Electronic Journal of Linear Algebra

Volume 34 *Volume 34 (2018)* Article 22

2018

Semipositivity of linear maps relative to proper cones in finite dimensional real Hilbert spaces

Chandrashekaran Arumugasamy

Sachindranath Jayaraman
IISER Trivandrum, sachindranathj@gmail.com

Vatsalkumar N. Mer

Recommended Citation

Arumugasamy, Chandrashekaran; Jayaraman, Sachindranath; and Mer, Vatsalkumar N.. (2018), "Semipositivity of linear maps relative to proper cones in finite dimensional real Hilbert spaces", *Electronic Journal of Linear Algebra*, Volume 34, pp. 304-319.
DOI: https://doi.org/10.13001/1081-3810, 1537-9582.3733

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

SEMIPOSITIVITY OF LINEAR MAPS RELATIVE TO PROPER CONES IN FINITE DIMENSIONAL REAL HILBERT SPACES*

CHANDRASHEKARAN ARUMUGASAMY†, SACHINDRANATH JAYARAMAN‡, AND VATSALKUMAR N. MER‡

Abstract. For a proper cone K in a finite dimensional real Hilbert space V, a linear map L is said to be K-semipositive if there exists $d \in K^{\circ}$, the interior of K, such that $L(d) \in K^{\circ}$. The aim of this manuscript is to characterize K-semipositivity of linear maps relative to a proper cone. Among several results obtained, K-semipositivity is characterized in terms of products of the form YX^{-1} for K-positive linear maps ($L(K \setminus \{0\}) \subseteq K^{\circ}$) with X invertible, semipositivity of matrices relative to the n-dimensional Lorentz cone $\mathcal{L}^n_+$ is characterized, semipositivity of the following three linear maps relative to the cone $\mathcal{S}^n_+$: $X \mapsto AXB$ (denoted by $M_{A,B}$), $X \mapsto AXB + B^tXA^t$ (denoted by $L_{A,B}$), where $A, B \in M_n(\mathbb{R})$, and $X \mapsto X - AXA^t$ (denoted by S_A, known as the Stein transformation) is characterized. It is also proved that $M_{A,B}$ is semipositive if and only if $B = \alpha A^t$ for some $\alpha > 0$, the map $L_{A,B}$ is semipositive if and only if $A(B^t)^{-1}$ is positive stable. A particular case of the new result generalizes Lyapunov's theorem. Decompositions of the above maps (when they are semipositive) in the form $L_1L_2^{-1}$, where L_1 and L_2 are both positive and invertible (assuming A is invertible in the case of S_A) are presented. Moreover, a question on invariance of the semipositive cone $\mathcal{K}_A$ of a matrix under A is partially answered.

Key words. Positivity and semipositivity of linear maps, Proper cones, Positive definite matrices, Positive stable matrices, Semidefinite linear complementarity problems, Lyapunov and Stein transformations, Semipositive cone.

AMS subject classifications. 15B48, 90C33.

1. Introduction, definitions and preliminaries. We work throughout with matrices or linear maps over the field $\mathbb{R}$ of real numbers. An $m \times n$ matrix A with real entries is said to be semipositive (with respect to the cone $\mathbb{R}^n_+$ of nonnegative vectors in $\mathbb{R}^n$) if there exists a $d > 0$ such that $Ad > 0$, where the inequalities are understood componentwise. Semipositivity characterizes invertible M-matrices within the class of Z-matrices (see Chapter 6 of [7]) and was also studied in the context of stability of matrices [8]. In [15], Fiedler and Ptak call the class of semipositive matrices as S-matrices. For interesting connections to game theory problems, refer to the first chapter of Bapat and Raghavan's book [4]. Semipositivity occurs naturally in optimization problems. A brief sketch is presented in the next section. Several interesting results have appeared recently concerning the structure as well the preserver properties of semipositive matrices (see for instance [1, 12, 13, 23, 33]).

The following notations will be used throughout. The letter V will denote a finite dimensional real Hilbert space. $M_{m,n}(\mathbb{R})$ will denote the vector space of $m \times n$ matrices over $\mathbb{R}$. This set will be denoted by $M_n(\mathbb{R})$ when $m = n$. The subspace of $M_n(\mathbb{R})$ consisting of symmetric matrices will be denoted by $\mathcal{S}^n$. For $A \in M_n(\mathbb{R})$, we shall denote by $\lambda(A)$ or simply λ, an eigenvalue of A. The set of all eigenvalues of A will be denoted by $\sigma(A)$.

*Received by the editors on February 16, 2018. Accepted for publication on May 7, 2018. Handling Editor: Michael Tsatsomeros. Corresponding Author: Sachindranath Jayaraman.

†Department of Mathematics, School of Mathematics and Computer Sciences, Central University of Tamil Nadu, Neelakudi, Kangalancherry, Thiruvarur – 610 101, Tamilnadu, India (chandrashekaran@cutn.ac.in, chandru1782@gmail.com).

‡School of Mathematics, Indian Institute of Science Education and Research Thiruvananthapuram, Maruthamala P.O., Vithura, Thiruvananthapuram – 695 551, Kerala, India (sachindranathj@gmail.com, sachindranathj@iisertvm.ac.in, vatsal.n15@iisertvm.ac.in, vnm232657@gmail.com).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

The organization and important results obtained in the manuscript are as follows. Section 1 is introductory, contains basic definitions, preliminaries, a brief introduction, basic facts about convex (proper) cones and positive/semipositive operators relative to such cones, basic facts about LCPs and generalizations. The main results are presented in Section 2. We start with a decomposition result for K-semipositive matrices, followed by semipositivity of matrices over the Lorentz cone $\mathcal{L}^n_+$. Semipositivity and decomposition results over the cone $\mathcal{S}^n_+$ of three specific maps $M_{A,B}$, $L_{A,B}$ and S_A are presented. These are followed by a few remarks and examples. A brief section on the semipositive cone of a matrix A and its invariance under A is also brought out. In Section 3 of the manuscript, we present a possible application to linear preserver problems. The manuscript ends with a few concluding remarks.

1.1. Convex cones, positive and semipositive operators. Let us recall that a subset K of a finite dimensional real Hilbert space V is a convex cone if $K + K \subseteq K$ and $\alpha K \subseteq K$ for all $\alpha \geq 0$. K is said to be proper if it is closed, pointed ($K \cap -K = \{0\}$) and has nonempty interior (denoted by K°). The dual, K^*, is defined as $K^* = \{y \in V : \langle y, x\rangle \geq 0 \ \forall x \in K\}$. When K is a convex cone in V such that $K = K^*$, then we say that K is a self-dual cone in V. Our focus is when V is either $\mathbb{R}^n$ with the standard Euclidean inner product or $M_n(\mathbb{R})$ with inner product $\langle X, Y\rangle = trace(Y^tX)$ or $\mathcal{S}^n$ with inner product $\langle X, Y\rangle = trace(XY)$.

A cone K is said to be polyhedral if $K = X(\mathbb{R}^m_+)$ for some $X \in M_{n\times m}(\mathbb{R})$ and simplicial if X is an invertible matrix. Three well known examples of proper self-dual convex cones are the following:

(1) $K = \mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n : x_i \geq 0 \ \forall\ 1 \leq i \leq n\}$, the nonnegative orthant in $\mathbb{R}^n$.

(2) $K = \mathcal{L}^n_+ = \{x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n : x_n \geq 0, x_n^2 \geq \sum_{i=1}^{n-1} x_i^2\}$, the Lorentz cone in $\mathbb{R}^n$.

(3) $K = \mathcal{S}^n_+ = \{A \in \mathcal{S}^n : A \text{ is positive semidefinite}\}$, the set of symmetric positive semidefinite matrices in $\mathcal{S}^n$.

In general, one can also consider the set $K_p = \{x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n : x_n \geq 0, x_n^p \geq \sum_{i=1}^{n-1} x_i^p\}$ for $1 \leq p < \infty$ and for $p = \infty$, $K_\infty = \{x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n : x_n \geq 0, x_n \geq |x_i|,\ i = 1, \ldots, n-1\}$. This set K_p, for $1 \leq p \leq \infty$, is also an example of a proper cone, called the p-norm cone. K_p is self-dual only when $p = 2$. Note that the cone K_∞ in $\mathbb{R}^2$ is the Lorentz cone. We shall need the notion of an ellipsoidal cone later on when we study semipositivity relative to the Lorentz cone. Let Q be a nonsingular symmetric matrix with inertia $(n-1, 0, 1)$. Let λ_n be the single negative eigenvalue of Q with a normalized eigenvector u_n. Define $K := K(Q, u_n) = \{x \in \mathbb{R}^n : x^tQx \leq 0,\ x^tu_n \geq 0\}$. It can be seen that $K(Q, \pm u_n)$ is a proper cone, known as an ellipsoidal cone. The Lorentz cone $\mathcal{L}^n_+$ is an example of an ellipsoidal cone. This can be seen by taking $Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}$ and $u_n = e_n$, the n^{th} unit vector in $\mathbb{R}^n$. The following result from [29] will be used later on.

