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RANK FUNCTION AND OUTER INVERSES*

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Abstract. For the class of matrices over a field, the notion of 'rank of a matrix' as defined by 'the dimension of subspace generated by columns of that matrix' is folklore and cannot be generalized to the class of matrices over an arbitrary commutative ring. The 'determinantal rank' defined by the size of largest submatrix having nonzero determinant, which is same as the column rank of given matrix when the commutative ring under consideration is a field, was considered to be the best alternative for the 'rank' in the class of matrices over a commutative ring. Even this determinantal rank and the McCoy rank are not so efficient in describing several characteristics of matrices like in the case of discussing solvability of linear system. In the present article, the 'rank–function' associated with the matrix as defined in [*Solvability of linear equations and rank–function*, K. Manjunatha Prasad, http://dx.doi.org/10.1080/00927879708825854] is discussed and the same is used to provide a necessary and sufficient condition for the existence of an outer inverse with specific column space and row space. Also, a rank condition is presented for the existence of Drazin inverse, as a special case of an outer inverse, and an iterative procedure to verify the same in terms of sum of principal minors of the given square matrix over a commutative ring is discussed.

Key words. rank–function, generalized inverse, outer inverse, Drazin inverse, matrix over a commutative ring

AMS subject classifications. 15A09, 15A15

1. Introduction. Given a matrix A over a field, the notion of $\mathrm{rank}(A)$ given by the dimension of subspace generated by the columns is well defined, folklore. Given a matrix A over real or complex field, the statements

(i) The linear system $Ax = b$ is consistent if and only if $\mathrm{rank}(A) = \mathrm{rank}\begin{pmatrix} A & b \end{pmatrix}$, where $\begin{pmatrix} A & b \end{pmatrix}$ is the matrix obtained by augmenting the column b with A;
(ii) A generalized inverse (similarly, an outer inverse) G is a reflexive generalized inverse if and only if $\mathrm{rank}(G) = \mathrm{rank}(A)$;
(iii) The matrix A has group inverse if and only if $\mathrm{rank}(A) = \mathrm{rank}(A^2)$; and

several other results depending on the notion of column rank are well established in the literature. But, whenever we consider the matrices with entries taken from an arbitrary commutative ring, all the properties discussed above fail to hold in general and in fact, the notion of column rank itself is not well defined. The determinantal rank, denoted by $\rho(A)$, is widely used as an alternative notion for $\mathrm{rank}(A)$ in several contexts, but still does not help in extending those few properties discussed above. Throughout this paper, $\mathcal{A}$ denotes an arbitrary commutative ring with identity, $\mathcal{E}$ denotes the set of all nonzero idempotents in $\mathcal{A}$.

Definition 1.1. *Given an $m \times n$ matrix A over a commutative ring $\mathcal{A}$, the determinantal rank of A, denoted by $\rho(A)$, is the size of largest submatrix of A with nonzero determinant.*

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In [4, 5, 9], attempts have been made for providing some necessary and some sufficient conditions for the solvability of linear system $Ax = b$ over a commutative ring. For the purpose, the notion of McCoy rank has been introduced.

Definition 1.2. *The McCoy rank of an $m \times n$ matrix A over a commutative ring, denoted by $\rho_M(A)$, is the largest t for which $\mathrm{Ann}(D_t(A)) = (0)$, where $D_t(A)$ is the ideal generated by $t \times t$ minors of A, and for $S \subset \mathcal{A}$, $\mathrm{Ann}(S) = \{a \in \mathcal{A} | a \cdot s = 0$ for all $s \in S\}$.*

From the definition, it is trivial that $\rho_M(A) \leq \rho(A)$. In spite of the usefulness of McCoy rank in establishing some important results, it remains unacceptable for several reasons. For example, observe that the nonzero matrix $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ over $\mathbb{Z}_6$ is with $\rho_M(A) = 0$, a deviation from very basic property of rank. In search of a suitable notion replacing 'column rank', the rank-function was introduced and several properties of the same were discussed in [13] and [8]. The rank–function associated with given matrix is an integer valued function on the set $\mathcal{E}$ of nonzero idempotents in the commutative ring $\mathcal{A}$.

