12	$a_{11} = -1, a_{12} = a_{23} = a_{34} = a_{41} = 1, a_{13} = -6/5, a_{22} = -5/6$
G3	24	$a_{22} = a_{13} = -1, a_{12} = a_{23} = a_{34} = a_{41} = 1, a_{44} = -2$
G4	24	$a_{22} = -1, a_{12} = a_{23} = a_{14} = a_{31} = 1, a_{43} = a_{44} = -2$
G5	34	$a_{21} = a_{33} = -1, a_{12} = a_{23} = a_{34} = 1, a_{41} = 2, a_{44} = -2$
G6	2	$a_{13} = a_{22} = a_{41} = -1, a_{12} = a_{23} = a_{34} = 1, a_{14} = 2$
G7	2	$a_{13} = a_{22} = -1, a_{12} = a_{23} = a_{34} = 1, a_{21} = -1/3, a_{41} = 1/27$
G8	1	$a_{11} = -1, a_{12} = a_{23} = a_{34} = a_{42} = 1, a_{21} = -1/2, a_{43} = -2$
G8	34	$a_{21} = a_{33} = a_{44} = -1, a_{12} = a_{23} = a_{34} = a_{43} = 1, a_{42} = 3/2$
G9	2	$a_{13} = a_{22} = -1, a_{12} = a_{23} = a_{34} = 1, a_{31} = 1/3, a_{41} = 1/27$
G11	34	$a_{21} = a_{33} = a_{43} = a_{44} = -1, a_{12} = a_{14} = a_{23} = a_{31} = 1$
G13	14	$a_{11} = a_{32} = a_{44} = -1, a_{12} = a_{23} = a_{34} = a_{42} = 1, a_{21} = -2$
G15	12	$a_{11} = a_{22} = a_{32} = a_{43} = -1, a_{12} = a_{23} = a_{34} = a_{21} = 1$
G15	14	$a_{11} = a_{21} = a_{32} = a_{44} = -1, a_{12} = a_{23} = a_{34} = a_{43} = 1$
G16	1	$a_{11} = -1, a_{12} = a_{23} = a_{34} = 1, a_{14} = -27, a_{32} = -1/27, a_{41} = 8/729$
G17	234	$a_{22} = a_{44} = -1, a_{12} = a_{13} = a_{14} = 1,$ $a_{33} = -2, a_{21} = -4, a_{31} = -3, a_{41} = 5.5$
G22	1	$a_{11} = -1, a_{12} = a_{23} = a_{34} = a_{24} = 1, a_{13} = 3, a_{32} = -1/3, a_{41} = 2/27$
G29	2	$a_{22} = a_{34} = -1, a_{13} = a_{24} = a_{32} = a_{21} = 1, a_{12} = -1/3, a_{41} = 7/9$

TABLE 2  
 Order 4 realizations with inertia (0, 3, 1).

**5. Zero-nonzero patterns that allow  $\mathbb{S}_4^*$ .** In this section, we describe all irreducible nonequivalent zero-nonzero patterns of order 4 that allow  $\mathbb{S}_4^*$ . In Appendix B, we list 38 of the 61 digraphs G1–G61 given in [1]; the remaining 23 digraphs are not needed here since their associated zero-nonzero patterns are superpatterns of patterns associated with the listed 38 digraphs. With the digraphs in Appendix B, Table 2 gives realizations  $A = [a_{ij}]$  of 19 different sign (hence, zero-nonzero) patterns. Entries not mentioned are taken to be 0. Each of these realizations has inertia (0, 3, 1) and allows a Jacobian matrix of full rank. By Theorem 2.2, any of these 19 zero-nonzero patterns and any superpattern of any of these 19 zero-nonzero patterns allow  $\mathbb{S}_4^*$ .