Lemma 1.1. (Lemma 2.7 of [29]) *A cone K is ellipsoidal if and only if $K = T(\mathcal{L}^n_+)$ for some invertible matrix T.*

Let K be a proper cone in V. When $x \in K^\circ$, we sometimes write $x > 0$. The following holds for any proper cone K in V.

Lemma 1.2. *Let K be a proper cone in a finite dimensional real Hilbert space V. If $x \in K^\circ, y \in K^*$ and $\langle x, y\rangle = 0$, then $y = 0$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

Let us denote by $\mathcal{L}(V_1, V_2)$, the set of all linear maps from V_1 to V_2. Relative to proper cones K_1 and K_2 in finite dimensional real Hilbert spaces V_1 and V_2, respectively, we have the following notions.

Definition 1.3. $L \in \mathcal{L}(V_1, V_2)$ is

1. nonnegative if $L(K_1) \subseteq K_2$.
2. positive if $L(K_1 \setminus \{0\}) \subseteq K_2^\circ$.
3. semipositive if there exists a $d \in K_1^\circ$ such that $L(d) \in K_2^\circ$.

It is customary to denote the set of all linear maps that are nonnegative relative to two proper cones by $\pi(K_1, K_2)$. When $K_1 = K_2 = K$, this will be denoted by $\pi(K)$. Let us also denote the set of all maps that are semipositive relative to K_1 and K_2 by $S(K_1, K_2)$. Once again, when $K_1 = K_2 = K$, this will be denoted by $S(K)$. When $K_1 = K_2 = K$, we shall use the term K-nonnegative and K-semipositive, respectively. Since we work with proper cones, in Definition 1.3 (3), an equivalant way to define K-semipositivity is to require that there exists a $d \in K$ such that $L(d) \in K^\circ$. It is obvious that K-positivity implies K-semipositivity, while the other inclusion is not true. An extensive study on the structure and properties of $\pi(K)$ can be found in [5, 6, 7, 30, 31] and the references cited therein. We state below only those results that will be used in a later section.

Remark 1.4. When $V_1 = \mathbb{R}^n, V_2 = \mathbb{R}^m$, we shall work with matrix representations of $L \in \mathcal{L}(V_1, V_2)$ with respespect to the standard orthonormal basis of V_1 and V_2, respectively, and we shall denote this matrix by A. Moreover, throughout this manuscript, semipositivity of a matrix A will always be relative to the nonnegative orthant $\mathbb{R}^n_+$ of $\mathbb{R}^n$. Semipositivity relative to any other proper cone K will be written as K-semipositive.

The following results are well known in the literature. The proofs may be found in [5, 6, 30, 31] etc.

Proposition 1.5. *For proper cones K_1 and K_2 in V_1 and V_2, respectively, and $L \in \mathcal{L}(V_1, V_2)$, the following hold.*

1. *$\pi(K_1, K_2)$ is a proper cone in $\mathcal{L}(V_1, V_2)$.*
2. *L is (K_1, K_2)-positive if and only if $L \in \pi(K_1, K_2)^\circ$.*
3. *$L \in \pi(K_1, K_2)$ if and only if $L^t \in \pi(K_2^*, K_1^*)$.*

We shall use the notion of isomorphic cones in the next section. We recall the definition below and list a few familiar examples.

Definition 1.6. Let K_1 and K_2 be (proper) cones in finite dimensional real Hilbert spaces V_1 and V_2, respectively. K_1 and K_2 are said to be isomorphic if there exists an invertible linear map $T : V_1 \to V_2$ such that $T(K_1) = K_2$. When $V_1 = V_2 = V$ and $K_1 = K_2 = K$, such a map is called an automorphism of the cone K.

We list below automorphisms of $\mathbb{R}^n_+, \mathcal{L}^n_+$ and $\mathcal{S}^n_+$.

Proposition 1.7. *Let K be one of $\mathbb{R}^n_+, \mathcal{L}^n_+$ or $\mathcal{S}^n_+$ and let T be an automorphism of K. We have the following.*

1. *If $K = \mathbb{R}^n_+$, then T is nonnegative and each row and column has exactly one nonzero entry.*
2. *If $K = \mathcal{L}^n_+$, then $T^t J_n T = \mu J_n$ for some $\mu > 0$, where $J_n = diag(-1, -1, \ldots, -1, 1)$.*
3. *If $K = \mathcal{S}^n_+$, then $T(A) = S^t A S$ for some invertible matrix $S \in M_n(\mathbb{R})$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

The proof of the first statement can be found in Chapter 5 of [7]. Proofs of the second and third statements can be found in Theorem 2.4 of [27] and Theorem 2 of [26], respectively.

1.2. Linear complementarity problems. Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional real inner product space and K be a proper cone in V with dual K^*. Given a linear transformation $L : V \to V$ and an element $q \in V$ the general linear complementarity problem $LCP(L, K, q)$ (or LCP) is to find a $x \in V$ such that

$$x \in K,\ L(x) + q \in K^* \text{ and } \langle x, L(x) + q\rangle = 0.$$

If such an x exists we call x as a solution to the problem $LCP(L, K, q)$. A linear transformation L is said to have the Q-property (globally uniquely solvable (GUS) property) if $LCP(L, K, q)$ has a solution (unique solution) for every $q \in V$. A linear map is said to have $\mathcal{Z}$-property with respect to a proper cone K if:

$$[x \in K, y \in K^* \text{ with } \langle x, y\rangle = 0] \Rightarrow \langle L(x), y\rangle \leq 0.$$

Notice that linear maps on $\mathbb{R}^n$ that have the $\mathcal{Z}$-property with respect to the nonnegative orthant are precisely those $n \times n$ real matrices with nonpositive off-diagonal entries, namely, Z-matrices.

These problems were motivated by various applications to Game Theory, Semidefinite Optimization etc., (see for details, [11, 24]). The question of characterizing Q and GUS properties were studied extensively and interesting connections with various positivity classes of matrices were obtained. For instance, the following is a well known resutlt in the case when $V = \mathbb{R}^n$ and $K = K^* = \mathbb{R}^n_+$ (this special case is called the standard linear complementarity problem, see [11]).

THEOREM 1.8. ([11]) *Let $A \in M_n(\mathbb{R})$ be a Z-matrix. Then the following are equivalent.*

1. *There exists $x \geq 0$ such that $Ax > 0$.*
2. *$LCP(A, \mathbb{R}^n_+, q)$ is globally uniquely solvable.*

Motivated by Theorem 1.8, semipositivity and its relationship with solvability of LCPs were studied in special cases like semidefinite linear complementarity problem ($SDLCP$), the second order cone LCP and the symmetric cone LCP (see for details, [3, 16, 17, 18, 20]).

Let $A \in M_n(\mathbb{R})$. The linear maps $L_A, S_A : \mathcal{S}^n \to \mathcal{S}^n$ defined as $L_A(X) := AX + XA^t$ and $S_A(X) := X - AXA^t$ are called Lyapunov and Stein transformations, respectively. The semidefinite linear complementarity problem is the following: Given a linear map L on $\mathcal{S}^n$ and a $Q \in \mathcal{S}^n$, find $X \in \mathcal{S}^n_+$ such that $Y = L(X) + Q \in \mathcal{S}^n_+$ and $\langle X, Y\rangle = 0$. This is usually abbreviated as $SDLCP(L, Q)$. This problem is the special case of the general LCP where $V = \mathcal{S}^n$ and $K = \mathcal{S}^n_+$.

The notion of semipositivity of linear maps on $\mathcal{S}^n$ is proved to be equivalent to the solvability of certain $SDLCP$'s and also to the asymptotic stability of certain dynamical systems. The following theorem and the note that follows is the summary of results proved in the papers by Gowda and Song [18] and Gowda and Parthasarathy [17] (see for details, [16]).

THEOREM 1.9. *Let $A \in M_n(\mathbb{R})$. Then the following are equivalent.*

1. *The system $\dot{x} + Ax = 0$ is asymptotically stable in $\mathbb{R}^n$ (that is, the trajectory of the system from any starting point in $\mathbb{R}^n$ converges to the origin as $t \to \infty$).*
2. *A is positive stable (that is, all the eigenvalues of A lie in the open right-half plane).*
3. *There exists $X \in \mathcal{S}^n$ such that X and $L_A(X)$ are positive definite (that is, L_A is semipositive with respect to the cone $\mathcal{S}^n_+$ of positive semidefinite matrices in $\mathcal{S}^n$).*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

4. *For every positive definite $Y \in \mathcal{S}^n$, the equation $L_A(X) = Y$ has a positive definite solution $X \in \mathcal{S}^n$.*
5. *L_A has Q-property.*

The equivalence of items (1)–(4) in the above theorem is Lyapunov's theorem. For a discrete dynamical system of the form $x(k+1) = Ax(k),\ k = 1, 2, \ldots$, statements similar to the above theorem can be formulated by replacing positive stability of A with Schur stability of A (which means that all the eigenvalues of A lie in the open unit disk) and L_A by S_A (see Theorem 11 of [17]). Note that L_A and S_A both have $\mathcal{Z}$-properties with reference to the cone $\mathcal{S}^n_+$ (see for details, [16]).