Definition 1.3 (Rank–Function, [13]). *The rank–function of an $m \times n$ matrix A, denoted by $\Re_A$, is the integer valued function*

$$\Re_A : \mathcal{E} \to \mathbb{Z}$$

such that $\Re_A(e) = \rho(eA)$ for all $e \in \mathcal{E}$.

It has been observed that rank–function is an effective alternative notion, particularly when the matrix under discussion is regular. In fact, the following theorem from [13] provides a necessary and sufficient condition for the solvability of linear system over a commutative ring, whenever the matrix involved is regular.

Theorem 1.4 (Theorem 2.2, [13]). *Let A be an $m \times n$ regular matrix such that there exists a matrix G satisfying $AGA = A$. Then the linear system $Ax = b$ is consistent if and only if $\Re_A = \Re_T$, where $T = \begin{pmatrix} A & b \end{pmatrix}$ is the matrix obtained by augmenting A with b.*

In [8], the authors have explored several interesting properties of rank–function and used them to study the notion of dimension–function for the modules over a commutative ring. In this paper, we further study the properties of generalized inverses and outer inverses with reference to the rank–function.

2. Rank–function and outer inverses. First, we introduce some notation and definitions. If $\mathcal{A}$ is a commutative ring with identity, $\mathcal{A}^{m \times n}$ denotes the set of all $m \times n$ matrices over $\mathcal{A}$.

Definition 2.1. *A matrix $A \in \mathcal{A}^{m \times n}$ is a* regular matrix *if there is a matrix $G \in \mathcal{A}^{n \times m}$ such that*

$$AGA = A,$$

in which case G is called a generalized inverse *or simply a* g-inverse *of A. A matrix H is an* outer inverse *of A if*

$$HAH = H.$$

A reflexive generalized inverse of A, if exists, is a matrix G satisfying

$$AGA = A, \quad GAG = A.$$

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$\mathfrak{R}^{m\times n}$ denotes the set of all $m\times n$ regular matrices over the commutative ring $\mathcal{A}$ and $\mathfrak{R}$ denotes the class of all regular matrices over the commutative ring. An arbitrary g-inverse of A is denoted by A^- and the class of all g-inverses is denoted by $\{A^-\}$. Also, an arbitrary outer inverse of A is denoted by $A^=$ and the class of all outer inverses is denoted by $\{A^=\}$. An arbitrary reflexive generalized inverse of A and the class of all reflexive generalized inverses are denoted by A_r^- and $\{A_r^-\}$ respectively. If $A\in\mathcal{A}^{m\times n}$, A^T denotes the transpose of A, $\mathcal{C}(A)$ denotes the submodule generated by the columns of A in $\mathcal{A}^m$, $\mathcal{R}(A)$ denotes the submodule generated by the rows of A in $\mathcal{A}^n$. We refer to [15, 3] and [6] for the basic properties of generalized inverses. If $I=\{i_1,i_2,\dots,i_r\}$ and $J=\{j_1,j_2,\dots,j_r\}$ are ordered r-element subsets of $\{1,2,...,m\}$ and $\{1,2,...,n\}$, respectively, then A_J^I denotes a submatrix of A determined by the rows indexed by I and columns indexed by J. If A is a square matrix, $\mathrm{Tr}(A)$ denotes the trace of matrix A, $|A|$ denotes the determinant of A, and if a_{ij} is the (i,j)th entry then $\frac{\partial}{\partial a_{ij}}|A|$ denotes the cofactor of a_{ij} in the expansion of $|A|$. If p and r are positive integers such that $r\le p$, then the set of all r elements subsets with lexicographical order is denoted by $\mathcal{Q}_{r,p}$. If A is an $m\ \times n$ matrix, then the r-th compound matrix of A, denoted by $C_r(A)$, is the $\binom{m}{r}\times\binom{n}{r}$ matrix whose rows and columns are indexed by $\mathcal{Q}_{r,m}$ and $\mathcal{Q}_{r,n}$, respectively, and (I,J)th element is $|A_J^I|$. If $J\in\mathcal{Q}_{r,m}$, $i\in J$ and j is any integer such that $1\le j\le m$, then the set obtained by replacing i by j in J is conveniently denoted by $J-\{i\}+\{j\}$. The ideal generated by an element $x\in\mathcal{A}$ is denoted by $\langle x\rangle$, the ideal generated by $a_1,\dots,a_k\in\mathcal{A}$ is denoted by $\langle a_1,\dots,a_k\rangle$, and for a matrix A over $\mathcal{A}$, $\langle A\rangle$ denotes the ideal generated by all the entries a_{ij} of A in $\mathcal{A}$. The matrices P and Q are said to be *space equivalent*, and denoted by $P\underset{sp}{\simeq}Q$, if $\mathcal{C}(P)=\mathcal{C}(Q)$ and $\mathcal{R}(P)=\mathcal{R}(Q)$.