EXAMPLE 5.1. To illustrate the use of the Jacobian, we give justification for one of the realizations. Consider the zero-nonzero pattern  $\mathcal{A}$  associated with digraph G3 with loops at vertices 2 and 4:

$$\mathcal{A} = \begin{bmatrix} 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & * \end{bmatrix}.$$



A matrix with an associated sign pattern and inertia  $(0, 3, 1)$  is

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 \end{bmatrix}.$$

In order to apply Theorem 2.2, we let

$$X = \begin{bmatrix} 0 & x_1 & x_2 & 0 \\ 0 & x_3 & x_4 & 0 \\ 0 & 0 & 0 & x_5 \\ x_6 & 0 & 0 & x_7 \end{bmatrix}.$$

The characteristic polynomial of  $X$  is

$$c_X(z) = z^4 + (-x_3 - x_7)z^3 + x_3x_7z^2 - x_2x_5x_6z + (x_2x_3x_5x_6 - x_1x_4x_5x_6)$$

and the Jacobian matrix  $J_{X_A}$  is

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & x_7 & 0 & 0 & 0 & x_3 \\ 0 & -x_5x_6 & 0 & 0 & -x_2x_6 & -x_2x_5 & 0 \\ -x_4x_5x_6 & x_3x_5x_6 & x_2x_5x_6 & -x_1x_5x_6 & x_2x_3x_6 - x_1x_4x_6 & x_2x_3x_5 - x_1x_4x_5 & 0 \end{bmatrix}.$$

Matrix  $J_{X_A}$  evaluated at  $x_1 = x_4 = x_5 = x_6 = 1, x_2 = x_3 = -1, x_7 = -2$ , gives the full rank matrix

$$J_{X_A} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, by Theorem 2.2,  $\mathcal{A}$  and any superpattern of  $\mathcal{A}$  allows  $\mathbb{S}_4^*$ .

In fact, any irreducible zero-nonzero pattern of order 4 that is not equivalent to a superpattern of one of these 19 in Table 2 does not allow  $\mathbb{S}_4^*$ . Tables 3–5 give justification for why each of these zero-nonzero patterns do not allow  $\mathbb{S}_4^*$ . Some zero-nonzero patterns are combinatorially singular while others are combinatorially nonsingular (see Tables 3 and 4). While this covers most patterns, there are three patterns that are neither combinatorially singular nor combinatorially nonsingular yet still do not allow  $\mathbb{S}_4^*$  (see Table 5). These patterns require a characteristic polynomial that has no quadratic term, and therefore, inertias  $(0, 3, 0)$  and  $(0, 2, 1)$  are not allowed.

**6. Concluding remarks.** As discussed in Section 1, there is no zero-nonzero pattern that requires  $\mathbb{S}_2^*$ . All zero-nonzero patterns that allow  $\mathbb{S}_3^*$  (from Section 4) and  $\mathbb{S}_4^*$  (from Section 5) have been checked, and each such pattern allows at least one additional inertia. Thus, no irreducible zero-nonzero patterns require  $\mathbb{S}_3^*$  or  $\mathbb{S}_4^*$ . We do not know if this remains true for  $n \geq 5$ .

However, note that the sign pattern  $\begin{bmatrix} - & - \\ - & - \end{bmatrix}$  requires  $\mathbb{S}_2$  and  $\begin{bmatrix} - & - & 0 \\ - & - & - \\ 0 & - & - \end{bmatrix}$  requires  $\mathbb{S}_3$  (see [15]).

These sign patterns can be used as follows to construct higher order reducible sign patterns that require  $\mathbb{S}_n$ .