Yet another motivation for us to take up this project concerns a recent result on the geometry of semipositive matrices obtained by Tsatsomeros, one of which says that any semipositive matrix A can be written as YX^{-1} for some positive matrices X and Y (Theorem 3.1 of [33]). In fact, a decomposition of the above form characterizes semipositivity over a general proper cone and we prove this in the next section (Theorem 2.3).

It is well known that any linear map L on $M_n(\mathbb{R})$ is of the form $L(X) = \sum_{i=1}^{n^2} A_i X B_i$, where $A_i, B_i \in M_n(\mathbb{R})$. It follows that a linear map on $\mathcal{S}^n$ can be expressed as $L(X) = \sum_{i=1}^{\frac{n(n+1)}{2}} A_i X B_i$, where $A_i, B_i \in M_n(\mathbb{R})$ (see for instance [25]). Note that in this case $\sum_{i=1}^{\frac{n(n+1)}{2}} A_i B_i$ is a symmetric matrix. We focus in this manuscript only on maps of the form $M_{A,B}(X) = AXB$, $L_{A,B}(X) = AXB + B^t X A^t$, where $A, B \in M_n(\mathbb{R})$ and $S_A(X) = X - AXA^t$. The motivation to study the map $L_{A,B}$ are the following: One, we obtain a generalization of the famous Lyapunov theorem. Second, this operator has properties similar to the Lyapunov operator and in the particular case when B is an orthogonal matrix, this operator serves as yet another example for which the conjecture $\mathcal{Z} \cap \mathcal{P} = \mathcal{Z} \cap \mathcal{S}$ holds. The general case will be taken up for future study.

2. Main results.

We begin by recalling a few preliminary results on semipositive matrices over the nonnegative orthant:

LEMMA 2.1. *The following statements hold.*

1. (Lemma 2.1 of [23]) $A \in M_{m,n}(\mathbb{R})$ *is semipositive if and only if there exists a $d > 0$ such that $Ad > 0$.*
2. *An invertible matrix A is semipositive if and only if A^{-1} is semipositive.*
3. *Every symmetric positive definite matrix is semipositive.*
4. (Theorem 4.3 of [23]) *Let $m \geq n$. Then $A \in M_{m,n}(\mathbb{R})$ is semipositive if and only if every $n \times n$ submatrix is also semipositive.*
5. *Let $A \in M_{m,n}(\mathbb{R})$ be semipositive. If $P \in M_m(\mathbb{R})$ is nonnegative and nonsingular, and $Q \in M_n(\mathbb{R})$ is inverse nonnegative, then PAQ is semipositive. On the other hand, if P is inverse nonnegative, and Q is nonnegative and nonsingular, then semipositivity of PAQ implies that of A.*
6. (Theorem 3.1 of [33]) $A \in M_n(\mathbb{R})$ *is semipositive if and only if there exist positive matrices X and Y with X invertible and $A = YX^{-1}$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

2.1. Semipositivity of matrices relative to a proper cone K. We take up in this section K-semipositivity of matrices in $\mathbb{R}^n$ and prove a decomposition result for such matrices relative to K. This result is similar to what has been stated in the above lemma. Our first result in this section is the following.

THEOREM 2.2. *Let K_1 and K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $A \in M_{m,n}(\mathbb{R})$. Then, $A \in S(K_1, K_2)$ if and only if there exists an invertible matrix $\widetilde{X} \in M_n(\mathbb{R})$ that is K_1-positive and $\widetilde{Y} \in M_{m,n}(\mathbb{R})$ that is (K_1, K_2)-positive with $A = \widetilde{Y}\widetilde{X}^{-1}$.*

Proof. The *if* statement is easy to prove. Suppose there exist matrices satisfying the required conditions such that $A = \widetilde{Y}\widetilde{X}^{-1}$. Let $u \in K_1^\circ$ be such that $\widetilde{Y}u \in K_2^\circ$. Set $v := \widetilde{X}u \in K_1^\circ$. Then, $Av = \widetilde{Y}\widetilde{X}^{-1}v = \widetilde{Y}u \in K_2^\circ$, thereby proving semipositivity of A with respect to K_1 and K_2.

Conversely, assume that A is semipositive relative to K_1 and K_2. Let $d \in K_1^\circ$ be such that $Ad \in K_2^\circ$. Define $X := de^t$ and $Y := AX$, where e is any element of $(K_1^*)^\circ$. Then for any $0 \neq s \in K_1$, we have $Xs = d\langle e, s\rangle \in K_1^\circ$ and $Ys = Ad\langle e, s\rangle \in K_2^\circ$. This means X and Y are positive relative to K_1 and (K_1, K_2), respectively. It follows from Proposition 1.5 (2) that X and Y are interior points of $\pi(K_1)$ and $\pi(K_1, K_2)$, respectively. Since in a small enough neighbourhood of X there exist invertible matrices, one can find (by a continuity argument) $\widetilde{X}$ and $\widetilde{Y}$ in the interior of $\pi(K_1)$ and $\pi(K_1, K_2)$ such that $\widetilde{X}$ is invertible and $A\widetilde{X} = \widetilde{Y}$. This completes the proof. ∎

In particular, when $K_1 = K_2 = K$ is a proper cone in $\mathbb{R}^n$, we have the following theorem. The proof is skipped.

THEOREM 2.3. *Let K be a proper cone in $\mathbb{R}^n$ and $A \in M_n(\mathbb{R})$. Then A is K-semipositive if and only if there exist $\widetilde{X}$, $\widetilde{Y} \in M_n(\mathbb{R})$ both K-positive with $\widetilde{X}$ invertible and $A = \widetilde{Y}\widetilde{X}^{-1}$.*

We shall exploit decompositions of the form YX^{-1} with Y semipositive and X positive, later on to characterize semipositivity of a specific linear map over the cone $\mathcal{S}_+^n$. We prove next that semipositivity relative a proper cone K_1 determines semipositivity with respect to another proper cone K_2 through any invertible linear map $S \in \pi(K_1, K_2)$. We shall use this result later on when we discuss semipositivity with respect to the Lorentz cone.

THEOREM 2.4. *For proper cones K_1, K_2 in $\mathbb{R}^n$, let $S \in \pi(K_1, K_2)$ be an invertible linear map on $\mathbb{R}^n$. If a matrix A is K_1-semipositive, then the matrix $B = SAS^{-1}$ is K_2-semipositive. Conversely, if the cones are self-dual and if C is K_2-semipositive, then there exists a K_1-semipositive matrix A such that $C = (S^t)^{-1}AS^t$.*

Proof. Suppose A is K_1-semipositive. Consider the matrix $B = SAS^{-1}$. By the definition of semipositivity, there exists a $x \in K_1^\circ$ such that $Ax \in K_1^\circ$. Let $y = Sx$ so that $y \in S(K_1^\circ) = [S(K_1)]^\circ \subseteq K_2^\circ$. Then, $By = SAS^{-1}y = SAx \in K_2^\circ$.

Conversely, suppose the cones are self-dual and that C is a K_2-semipositive matrix. Since $S(K_1) \subseteq K_2$, $S^t(K_2) \subseteq K_1$ as they are both self-dual cones. Therefore, $K_2 \subseteq (S^t)^{-1}(K_1)$. Get a $x \in K_2^\circ$ such that $Cx \in K_2^\circ$. Let $y = S^tx \in S^t(K_2^\circ) = [S^t(K_2)]^\circ \subseteq K_1^\circ$. Let $A := S^tC(S^t)^{-1}$. Then, $Ay = S^tC(S^t)^{-1}y = S^tCx \in K_1^\circ$. ∎

The following theorem, which says that semipositivity is preserved with respect to isomorphic cones, follows from the above theorem. Notice that in this case, we do not require the cones to be self-dual.

THEOREM 2.5. *Let K_1 and K_2 be two proper cones in finite dimensional real Hilbert spaces V_1 and V_2, respectively. Let $T : V_1 \to V_2$ be an invertible linear map such that $T(K_1) = K_2$. If a matrix A is*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

K_1-semipositive, then TAT^{-1} is K_2-semipositive. Conversely, if B is K_2-semipositive, then there exists a K_1-semipositive matrix A such that $B = TAT^{-1}$.