In the following, we present some notions related to Rao–regularity and some results from [12] before addressing the main results of the paper.

Definition 2.2 (Rao–regular matrix, [12]). *An $m\times n$ matrix A over a commutative ring $\mathcal{A}$ is* Rao–regular *if there exists an idempotent $e\in\mathcal{A}$ obtained by a linear combination of $r\times r$ minors of matrix A such that $e\cdot A=A$, where r is the determinantal rank of matrix A. Such an idempotent e, whenever exists, is called* Rao–idempotent *of A and denoted by $\mathcal{I}(A)$.*

In the case that A is a Rao–regular matrix, then $\mathcal{I}(A)$ is an idempotent e such that $\langle e\rangle=\langle(|A_J^I|)\rangle=\langle(a_{ij})\rangle$, in other words $\langle e\rangle=\langle C_r(A)\rangle=\langle A\rangle$. For all the basic properties required for our discussion, readers are referred to [12, 8]. The following lemma is immediate consequence of Theorem 2.2 of [13] and the definition of rank–function.

Lemma 2.3. *Let A be a regular matrix over a commutative ring $\mathcal{A}$. Then,*

(i) $\mathcal{C}(X)\subseteq\mathcal{C}(A)$ *if and only if* $\mathfrak{R}_A=\mathfrak{R}_{T_1}$, *where* $T_1=\begin{pmatrix}A & X\end{pmatrix}$,

(ii) $\mathcal{R}(Y)\subseteq\mathcal{R}(A)$ *if and only if* $\mathfrak{R}_A=\mathfrak{R}_{T_2}$, *where* $T_2=\begin{pmatrix}A\\ Y\end{pmatrix}$.

Suppose G is a reflexive generalized inverse of A such that $AGA=A$ and $GAG=G$. Using the Cauchy-Binet formula, it is easily verified that $\rho(eA)=\rho(eG)$ and then referring to the definition of rank–function, we get

$$G\in\{A_r^-\}\implies\mathfrak{R}_A=\mathfrak{R}_G. \tag{2.1}$$

The following theorem provides a necessary and sufficient condition, in terms of rank–function, for an outer inverse (similarly, for a generalized inverse) to be a reflexive generalized inverse.

Theorem 2.4. *Let $A\in\mathcal{A}^{m\times n}$. Then the following assertions hold.*

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(i) *An outer inverse G is a reflexive generalized inverse if and only if* $\mathfrak{R}_A = \mathfrak{R}_G$.
(ii) *A generalized inverse H is a reflexive generalized inverse if and only if* $\mathfrak{R}_A = \mathfrak{R}_H$.

Proof. Let $G \in \{A^{=}\}$ and $\mathfrak{R}_A = \mathfrak{R}_G$. Define $B = AGA$. It is easily verified that B is matrix such that

(a) B is regular with G as a reflexive generalized inverse, and
(b) $\mathcal{C}(B) \subseteq \mathcal{C}(A)$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.

From (a) and the equation (2.1), it is clear that $\mathfrak{R}_B = \mathfrak{R}_G = \mathfrak{R}_A$. Now, for $T = \begin{pmatrix} B & A \end{pmatrix} = \begin{pmatrix} AGA & A \end{pmatrix}$, we have that $\mathfrak{R}_T = \mathfrak{R}_A$ and therefore $\mathfrak{R}_T = \mathfrak{R}_B$. Now from (i) of Lemma 2.3, we get that $\mathcal{C}(A) \subseteq \mathcal{C}(B)$, and therefore (b) implies $\mathcal{C}(A) = \mathcal{C}(B)$ and hence $BGA = A$. Now note that

$$\begin{aligned} A &= BGA \quad \because G \in \{B_r^-\} \\ &= (AGA)GA \quad \because AGA = B \\ &= A(GAG)A = AGA = B \quad \because G \in \{A^{=}\}. \end{aligned}$$

Therefore $G \in \{A_r^-\}$, thus proving the 'if part' of (i). The 'only if' part is trivial from (2.1).