Digraph(s)	Loop location(s) with combinatorially nonsingular pattern
G1	1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234
G2	1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 234
G3, G6	1, 3, 4, 13, 14, 34, 134
G4, G10	2, 4, 12, 14, 23, 34, 123, 134
G5	1, 2, 3, 4, 12, 13, 14, 23, 24, 123, 124
G7	1, 3, 4, 13, 14
G8, G28, G45	2, 3, 4, 23, 24
G9, G19, G24, G27, G35, G38	1, 3, 13
G11	2, 4, 14, 23, 123
G12	1, 2, 3, 4, 12, 13, 23, 24, 123
G13	1, 12, 13, 34, 123, 234
G14, G18, G21	1, 4, 14
G15	1, 2, 3, 4, 13, 23, 24
G17	23, 24, 34, 123, 124, 134
G23	1, 2, 3, 12, 13, 23, 123
G26	1, 2, 3, 12, 13
G29, G33	1
G30	2, 4, 14, 23, 34, 134
G31	2, 4, 12, 14
G32	1, 12, 34, 234
G42, G44	2, 4, 24
G47	1, 4, 12, 24
G51	2

TABLE 3

Digraphs of order 4 with combinatorially nonsingular associated patterns.

Digraph(s)	Loop location(s) with combinatorially singular pattern
G4, G10, G30, G31	1, 3, 13
G11	1, 3, 12, 13
G13, G32	2, 3, 4, 23, 24
G17	1, 2, 3, 4, 12, 13, 14
G47	2, 3, 23

TABLE 4

Digraphs of order 4 with combinatorially singular associated patterns.

OBSERVATION 6.1. Consider the reducible sign pattern

$$\mathcal{G}_{m+n} = \begin{bmatrix} \mathcal{A}_n & \# \\ 0 & \mathcal{P}_m \end{bmatrix},$$

with  $\#$  an arbitrary  $n \times m$  sign pattern.

(a) If  $\mathcal{A}_n$  allows  $\mathbb{S}_n$  and  $\mathcal{P}_m$  allows inertia  $(0, m, 0)$ , then  $\mathcal{G}_{m+n}$  allows  $\mathbb{S}_{m+n}$ .

(b) If  $\mathcal{A}_n$  requires  $\mathbb{S}_n$  and  $\mathcal{P}_m$  is sign stable (i.e.,  $i(\mathcal{P}_m) = (0, m, 0)$  for all  $\mathcal{P}_m \in Q(\mathcal{P}_m)$ ), then  $\mathcal{G}_{m+n}$  requires  $\mathbb{S}_{m+n}$ .

Digraph(s)	Loop location(s) that do not allow $(0, 4, 0)$
G3	2
G14	2, 3

TABLE 5

Other digraphs of order 4 with associated patterns that do not allow inertia  $(0, 4, 0)$ .

As mentioned in Section 1, motivation for  $\mathbb{S}_n$  comes from considering the onset of instability in dynamical systems. Consider an ecological example given by Weisser et al. [17, equation (5)] for a host-parasitoid system with a type II functional response of the parasitoids. For this system, the Jacobian matrix at the positive equilibrium has the sign pattern

$$\mathcal{A} = \begin{bmatrix} + & - & 0 \\ + & - & + \\ 0 & + & - \end{bmatrix}.$$

As shown in [5, 15],  $\mathcal{A}$  allows  $\mathbb{H}_3$ . In addition, the following three realizations show that  $\mathcal{A}$  allows  $\mathbb{S}_3$ :

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & -3 & 1 \\ 0 & 2 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ 0 & 2 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 \\ 1 & -3 & 1 \\ 0 & 2 & -2 \end{bmatrix}.$$

Thus, for this sign pattern, instability of the positive equilibrium can occur through either a pair of nonzero complex eigenvalues or a zero eigenvalue, depending on the numerical values of a matrix realization.

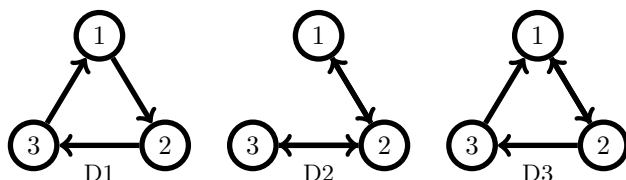
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**Appendix A. Non-equivalent strongly connected digraphs on 3 vertices used in Section 4 (retaining the labels from [1]).**



**Appendix B. Non-equivalent strongly connected digraphs on 4 vertices used in Section 5 (retaining the labels from [1]).**