Proof. The proof follows from Theorem 2.4 as $T \in \pi(K_1, K_2)$ and $T^{-1} \in \pi(K_2, K_1)$ are invertible maps. Alternatively, a direct proof can be given by observing that interior points are mapped to interior points by T and T^{-1}. If A is K_1-semipositive, then there exists $x \in K_1^\circ$ such that $Ax \in K_1^\circ$. Let $y \in V_2$ be such that $T^{-1}y = x$. Note that $y \in K_2^\circ$. Then, $Ax \in K_1^\circ$ and $Ax = AT^{-1}y$. It now follows easily that $TAT^{-1}y = TAx \in (K_2)^\circ$. This proves K_2-semipositivity of TAT^{-1}. Conversely, if B is K_2-semipositive, then take $A = T^{-1}BT$. Then, a similar argument as above proves that A is K_1-semipositive. □

The following corollary is immediate.

Corollary 2.6. *Let K_1, K_2 and T be as in Theorem 2.5. If A is K_1-semipositive and has a decomposition $A = YX^{-1}$, where X, Y are K_1-positive with X invertible, then TAT^{-1} has the decomposition $Y_1X_1^{-1}$, where $Y_1 = TYT^{-1}$ and $X_1 = TXT^{-1}$.*

Proof. Since T is an isomorphism between K_1 and K_2, it follows that X_1 and Y_1 are K_2-positive. The rest is obvious. □

2.2. Semipositivity of matrices relative to the Lorentz cone $\mathcal{L}^n_+$. We consider in this section semipositivity of matrices with respect to the n-dimensional Lorentz cone $\mathcal{L}^n_+$ in $\mathbb{R}^n$. Before proceeding with semipositivity over the Lorentz cone, we state a classical result on characterization of nonnegativity over the Lorentz cone due to Loewy and Schneider. Let J_n denote the diagonal matrix $diag(-1, -1, \ldots, -1, 1)$. Recall that for $A, B \in \mathcal{S}^n, A \succeq B$ if and only if $A - B \in \mathcal{S}^n_+$ defines a partial order on $\mathcal{S}^n$, known as the Löwner partial order.

Theorem 2.7. (Theorem 2.2 of [27]) *Let $Y \in M_n(\mathbb{R})$. If $Y(\mathcal{L}^n_+) \subseteq \mathcal{L}^n_+$ or $-Y(\mathcal{L}^n_+) \subseteq \mathcal{L}^n_+$, then $Y^tJ_nY \succeq \mu J_n$ for some $\mu \geq 0$. Conversely, if rank $Y \neq 1$ and there is a $\mu \geq 0$ such that $Y^tJ_nY \succeq \mu J_n$ then $Y(\mathcal{L}^n_+) \subseteq \mathcal{L}^n_+$ or $-Y(\mathcal{L}^n_+) \subseteq \mathcal{L}^n_+$.*

Recall that $\mathcal{L}^2_+$ is a simplicial cone, whereas for $n \geq 3$, $\mathcal{L}^n_+$ is non-polyhedral. We can therefore write down the structure of an $\mathcal{L}^2_+$-semipositive matrix easily. We record this below. Moreover, a decomposition in the form YX^{-1} follows easily from Corollary 2.6. We then determine the structure of a semipositive matrix relative to $\mathcal{L}^n_+$ for $n \geq 3$ and also write down an explicit decomposition of such a matrix as a product YX^{-1} with X, Y both $\mathcal{L}^n_+$-positive and X invertible.

Theorem 2.8. *If a 2×2 matrix A is semipositive, then TAT^{-1} is $\mathcal{L}^2_+$-semipositive, where $T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Conversely, if B is $\mathcal{L}^2_+$-semipositive, then there exists a semipositive matrix A such that $B = TAT^{-1}$.*

Proof. The matrix T is invertible with $T^{-1} = (1/2)T^t$. Moreover, $T(\mathbb{R}^2_+) \subseteq \mathcal{L}^2_+$ and so $T^t(\mathcal{L}^2_+) \subseteq \mathbb{R}^2_+$ (Refer Proposition 1.5 (3)). The rest of the proof is similar to that of Theorem 2.5. □

We now characterize semipositivity over the Lorentz cone when $n \geq 3$.

Theorem 2.9. *Let $n \geq 3$. If a matrix A is semipositive, then the matrix $B = SAS^{-1}$ is $\mathcal{L}^n_+$-semipositive for some invertible matrix $S \in \pi(\mathbb{R}^n_+, \mathcal{L}^n_+)$. Conversely, if C is $\mathcal{L}^n_+$-semipositive, then there exists a semipositive matrix A such that $C = (S^t)^{-1}AS^t$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

Proof. Take S on $\mathbb{R}^n$ defined by $x = (x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, x_1 + \cdots + x_n)$. This map S is an invertible linear map and $S \in \pi(\mathbb{R}^n_+, \mathcal{L}^n_+)$. Moreover, both these cones are self-dual. The conclusion now follows from Theorem 2.4. ◻

We are now ready to exhibit a decomposition of $\mathcal{L}^n_+$-semipositive matrices in the form YX^{-1} with X, Y $\mathcal{L}^n_+$-positive and X invertible.

THEOREM 2.10. *$A \in M_n(\mathbb{R})$ is $\mathcal{L}^n_+$-semipositive if and only if there exist $X, Y \in M_n(\mathbb{R})$ with X invertible, X, Y $\mathcal{L}^n_+$-positive and $A = YX^{-1}$.*

Proof. We discuss two cases here.

Case 1: rank$(A) = 1$. Let $x \in \mathcal{L}^n_+$ be such that $Ax \in (\mathcal{L}^n_+)^\circ$. In this case, we can write $A = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ 1 \end{bmatrix} p^t$, where $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ 1 \end{bmatrix} \in (\mathcal{L}^n_+)^\circ$ and $p \in \mathbb{R}^n$. Choose $\beta > 0$ such that $(\beta p^t x)^2 - p_1^2 - p_2^2 - \cdots - p_{n-1}^2 > 0$ and $\gamma := \beta^2(x_n^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2) > 1$.
Define $X := \begin{bmatrix} 1 & 0 & \cdots & \beta x_1 \\ 0 & \ddots & \cdots & \beta x_2 \\ 0 & 0 & 1 & \vdots \\ 0 & \cdots & 0 & \beta x_n \end{bmatrix}$. Notice that X is invertible. Now $X^t J_n X - J_n = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \gamma - 1 \end{bmatrix} \succeq 0$, and so by Theorem 2.7, we have $X(\mathcal{L}^n_+) \subseteq \mathcal{L}^n_+$. Note that $X(\mathcal{L}^n_+ \setminus \{0\}) \subseteq (\mathcal{L}^n_+)^\circ$, since $\mu = 1$. Define $Y := AX$. We have $Y = uv^t$, where $u = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} p_1 \\ \vdots \\ p_{n-1} \\ \beta p^t x \end{bmatrix}$. Since $v, v \in (\mathcal{L}^n_+)^\circ$, it follows that $Y(\mathcal{L}^n_+ \setminus \{0\}) \subseteq (\mathcal{L}^n_+)^\circ$.

Case 2: rank$(A) > 1$. Let $A \in M_n(\mathbb{R})$ be an $\mathcal{L}^n_+$-semipositive matrix. Let $x \in \mathcal{L}^n_+$ be such that $y = Ax \in (\mathcal{L}^n_+)^\circ$. Let us a take the vector $j^t = (0, \dots, 0, 1) \in \mathcal{L}^n_+$. Define $X := xj^t + \epsilon I$, where $\epsilon > 0$. Notice that $X = \begin{bmatrix} \epsilon & 0 & \cdots & x_1 \\ 0 & \epsilon & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \epsilon + x_n \end{bmatrix}$ is an invertible matrix since the columns of X are linearly independent. If $z \in \mathcal{L}^n_+, Xz = xj^t z + \epsilon z$. Since $j^t z = z_n \geq 0$ and $\mathcal{L}^n_+$ is convex cone, $Xz \in \mathcal{L}^n_+$. Thus, X is $\mathcal{L}^n_+$-nonnegative. In fact, X maps the interior of $\mathcal{L}^n_+$ into itself. Let $Y := AX$. It can be verified that $Y = yj^t + \epsilon A$. Define a map $f : \mathbb{R} \to M_n(\mathbb{R})$ by $f(\epsilon) = (yj^t + \epsilon A)^t J_n (yj^t + \epsilon A) - \mu J_n$, where $0 < \mu < (y_n^2 - y_1^2 - y_2^2 - \cdots - y_{n-1}^2)$. Since f is continuous and $f(0) \succ 0$, there exists $\delta > 0$ such that for all $\beta \in (0, \delta)$, $f(\beta) \succ 0$. Choose one such $\epsilon \in (0, \delta)$, so that $f(\epsilon) \succ 0$. We then have $Y(\mathcal{L}^n_+) \subseteq \mathcal{L}^n_+$. Note that $Y(\mathcal{L}^n_+ \setminus \{0\}) \subseteq (\mathcal{L}^n_+)^\circ$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

The proof of the converse is similar to that of Theorem 2.3. ∎

THEOREM 2.11. *$A \in M_n(\mathbb{R})$ is $\mathcal{L}^n_+$-semipositive if and only if there exists a proper ellipsoidal cone $K_1 \subseteq \mathcal{L}^n_+$ and a closed cone $K_2 \subseteq (\mathcal{L}^n_+)^\circ \cup \{0\}$ such that $A(K_1) = K_2$.*

Proof. Let A be $\mathcal{L}^n_+$-semipositive. Consider matrices X, Y as in the proof Theorem 2.10 such that $A = YX^{-1}$. Let $K_1 = X(\mathcal{L}^n_+)$ and $K_2 = Y(\mathcal{L}^n_+)$. Since X is invertible, it follows from Lemma 1.1 that K_1 is an ellipsoidal cone. Moreover, it follows from $X(\mathcal{L}^n_+ \setminus \{0\}) \subseteq (\mathcal{L}^n_+)^\circ$, that K_1 is proper and contained in $\mathcal{L}^n_+$. Again, as $Y(\mathcal{L}^n_+ \setminus \{0\}) \subseteq (\mathcal{L}^n_+)^\circ$, we see that K_2 is a closed cone in $(\mathcal{L}^n_+)^\circ \cup \{0\}$. Finally, we have $A(K_1) = YX^{-1}X(\mathcal{L}^n_+) = Y(\mathcal{L}^n_+) = K_2$.