Part (ii) is proved by taking $G = HAH \in \{A_r^-\}$ and then proving $G = H$ in the earlier lines. ∎

Now, we will provide an interesting necessary and sufficient condition for the existence of outer inverse with specific column space and row space in terms of the rank–function and regularity.

Theorem 2.5. *Given $A \in \mathcal{A}^{m\times n}$, $X \in \mathcal{A}^{n\times p}$ and $Y \in \mathcal{A}^{r\times m}$, the following statements are equivalent.*

(i) *There exists an outer inverse G such that $\mathcal{C}(G) = \mathcal{C}(X)$ and $\mathcal{R}(G) = \mathcal{R}(Y)$ (known as an (X,Y)-inverse).*
(ii) *The matrix YAX is regular and $\mathfrak{R}_X = \mathfrak{R}_Y = \mathfrak{R}_{YAX}$.*

Proof. Let G be an (X,Y)-inverse of A satisfying (i). From Lemma 8 of [1], the existence of (X,Y)-inverse implies that XAY is regular, and with the property that $\mathcal{C}(YAX) = \mathcal{C}(Y)$ and $\mathcal{R}(YAX) = \mathcal{R}(X)$. Referring to (i) of Lemma 2.3, for $T = \begin{pmatrix} YAX & Y \end{pmatrix}$ we get that $\mathfrak{R}_T = \mathfrak{R}_{YAX}$. Since $T = \begin{pmatrix} YAX & Y \end{pmatrix} = Y \begin{pmatrix} AX & I \end{pmatrix}$ and $\begin{pmatrix} AX & I \end{pmatrix}$ has right inverse, we get that $\mathfrak{R}_T = \mathfrak{R}_Y$. Therefore $\mathfrak{R}_Y = \mathfrak{R}_{YAX}$. Since $\begin{pmatrix} YAX \\ X \end{pmatrix} = \begin{pmatrix} YA \\ I \end{pmatrix} X$, $\mathfrak{R}_X = \mathfrak{R}_{YAX}$ is similarly proved with the help of (ii) of Lemma 2.3. Hence (i) implies (ii).

Consider the matrices X, Y and A satisfying (ii). For the matrix T as defined above, we know that $\mathfrak{R}_T = \mathfrak{R}_Y$. Now from (ii), we have that $\mathfrak{R}_Y = \mathfrak{R}_{YAX}$ and therefore $\mathfrak{R}_T = \mathfrak{R}_{XAY}$. So from (i) of Lemma 2.3, we get that $\mathcal{C}(Y) \subseteq \mathcal{C}(YAX)$. $\mathcal{R}(X) \subseteq \mathcal{R}(YAX)$ is similarly proved by considering $S = \begin{pmatrix} YAX \\ X \end{pmatrix}$ and referring to (ii) of Lemma 2.3. Hence $\mathcal{C}(Y) = \mathcal{C}(YAX)$ and $\mathcal{R}(X) = \mathcal{R}(YAX)$. Now, it is easily proved that the matrix

$$X(YAX)^-Y \tag{2.2}$$

is the outer inverse satisfying (i), as in Lemma 8 of [1]. Thus (ii) implies (i). ∎

The above result reduces to the well known result, given in the following corollary, whenever the commutative ring $\mathcal{A}$ is a field.

Corollary 2.6. *Let X, Y and A be the matrices as considered in Theorem 2.5, and $\mathcal{A}$ be a field. Then A has (X,Y)-inverse if and only if* $\operatorname{rank}(X) = \operatorname{rank}(Y) = \operatorname{rank}(YAX)$, *where* rank *represents the well known column rank of matrix over field.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
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3. Characterization of Drazin inverse in terms of minors. In n [7], M. P. Drazin i introduced a typical outer inverse for an element from an associative ring or semigroup. The same outer inverse, whenever it exists, has been popularly named after Drazin and called the Drazin inverse. Drazin inverses of matrices have interesting applications in the branches of applied mathematics. First, we provide a definition of Drazin inverse in the context of square matrices over a commutative ring.