Conversely, suppose there exists a proper ellipsoidal cone $K_1 \subseteq \mathcal{L}^n_+$ and a closed cone $K_2 \subseteq (\mathcal{L}^n_+)^\circ \cup \{0\}$ such that $AK_1 = K_2$. Since $K_1 \subseteq \mathcal{L}^n_+$ is proper and $A(K_1) = K_2$, there is $x \in K_1^\circ$ such that $Ax \in K_2 \setminus \{0\}$. Thus, A is $\mathcal{L}^n_+$-semipositive. ∎

REMARK 2.12. It follows from Lemma 1.1 and Theorem 2.5 that the results presented in this section hold for any ellipsoidal cone.

2.3. Semipositivity of maps relative to $\mathcal{S}^n_+$. Let us observe that if $L(X) = \sum_{i=1}^{\frac{n(n+1)}{2}} A_i X B_i$ is such that $\sum_{i=1}^{\frac{n(n+1)}{2}} A_i B_i$ is positive definite, then by taking $X = I$, it follows that L is semipositive. Observe that the cone $\mathcal{S}^2_+$ is isomorphic to the Lorentz cone $\mathcal{L}^3_+$ in $\mathbb{R}^3$. The following map gives an isomorphism between these cones. Define $F : \mathbb{R}^3 \to \mathcal{S}^2$ as follows:

$$x = (x_1, x_2, x_3)^t \mapsto A = \begin{bmatrix} x_3 - x_1 & x_2 \\ x_2 & x_3 + x_1 \end{bmatrix}.$$

It can be easily seen that F is an invertible linear map such that $F(\mathcal{L}^3_+) = \mathcal{S}^2_+$. In other words, $\mathcal{S}^2_+$ is an ellipsoidal cone. Using this map F and Theorem 2.5 one can easily characterize semipositivity of any linear map L on $\mathcal{S}^2$ relative to $\mathcal{S}^2_+$. We shall use this idea in Example 2.28 later on. We now take up semipositivity of three specific maps on $\mathcal{S}^n$.

2.4. Semipositivity of the maps $M_{A,B}, L_{A,B}$ and S_A. The following theorem characterizes semipositivity of the map $M_{A,B}$.

THEOREM 2.13. *Let $A, B \in M_n(\mathbb{R})$. Then $M_{A,B}$ is $\mathcal{S}^n_+$-semipositive if and only if $B = \alpha A^t$ for some $\alpha > 0$. Consequently, the map $M_{A,B}$ is semipositive if and only if it is an automorphism of the cone $\mathcal{S}^n_+$.*

Proof. The *if* part follows from the previous paragraph about general linear maps on $\mathcal{S}^n$. For the *only if* part, observe that $\mathcal{S}^n_+$-semipositivity of the map implies that both A and B are invertible. Now consider $C := A(B^t)^{-1}$. Since the map is defined on $\mathcal{S}^n$, we have $AXB = B^tXA^t$ for every $X \in \mathcal{S}^n$. From this, it follows that $CX = XC^t$ for all $X \in \mathcal{S}^n$. By varying X, we see that $C = \alpha I$ and so $B = \alpha A^t$. Thus, the map is of the form $M_{A,B}(X) = M_{A,\alpha A^t}(X) = \alpha AXA^t$ for all $X \in \mathcal{S}^n$. Once again using semipositivity of the map, we conclude that $\alpha > 0$. The second statement follows from Proposition 1.7 (3) (see also Theorem 2 of [26]). ∎

We now move on to semipositivity of the map $L_{A,B}(X) = AXB + B^tXA^t$ for $A, B \in M_n(\mathbb{R})$. The motivation to study this map has already been mentioned earlier. Let us call a square matrix $A \in M_n(\mathbb{R})$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

generalized positive definite if $A + A^t$ is positive definite. It was proved by Duan and Patton that a matrix A is positive stable if and only if $A = GP$, where G is a generalized positive definite matrix and P is a (symmetric) positive definite matrix. The proof can be found in Theorem 3.1 of [14]. We then have the following result.

Theorem 2.14. *Consider the map $L_{A,B}$ on $\mathcal{S}^n$, where A is generalized positive definite and B is symmetric positive definite. Then, $L_{A,B}$ is $\mathcal{S}^n_+$-semipositive.*

Proof. Take $X = B^{-1}$ and see that $L(X) = A + A^t$, which is positive definite. □

Our next result concerning semipositivity of the map $X \mapsto AXB + B^tXA^t$ is the following. We record a useful observation that will be used below.

Observation 2.15. *If the map $L_{A,B}$ is semipositive, then A and B are invertible matrices. Moreover, $L_{A,B} = L_C L_2^{-1}$, where L_C is the Lyapunov map induced by $C = A(B^t)^{-1}$ and $L_2^{-1}(X) = B^tXB$.*

Proof. If there exists a positive definite X_0 such that $L_{A,B}(X_0)$ is positive definite, then AX_0B is positive stable, thereby proving invertibility of A and B. The rest is easy computation. □

Theorem 2.16. *Consider the map $L_{A,B}$, $A, B \in M_n(\mathbb{R})$, where $AB = BA, B \in \mathcal{S}^n$. Assume further that A has only real eigenvalues. Then AB is positive stable, if and only if $L_{A,B}$ is $\mathcal{S}^n_+$-semipositive.*

Proof. Let us first observe that the map can be assumed to be of the form $L_{T,D}(X) = TXD + DXT^t$, where T is an upper triangular matrix and D is a diagonal matrix, both with real entries. This can be proved as follows: Commutativity and symmetry of B ensures that both A and B have the same set of eigenvalues that are also real. Now, simultaneously orthogonally triangularize A and B and once again use symmetry of B to get the required form.

Since AB is positive stable, so is TD, which implies that all the diagonal entries of the upper triangular matrix TD are positive. Choose positive real numbers $x_1, \ldots, x_n$ such that $x_1 > x_2 \cdots > x_n$ and set $X_0 = diag(x_1, x_2, \ldots, x_n)$. It can be verified by induction that all the principal minors of the symmetric matrix $L(X_0)$ are positive (by the choice of x_is). This proves semipositivity of the map $L_{T,D}$ and hence that of $L_{A,B}$ relative to $\mathcal{S}^n_+$.

Conversely, if $L_{T,D}$ is semipositive with respect to $\mathcal{S}^n_+$, then T and D are invertible matrices. Then, Observation 2.15, that $L_2(X) = DXD$ (as in Observation 2.15) is an automorphism of $\mathcal{S}^n_+$ and equivalence of items (2) and (3) of Theorem 1.9 imply that $C = TD^{-1}$ is positive stable. Since $AB = BA$, we see that $TD = DT$. It now follows that TD is positive stable. This is the same as saying that AB is positive stable. □

The following is a more general result and different from Theorem 2.16 on semipositivity of the above map, without any assumptions on A and B.

Theorem 2.17. *The map $L_{A,B}$ is semipositive with respect to the cone $\mathcal{S}^n_+$ if and only if $A(B^t)^{-1}$ is positive stable.*

Proof. Suppose $L_{A,B}$ is semipositive. Then, Observation 2.15, that L_2 (as in Observation 2.15) is an automorphism of $\mathcal{S}^n_+$ and equivalence of items (2) and (3) of Theorem 1.9 imply that $C = A(B^t)^{-1}$ is positive stable.

Conversely, suppose $C := A(B^t)^{-1}$ is positive stable. Then, there exists a positive definite matrix Y such that $CY + YC^t$ is positive definite. Let X be a positive definite matrix such that $Y = B^tXB$. Then, it is easy to verify that $L_{A,B}(X) = AXB + B^tXA^t = CY + YC^t$, which is positive definite. □

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

The following corollaries are immediate.

COROLLARY 2.18. *Consider the map $L_{A,B}$. Assume that either A or B is orthogonal. Then, the map $L_{A,B}$ is semipositive with respect to $\mathcal{S}^n_+$ if and only if AB is positive stable.*

COROLLARY 2.19. *Let $A, B \in M_n(\mathbb{R})$ with A or B orthogonal. Then the following are equivalent.*

1. *AB is positive stable.*
2. *The Lyapunov map $L_{AB} = ABX + X(AB)^t$ is $\mathcal{S}^n_+$-semipositive.*
3. *For each positive definite $Q \in \mathcal{S}^n$, there exists a positive definite $X \in \mathcal{S}^n$ such that $L_{AB}(X) = Q$.*
4. *The map $L_{A,B}(X) = AXB + B^tXA^t$ is $\mathcal{S}^n_+$-semipositive.*
5. *For each positive definite $Q \in \mathcal{S}^n$, there exists a positive definite $X \in \mathcal{S}^n$ such that $L_{A,B}(X) = Q$.*

Proof. We only need to prove the equivalence of the first and last statements. AB is positive stable if and only if there exists X (symmetric) positive definite such that $L_{AB}(X) = Q$ for each positive definite Q. Then, $Q = ABX + XB^tA^t = AYB + B^tYA^t$, where $Y = BXB^t$ is symmetric and positive definite. □

A few remarks are in order.

REMARK 2.20.

1. Taking $C = A(B^t)^{-1}$ and $Y = B^tXB$, we see that $AXB + B^tXA^t = CY + YC^t = L_C$, the Lyapunov map induced by C. In this context, it is pertinent to point out the following result due to Loewy (Theorem 1, [26]) concerning ranges of real Lyapunov mappings. Loewy proved that if the equation $AS + SA^t = DSC^t + CSD^t$ holds for all $S \in \mathcal{S}^n$ and real matrices A, C, D, then in the event that the left hand side of the above equation (which is the Lyapunov map induced by A) is invertible, either C or D is a scalar matrix.
2. Notice that $L_{A,B}(Y) = CY - Y(-C^t)$. Thus, $L_{A,B}$ is a special case of the Sylvester equation $S(X) = AX - XB$. Then $L_{A,B}$ is invertible if and only if $\sigma(C) \cap \sigma(-C^t) = \emptyset$ (see Section 2.4.4 of [21]), which is the same as saying that C does not have any pair of eigenvalues that are negatives of each other. In particular, if C is positive stable, the map $L_{A,B}$ is an invertible map. We thus have the following theorem.

   THEOREM 2.21. *If the map $L_{A,B}$ is semipositive with respect to $\mathcal{S}^n_+$, then it is an invertible map on $\mathcal{S}^n$. The converse is not true.*

   Proof. Semipositivity of the map $L_{A,B}$ implies it is invertible (see the previous remark). That the converse is not true follows from Example 2.26 given below. □
3. In Theorem 2.17, if BA is also symmetric, then AB is positive stable. Note that $BA = BA(B^{-1})^tB^t$ and that $A(B^{-1})^t$ is symmetric if and only if BA is symmetric. It follows that both BA and $BA(B^{-1})^tB^t$ will have the same eigenvalues. Since $A(B^{-1})^t$ is positive stable, we see that BA, and hence, AB is positive stable.
4. Any matrix X and its transpose have the same set of eigenvalues. It follows that if $L_{A,B}$ is semipositive, then both $A(B^t)^{-1}$ and $B^{-1}A^t$ are positive stable. However, $A(B^t)^{-1} + B^{-1}A^t$ need not be positive definite. There are pairs of matrices A, B such that AB is positive stable, but the map $L_{A,B}$ is not semipositive (Example 2.26) and there are also pairs of matrices A, B such that $L_{A,B}$ is semipositive but AB is not positive stable (Example 2.27).

We now take up semipositivity of the map S_A (also known as the Stein transformation) on $\mathcal{S}^n$ defined by $S_A(X) = X - AXA^t,\ A \in M_n(\mathbb{R})$. It is obvious that if $I - AA^t$ is positive definite, then the map S_A is

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

semipositive. The converse, however, is not true. This can be seen by considering the Stein transformation on $\mathcal{S}^2$ induced by the matrix $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. It is easy to see that $I - AA^t$ is not positive definite. Now, $S_A(X_0) = I$, where $X_0 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$. Semipositivity of the Stein transformation was characterized by Gowda and Parthasarathy in Theorem 11 of [17].

2.5. Decompositions of semipositive maps relative to $\mathcal{S}^n_+$. We now discuss decomposition results for the above three maps with respect to the cone $\mathcal{S}^n_+$. It is obvious that the map $M_{A,B}$ can be written as a product $L_1 L_2^{-1}$, where both L_1 and L_2 are positive and invertible. For the map $L_{A,B}$, a decomposition was already there in the proof of Theorem 2.17. We make this explicit below. The following result will be used in the theorem that follows.

THEOREM 2.22. *Let K be a proper cone in V and $L \in \mathcal{L}(V)$. Then L is K-semipositive if and only if there exist $L_1, L_2 \in \mathcal{L}(V)$ both K-positive with L_1 invertible and $L = L_2 L_1^{-1}$.*

Proof. Suppose L is K-semipositive. Let $d \in K^\circ$ be such that $L(d) \in K^\circ$. Let $L_1(s) = d\langle e, s\rangle, s \in V$ be the rank one operator on V, where $e \in (K^*)^\circ$. Let $L_2 = LL_1$. Rest of the proof follows similar to that of Theorem 2.2 with $K_1 = K_2 = K$. ◻

THEOREM 2.23. *Consider the map $L_{A,B}(X) = AXB + B^t X A^t$, $A, B \in M_n(\mathbb{R})$. If $L_{A,B}$ is $\mathcal{S}^n_+$-semipositive, then there exist maps T_1 and T_2, both $\mathcal{S}^n_+$-positive with T_2 invertible such that $L = T_1 T_2^{-1}$.*

Proof. Recall that $L_{A,B}$ is semipositive with respect to $\mathcal{S}^n_+$ if and only if $A(B^t)^{-1}$ is positive stable. Moreover, by taking $C = A(B^t)^{-1}$, $L_1(X) = L_C(X)$, $L_2^{-1}(X) = B^t X B$, we see that $L_{A,B} = L_1 L_2^{-1}$. Notice that L_1 is semipositive as C is positive stable and positivity of L_2 is obvious (both with respect to the cone $\mathcal{S}^n_+$). Since C is positive stable, the map L_1 is an invertible map. Using Theorem 2.22, we can write $L_C = L_3 L_4^{-1}$, where L_3 and L_4 are both $\mathcal{S}^n_+$-positive with L_4 invertible. Now set $T_1 = L_3$ and $T_2 = L_2 L_4$ and note that T_2 is a $\mathcal{S}^n_+$-positive. ◻

REMARK 2.24.

1. Notice that in Theorem 2.23, the map L_1, which is the Lyapunov map induced by $C = A(B^t)^{-1}$, is invertible. Moreover, L_C^{-1} is a positive map. This is because L_C has the $\mathcal{Z}$-property (see for instance [16, 19]). It is however not an automorphism of the cone $\mathcal{S}^n_+$.
2. A linear map L on $\mathcal{S}^n$ is said to have the $\mathcal{P}$-property relative to the cone $\mathcal{S}^n_+$ if X and $L(X)$ commute and $XL(X)$ negative semidefinite implies $X = 0$ (see Definition 2 of [18, page 578]). It can be proved that the Lyapunov map L_C induced by a matrix C has the $\mathcal{P}$-property if and only if C is positive stable (Theorem 5 of [18]). We also know that the map L_C is $\mathcal{S}^n_+$- semipositive if and only if C is positive stable. Now consider the map $L_{A,B}$, with B an orthogonal matrix. We know that $L_{A,B}$ can be decomposed as $L_C L_2^{-1}$, where L_C is the Lyapunov map induced by $C = A(B^t)^{-1} = AB$ and $L_2^{-1}(X) = B^t X B$, an automorphism of $\mathcal{S}^n_+$. Since the map L_C has the $\mathcal{Z}$-property relative to the cone $\mathcal{S}^n_+$, it is easy to verify that the map $L_{A,B}$ also has the $\mathcal{Z}$-property when B is orthogonal. It follows from the above comments and the results proved earlier that when B is an orthogonal matrix such that $C = AB$ is positive stable, the map $L_{A,B}$ is yet another example of an instance where the conjecture $\mathcal{Z} \cap \mathcal{P} = \mathcal{Z} \cap \mathcal{S}$ holds.