DEFINITION 3.1 (Drazin inverse). *Let A be a square matrix over a commutative ring. The* Drazin inverse *of A, whenever it exists, is the matrix G satisfying the following properties.*

$$GAG = G, \quad AG = GA, \quad GA^{k+1} = A^k \quad \text{for some positive integer} \quad k. \tag{3.3}$$

The smallest integer k satisfying the above equation is called the Drazin index *of A, and denoted by $ind(A)$. The Drazin inverse of A, whenever it exists, is unique and denoted by A^D.*

Whenever the smallest positive integer k in the equation (3.3) is one, the Drazin inverse of A is known as group inverse and is denoted by $A^{\#}$. The case of A being a nilpotent matrix, i.e., $A^k = 0$ for some positive integer k, may be considered to be a trivial case and treat $A^D = 0$, and the index is the order of nilpotency of A.

In order to continue with the characterization of matrices having Drazin inverse in terms of rank–function and then in terms of minors, first we recall some interesting properties of Rao–regular matrices and its role in discussing the Drazin inverse.

From the characterization of regular matrices as given in Lemma 4 of [12], we have that if A is an $m \times n$ regular matrix with $\rho(A) = r$, then A can be written as

$$A = A_r + A_{r-1} + \cdots + A_1, \quad \text{(Rao–decomposition)} \tag{3.4}$$

where each of A_k $(1 \leq k \leq r)$ is Rao–regular with $\rho(A_k) = k$ unless A_k is the zero matrix, and $\mathcal{I}(A_k) = e_k$ such that $e_i e_j = 0$ for $i \neq j$. It was also noted from [12] that the Rao–decomposition of a regular matrix is unique and in fact, A_r is a strictly nonzero matrix with e_r as a generator of the ideal of all $r \times r$ minors of A. Further, we say two regular matrices A and B have *similar Rao–decomposition* if the matrices A_i and B_i in the respective Rao–decomposition have same rank and have same Rao-idempotents.

The Drazin inverse A^D of the matrix A, whenever it exists, is an outer inverse and therefore it is a regular matrix. Let $G_k + G_{k-1} + \cdots + G_1$ be the Rao–decomposition of Drazin inverse, where G_i are Rao–regular with Rao–idempotents e_i for all $i = 1, 2, \ldots k$. In such a case the matrix A has a decomposition (need not be a Rao–decomposition) given by

$$A = A_k + A_{k-1} + \cdots + A_1 + A_0,$$

where $A_i = e_i A$ for $1 \leq i \leq k$, and $A_0 = A - (A_k + \cdots + A_1)$. Since the idempotents e_i are orthogonal to each other and A^D is the Drazin inverse of A, it is easily verified that G_i is the Drazin inverse of A_i for $1 \leq i \leq k$.

REMARK 3.2. *In the following, we provide some useful and interesting observations on regular matrices:*

(i) *If P and Q are the matrices such that $\mathcal{C}(P) = \mathcal{C}(Q)$ (or, $\mathcal{R}(P) = \mathcal{R}(Q)$), then the regularity of P implies the regularity of Q (Lemma 9, [1]). Further P and Q have similar Rao–decompositions.*
(ii) *If G is the (X, Y)-inverse of A, then X, Y, YAX and G have similar Rao–decompositions, and further $\mathfrak{R}_X = \mathfrak{R}_Y = \mathfrak{R}_{YAX} = \mathfrak{R}_G$. Proof follows from Theorem 2.5 and (i) above.*

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(iii) *If $G = A^D$ and d is the index of A, then G, G^d, A^d, G^k, and A^k $(k \geq d)$ have similar Rao–decompositions and have same rank–functions. Proof is easy, with the observation that G^k is the group inverse of A^k for all $k \geq d$, and those matrices are space equivalent (having same column space and same row space).*

The following theorem, which characterizes the matrices having Drazin inverse, is an immediate consequence of Theorem 2.5.