Now consider the Stein transformation S_A induced by a matrix A. We have the following decomposition of S_A.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

Theorem 2.25. *Consider the Stein transformation S_A induced by an invertible matrix A. Suppose S_A is semipositive with respect to the cone $\mathcal{S}^n_+$. Then, there exist positive maps L_1 and L_2 (relative to $\mathcal{S}^n_+$) with L_2 invertible and $S_A = L_1 L_2^{-1}$.*

Proof. Define $L_1(X) = AXA^t$. Then, L_1 is a positive map on $\mathcal{S}^n$. We know from Theorem 11 of [17] that semipositivity of S_A is equivalent to $\rho(A) < 1$. A well known result of Stein says that the transformation $(S_A)^{-1}$ maps the set of positive definite matrices into itself if and only if $\rho(A) < 1$. Now define $L_2 = (S_A)^{-1}L_1$, so that $S_A = L_1 L_2^{-1}$. Since both L_1 and $(S_A)^{-1}$ map the set of positive definite matrices into itself, it follows that L_2 also maps the set of positive definite matrices into itself. Invertibility of the map L_2 is obvious. Note that we can actually write down the map L_2^{-1} as $L_2^{-1}(X) = A^{-1}X(A^{-1})^t - X$. ∎

2.6. Examples. We present a few examples of our results.

Example 2.26. Let $A = \begin{bmatrix} -1 & 0 \\ -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Note that $\sigma(AB) = \{0.2679, 3.7320\}$ and $\sigma(A(B^t)^{-1}) = \{-1, -1\}$, which implies that $A(B^t)^{-1}$ is not positive stable. It therefore follows from Theorem 2.17 that $L_{A,B}$ is not semipositive. It is also easy to see that the map $L_{A,B}(X)$ is given by $X \mapsto \begin{bmatrix} -2x_1 & -x_1 - 2x_2 \\ -x_1 - 2x_2 & 12x_1 - 2x_2 - 2x_3 \end{bmatrix}$, where $X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}$. This is clearly not semipositive. Note that neither A nor B is orthogonal, $AB \neq BA, BA$ is not symmetric.

Example 2.27. Let $A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$. It can be easily computed that the eigenvalues of AB are $\pm i$, whereas the eigenvalues of $A(B^t)^{-1}$ is 1 repeated twice. Thus, AB is not positive stable and $A(B^t)^{-1}$ is positive stable, thereby proving that the map $L_{A,B}$ is semipositive. Once again, one can see that $L_{A,B}(X) = \begin{bmatrix} 2x_1 & -x_1 - 2x_2 \\ -x_1 - 2x_2 & -4x_1 + 2x_2 + 2x_3 \end{bmatrix}$. By taking $X_0 = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}$, we see that $L(X_0) = \begin{bmatrix} 2 & -1 \\ -1 & 16 \end{bmatrix}$ is positive definite.

In our next example, we write down an explicit decomposition of a semipositive map of the form $L_{A,B}$ as a product $L_1 L_2^{-1}$ with L_1 and L_2 positive (all of these relative to the cone $\mathcal{S}^n_+$).

Example 2.28. Consider the map $L_{A,B}$ in Example 2.27 and let $F : \mathbb{R}^3 \to \mathcal{S}^2$ be the map $x = (x_1, x_2, x_3)^t \mapsto \begin{bmatrix} x_3 - x_1 & x_2 \\ x_2 & x_3 + x_1 \end{bmatrix}$. We know that F is an isomorphism between $\mathcal{L}^3_+$ and $\mathcal{S}^2_+$. Consider the matrix representation Y (with respect to the standard basis) of the linear map $F^{-1}L_{A,B}F$ on $\mathbb{R}^3$. By taking $x_0 = (9/2, 0, 11/2)^t \in (\mathcal{L}^3_+)^\circ$, we see that $Yx_0 \in (\mathcal{L}^3_+)^\circ$, thereby proving that Y is $\mathcal{L}^3_+$-semipositive. Now form the matrices $\widetilde{X} = x_0 j^t + I$, and $\widetilde{Y} = Y\widetilde{X}$, where $j^t = (0, 0, 1)$. We know from the proof of Theorem 2.10 that $\widetilde{X}$ is $\mathcal{L}^3_+$-positive and also invertible. Consequently, the matrix $\widetilde{Y}$ is an invertible matrix (recall that $L_{A,B}$ is an invertible map as it is semipositive). It is also $\mathcal{L}^3_+$-positive as $(\widetilde{Y})^t J_3 \widetilde{Y} - 14J_3 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 10 & 8 \\ -4 & 8 & 56 \end{bmatrix}$ is positive definite (see Theorem 2.7). Set $L_1 = FYF^{-1}$ and $L_2 = FXF^{-1}$, where X and Y are the linear maps corresponding to the matrices $\widetilde{X}$ and $\widetilde{Y}$, respectively. Then, L_1 and L_2 are $\mathcal{S}^2_+$-positive, L_2 invertible and by construction $L_{A,B}L_2 = L_1$. It can be seen that $L_1(X) = \begin{bmatrix} 3x_1 + x_2 & -1.5x_1 - 2x_2 - 0.5x_3 \\ -1.5x_1 - 2x_2 - 0.5x_3 & 4x_1 + 2x_2 + 10x_3 \end{bmatrix}$ and $L_2(X) = \begin{bmatrix} 1.5x_1 + 0.5x_3 & x_2 \\ x_2 & 5x_1 + 6x_3 \end{bmatrix}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

2.7. The semipositive cone and related questions. We discuss in this section briefly the semipositive cone of a matrix relative to a proper cone K in $\mathbb{R}^n$. Most of the results are similar to those proved recently by Sivakumar and Tsatsomeros over the nonnegative orthant (see for instance Theorem 3.3 and Corollaries 3.4 and 3.5 of [28]), and can be proved for any proper cone K, possibly self-dual. The proofs are therefore omitted. Nevertheless, we give a partial answer to a question raised by Sivakumar and Tsatsomeros as to when the semipositive cone will be invariant under A (so that $\mathcal{K}_A$ will contain an eigenvector corresponding to the spectral radius of A).

Definition 2.29. Let $A \in M_n(\mathbb{R})$. The semipositive cone of A is the set

$$\mathcal{K}_A = \{x \in K : Ax \in K\}.$$

Notice that $\mathcal{K}_A = K \cap A^{-1}(K)$, where $A^{-1}(K) = \{x : Ax \in K\}$. As in [28], when $K = \mathbb{R}^n_+$, this set will be denoted by K_A.

Theorem 2.30. *Let $A \in M_n(\mathbb{R})$ be K-semipositive. If rank$(A) = 1$, then $A = yx^t$ for some $y \in K^\circ$ and $x \in \mathbb{R}^n$. In this case, $A(\mathcal{K}_A) \subseteq \mathcal{K}_A$ if and only if $x^t y \geq 0$.*

Proof. Suppose A is a rank one K-semipositive matrix. Then $A = pq^t$ for some $p, q \in \mathbb{R}^n$. Choose a vector $u \in K^\circ$ such that $Au = pq^t u \in K^\circ$. If $q^t u > 0$, then $p \in K^\circ$. In this case, take $y = p$ and $x = q$. If $q^t u < 0$, then $-p \in K^\circ$. Now take $y = -p$ and $x = -q$. This proves the first statement.

Assume that $A(\mathcal{K}_A) \subseteq \mathcal{K}_A$. There exists $u \in \mathcal{K}_A$ such that $Au \in K^\circ$, since A is K-semipositive. Now, $Au = (yx^t)u = (x^t u)y \in K^\circ$. Since $y \in K^\circ, x^t u > 0$ so that $(x^t u)y$, and hence, $y \in \mathcal{K}_A$. Finally, $Ay = yx^t y \in \mathcal{K}_A$ implies that $x^t y \geq 0$.

Conversely, suppose $x^t y \geq 0$. Then $Ay = (yx^t)y = (x^t y)y \in K$. It follows that $y \in \mathcal{K}_A$. Let $u \in \mathcal{K}_A$, so that $u \in K$ with $Au \in K$. Now, $Au = (yx^t)u = (x^t u)y$. Since $Au \in K$ and $y \in \mathcal{K}_A$, we must have $x^t u \geq 0$. Therefore, $Au \in \mathcal{K}_A$. ∎

We now discuss the case when A is an invertible matrix.

Theorem 2.31. *Let $A \in M_n(\mathbb{R})$ be invertible and let K be a proper cone in $\mathbb{R}^n$. If $A^2(K) \subseteq K$, then $A(\mathcal{K}_A) \subseteq \mathcal{K}_A$. The converse is not true.*

Proof. If $A^2(K) \subseteq K$, then $A(K) \subseteq A^{-1}(K)$, which then implies that $K \cap A(K) \subseteq K \cap A^{-1}(K)$. Now $A(\mathcal{K}_A) = A(K \cap A^{-1}(K)) \subseteq A(K) \cap A(A^{-1}(K)) = K \cap A(K)$. Thus, $A(\mathcal{K}_A) \subseteq \mathcal{K}_A$. That the converse fails to hold follows from by taking $K = \mathbb{R}^2_+$ and $A = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$. The cone $K_A = \{0\}$ and so obviously, $A(K_A) \subseteq K_A$. However, $A^2 \ngeq 0$. ∎

Suppose $A \in M_n(\mathbb{R})$ is a semipositive matrix. We know that the cone K_A and its dual K_A^* are proper polyhedral cones. Let $K_A = P(\mathbb{R}^n_+)$ and $K_A^* = Q(\mathbb{R}^n_+)$. We then have the following.