Theorem 3.3. *Let A be a square matrix over a commutative ring and not a nilpotent matrix. Then the following statements are equivalent.*

(i) *A has Drazin inverse*
(ii) *There exists a positive integer k such that A^{2k+1} is regular and $\mathfrak{R}_{A^k} = \mathfrak{R}_{A^{2k+1}}$*
(iii) *There exists a positive integer k such that A^{k+m} is regular and $\mathfrak{R}_{A^k} = \mathfrak{R}_{A^{k+m}}$ for $m = 0, 1, 2, \ldots$.*

Proof. Note that A^D, whenever exists, is the outer inverse G such that $\mathcal{C}(G) = \mathcal{C}(A^d)$ and $\mathcal{R}(G) = \mathcal{R}(A^d)$, where d is the Drazin index of A. Now (i) $\Leftrightarrow$(ii) is an immediate consequence of Theorem 2.5.

(iii) $\implies$ (ii) is trivial. If k is an integer satisfying (ii), then we get that $\mathfrak{R}_{A^{2k+1}} = \mathfrak{R}_T$, where $T = \begin{pmatrix} A^{2k+1} & A^k \end{pmatrix}$. Then referring to Lemma 2.3, we get that $\mathcal{C}(A^k) \subseteq \mathcal{C}(A^{2k+1})$, which in turn implies that $\mathcal{C}(A^k) = \mathcal{C}(A^{k+m})$, for all $m = 0, 1, 2, \ldots$. Similarly, we prove that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+m})$, for all $m = 0, 1, 2, \ldots$. Therefore $\mathfrak{R}_{A^k} = \mathfrak{R}_{A^{k+m}}$ for all nonnegative integer m and the regularity of A^{k+m} follows from the regularity of A^{2k+1} and (i) of Remark 3.2. ∎

From the equivalence of (i) and (ii) in the Theorem 3.3, we give following characterization of the Drazin index:

> The Drazin index of square matrix A over a commutative ring is the least positive integer k such that A^{k+m} is regular and $\mathfrak{R}_{A^k} = \mathfrak{R}_{A^{k+m}}$ for $m = 0, 1, 2, \ldots$.

If the commutative ring under discussion is the real or complex field, then $\text{rank}(A^k) = \text{rank}(A^{k+m})$ for $m = 0, 1, 2, \ldots$ is equivalent to saying that $\text{rank}(A^k) = \text{rank}(A^{k+1})$. Therefore the Drazin index of a square matrix is the least positive integer k for which $\text{rank}(A^{k+1}) = \text{rank}(A^k)$.

The following theorem from [1] provides necessary and sufficient condition for a matrix to have Rao–regular Drazin inverse.

Theorem 3.4 (Theorem 14, [1]). *Given a matrix $A \in \mathcal{A}^{n\times n}$ has Rao–regular Drazin inverse if and only if there exists an integer k such that the following hold:*

(i) *$C_p(A)$ are nilpotent for all $p \geq k$,*
(ii) *$\rho(A^d) = k$ and $\lambda = \text{Tr}(C_k(A^d))$ is π-regular (i.e., $\beta = \lambda^p$ is regular for some positive integer p) for some positive integer d, and*
(iii) *$(1 - \beta\beta^-)A$ is nilpotent.*

In the above case, $\beta\beta^-$ is the Rao–idempotent of Drazin inverse.

Now we will prove a result which leads to an iterative method to verify if the matrix given has a Drazin inverse. Given a square matrix A, let $\hat{k}$ represent the determinantal rank of A^k, i.e. $\hat{k} = \rho(A^k)$.

Theorem 3.5. *Given a square matrix A over a commutative ring, let k be the least positive integer for*

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which $\mathrm{Tr}(C_{\hat{k}}(A^k))$ *is not nilpotent. Then* A *has Drazin inverse if and only if the following hold.*

(i) $\mathrm{Tr}(C_{\hat{k}}(A^k))$*(say,* u*) is* π*-regular such that* u^m *is regular for positive integer* m*.*
(ii) $F = (1-e)A$ *has Drazin inverse of rank less than* $\hat{k}$*, unless* F *is nilpotent.*

In the above case, $\hat{k}$ *is exactly the rank of the Drazin inverse* A^D *of* A *and* e *is the Rao–idempotent of first factor in the Rao–decomposition of* A^D*.*

Proof. Let A be a matrix with Drazin inverse, say G, and with index d. If t is the rank of G, let

$$G = G_t + G_{t-1} + \cdots + G_1 \tag{3.5}$$

be the Rao–decomposition and each of G_i with Rao–idempotent e_i. From the properties of compound matrices and the orthogonality of Rao–idempotents, it is clear that $C_t(G)$ is the Drazin inverse of $C_t(A)$, $\rho(C_t(G)) = 1$, and $C_t(G) = C_t(G_t)$. If d is the index of $C_t(A)$, then $\rho((C_t(A))^d) = \rho(C_t(A^d)) = \rho(C_t(G^d)) = 1$ ($\because G^d = (A^d)^{\#}$). From (i) $\implies$ (iii) of Theorem 12 from [1], it is known that if X ($X = C_t(A)$ in this case) is a matrix such that $\rho(X^d) = 1$ and X has Drazin inverse, then for $w = \mathrm{Tr}(X^d)$ is regular satisfying $ww^- = \mathcal{I}(X^d)$. Further, note that $\mathrm{Tr}(X^{nd}) = (\mathrm{Tr}(X^d))^n$ (because $\rho(X^d) = 1$) is regular such that $vv^- = ww^- = \mathcal{I}(A^d)$, where $v = \mathrm{Tr}(X^{nd})$.

Again from the properties of compound matrices, we have that $C_{\hat{i}}(G)$ is the Drazin inverse of $C_{\hat{i}}(A)$ for every i, and therefore we get that $C_{\hat{i}}(A)$ is a nilpotent matrix for $\hat{i} > \hat{k}$. In fact, t is the least positive integer for which $C_t(A)$ is not nilpotent. Now from the definitions of k and $\hat{k}$, we have that $t = \hat{k}$ and $k \leq d$. Therefore, the matrix $P = C_t(A)$ is such that $P^k = (C_t(A))^k = C_t(A^k)$ is of rank one. Choose n such that k divides nd and $nd = mk$ to get that $\mathrm{Tr}(C_t(A^{nd})) = \mathrm{Tr}(C_t(A^{mk})) = (\mathrm{Tr}(C_t(A^k)))^m = u^m$ is regular, thus proving (i).

Clearly, $(1-e)G$ is the nonzero Drazin inverse of $F = (1-e)A$, unless B is nilpotent. Since $u^m(u^m)^- = ww^- = e$ is the Rao–idempotent of $X^n = P^m$, for $s = mk = nd$ we get that $\hat{s} = t$ and $eC_t(A^s) = C_t(A^s)$. This in turn gives that $C_t(F^s) = (1-e)C_t(A^s) = 0$ and therefore, the determinantal rank of F^s is strictly less than t. Therefore $\rho((1-e)G)$ is strictly less than t, thus proving (ii).

Conversely, assume (i) and (ii) hold. Define $E = eA$. Note that the matrix E inherits all the properties of A stated in the theorem and, in addition, we observe that E is Rao–regular with e being its Rao–idempotent and $(1-e)E = 0$. Since E satisfies the sufficient conditions given in Theorem 3.4 (Theorem 14, [1]), E has Rao–regular Drazin inverse, say G_1. If G_2 is the Drazin inverse of $F = (1-e)A$, then it is easily verified that $G_1 + G_2$ is the Drazin inverse of $A = E + F$. ∎

Remark 3.6. *In [1], the authors have given a determinantal formula for a Rao-regular Drazin inverse of a square matrix, whenever it exists. Readers are referred to [17] for a full-rank determinantal representation of Drazin inverse of a square matrix over real field, with the observation that a matrix need not have a rank factorization over an arbitrary commutative ring. Many efforts have been made to extend the determinantal formula for different types of generalized inverse over a field or integral domain. Some references for such work are [16, 2, 14, 10] and [11]. Obtaining a necessary and sufficient condition for the existence of* (X,Y)*-inverse of a matrix* A *appears to be a continuation of the present work, and finding determinantal formula for Rao–regular outer inverse could also be a future research problem. Such a work could extend the work of Yu and Wei [19], obtaining the determinantal representation of the generalized inverse* $A^{(2)}_{T,S}$ *over integral domains and its applications, to the case of matrices over commutative ring. We refer to [18] for the initial work on* $A^{(2)}_{T,S}$ *generalized inverse. Since the present work in not dealing with determinantal formula, we deviate from that problem and leave the same for future work.*

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