Theorem 2.32. *Let $A \in M_n(\mathbb{R})$ be semipositive. Then, $A(K_A) \subseteq K_A$ if and only if $Q^t AP$ is a nonnegative matrix, where P and Q are generating matrices for K_A and K_A^*, respectively.*

Proof. The proof follows from Theorem 4.1 of [9], and that both K_A as well as K_A^* are proper polyhedral cones. ∎

3. An application. We end with an interesting application. For a field $\mathbb{F}$ and the set $M_{m,n}(\mathbb{F})$ of $m \times n$ matrices over $\mathbb{F}$, a linear preserver ϕ is a linear map $\phi : M_{m,n}(\mathbb{F}) \longrightarrow M_{m,n}(\mathbb{F})$ that preserves a certain

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

property or a relation. Instead of looking at linear maps on $M_{m,n}(\mathbb{F})$, one can also look at linear preservers on $\mathcal{S}^n$; in particular, we can look at linear preservers of $\mathcal{S}^n_+$. This problem, to our knowledge, is still open in its full generality. The following result was proved recently (Theorem 11 of [32]).

THEOREM 3.1. *If L_1 defined on $\mathcal{S}^n$ is such that $L_1(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$ and $rank(L_1(E_{ii})) = 1$ for all i, then $L_1(X) = W(H \circ X)W^t$ for all $X \in \mathcal{S}^n_+$ for some invertible matrix $W \in M_n(\mathbb{R})$ and some $H \in \mathcal{S}^n_+$ with $diag(H) = diag(I)$ (the notation $\circ$ denotes the Hadamard product).*

We thus have the following result on the structure of semipositive maps of the form $L_1L_2^{-1}$ (the map L_1 is positive, L_2 is invertible and positive) with the assumption that $rank(L_1(E_{ii})) = 1$ for all i. The proof follows easily from Theorem 3.1 and we skip it.

THEOREM 3.2. *Let L_1 and L_2 be maps defined on $\mathcal{S}^n$ such that (1) $L_i(\mathcal{S}^n_+) \subseteq \mathcal{S}^n_+$ for $i = 1, 2$, (2) $rank(L_1(E_{ii})) = 1$ for $i = 1, \ldots, n$ and (3) L_2 invertible. Then any map L on $\mathcal{S}^n$ of the form $L_1L_2^{-1}$ (and hence is semipositive relative to $\mathcal{S}^n_+$) can be written as $L(X) = W(H \circ L_2^{-1}(X))W^t$ for some invertible matrix W and some $H \in \mathcal{S}^n_+$ with $diag(H) = diag(I)$.*

4. Concluding remarks. We have studied in this paper K-semipositivity for a proper cone K in a finite dimensional real Hilbert space V and have characterized K-semipositivity in terms of products of the form YX^{-1} with X and Y K-positive and X invertible. We also focus our attention to two frequently used cones in the optimization literature, namely, the Lorentz cone $\mathcal{L}^n_+$ in $\mathbb{R}^n$ and the cone of symmetric positive semidefinite matrices $\mathcal{S}^n_+$ in the space of real symmetric matrices $\mathcal{S}^n$. Semipositivity of matrices over the Lorentz cone is completely characterized, in addition to an explicit product decomposition of the form YX^{-1} with positive matrices (relative to this cone). Over the cone $\mathcal{S}^n_+$, we defer characterizing semipositivity of linear maps for a future study, although a decomposition of the form $L_1L_2^{-1}$ certainly exists with both L_1 and L_2 being positive maps. We however consider three specific maps over $\mathcal{S}^n$ and characterize their semipositivity. These maps are $M_{A,B}, L_{A,B}$ and S_A and write down explicit decompositions of the above form. Interesting connections to the optimization literature such as the Lyapunov theorem and the $\mathcal{Z} \cap \mathcal{S} = \mathcal{Z} \cap \mathcal{P}$ conjecture are pointed out. We end by pointing out the following interesting result of the authors obtained recently.

THEOREM 4.1. (Theorem 2.3 of [2]) *Let $A \in M_{m,n}(\mathbb{R})$ and let K_1, K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If $A + B \in S(K_1, K_2)$ for every $B \in S(K_1, K_2)$ then $A \in \pi(K_1, K_2)$.*

Acknowledgements. We thank the anonymous referee for a critical reading of the manuscript and useful suggestions, which has improved the manuscript considerably. We dedicate this manuscript to Professor T. Parthasarathy of the Indian Statistical Institute, Chennai Centre, on his 75^{th} birthday as a mark of respect. We thank Professor M. Seetharama Gowda for his help, suggestions, discussions and encouragement. We also thank him for discussions and help with the proof of the decomposition theorem in the general proper cones setting. The first author acknowledges the SERB, Government of India, for partial support in the form of a grant (Grant no. ECR/2017/000078). The third author acknowledges the Council of Scientific and Industrial Research (CSIR), India, for support in the form of a Junior and Senior Research Fellowships (Award no. 09/997(0033)/2015-EMR-I).

REFERENCES

[1] Ch. Arumugasamy and S. Jayaraman. Full column rank preservers that preserve semipositivity of matrices. *Special Matrices*, 6:37–45, 2018.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 34, pp. 304-319, July 2018.
http://repository.uwyo.edu/ela

[2] Ch. Arumugasamy, S. Jayaraman and V.N. Mer. A characterization of nonnegativity relative to proper cones. Preprint, arXiv:1801.09849, 2018.

[3] R. Balaji. Linear complementarity results for Z-matrices on Lorentz cone. *Linear Algebra and its Applications*, 480(1):107–114, 2015.

[4] R.B. Bapat and T.E.S. Raghavan. Nonnegative Matrices and Applications. *Encyclopedia of Mathematics and its Applications*, Vol. 64, Cambridge University Press, Cambridge, 1997.

[5] G.P. Barker. On matrices having an invariant cone. *Czechoslovak Mathematical Journal*, 22(97):49–68, 1972.

[6] G.P. Barker. Theory of cones. *Linear Algebra and its Applications*, 39:263–291, 1981.

[7] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* SIAM Classics in Applied Mathematics, Philadelphia, 1994.

[8] A. Berman and R.C. Ward. Stability and semipositivity of real matrices. *Bulletin of the American Mathematical Society*, 87(2):262–262, 1977.

[9] F. Burns, M. Fiedler, and E. Haynsworth. Polyhedral cones and positive operators. *Linear Algebra and its Applications*, 8:547–559, 1974.

[10] B. Cain, D. Hershkowitz, and H. Schneider. Theorems of the alternative for cones and Lyapunov regularity of matrices. *Czechoslovak Mathematical Journal*, 47(122):487–499, 1997.

[11] R.W. Cottle, J.-S. Pang, and R.E. Stone. *The Linear Complementarity Problem.* SIAM Classics in Applied Mathematics, Philadelphia, 1992.

[12] J. Dorsey, T. Gannon, N. Jacobson, C.R. Johnson, and M. Turnansky. Linear preservers of semi-positive matrices. *Linear and Multilinear Algebra*, 64(9):1853–1862, 2016.

[13] J. Dorsey, T. Gannon, C.R. Johnson, and M. Turnansky. New results about semi-positive matrices. *Czechoslovak Mathematical Journal*, 66(141):621–632, 2016.

[14] G.-R. Duan and R.J. Patton. A note on Hurwitz stability of matrices. *Automatica*, 34(4):509–511, 1998.

[15] M. Fiedler and V. Ptak. Some generalizations of positive definiteness and monotonicity. *Numerische Mathematik*, 9:163–172, 1966.

[16] M. Seetharama Gowda. On copositive and completely positive cones, and Z-transformations. *Electronic Journal of Linear Algebra*, 23:198–211, 2012.

[17] M. Seetharama Gowda and T. Parthasarathy. Complementarity forms of theorems of Lyapunov and Stein and related results. *Linear Algebra and its Applications*, 320:131–144, 2000.

[18] M. Seetharama Gowda and Y. Song. On semidefinite linear complementarity problems. *Mathematical Programming, Series A*, 88:575–587, 2000.

[19] M. Seetharama Gowda and J. Tao. Z-transformations on proper and symmetric cones: Z-transformations. *Mathematical Programming, Series B*, 117(12):195-221, 2009.

[20] M. Seetharama Gowda, R. Szajder, and J. Tao. Some P-properties for linear transformations on Euclidean Jordan algebras. *Linear Algebra and its Applications*, 393:203–232, 2004.

[21] R.A. Horn and C.R. Johnson. *Matrix Analysis*, second edition. Cambridge University Press, Cambridge, 2012.

[22] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1994.

[23] C.R. Johnson, M.K. Kerr, and D.P. Stanford. Semipositivity of matrices. *Linear and Multilinear Algebra*, 37:265–271, 1994.

[24] M. Kojima, S. Shindoh, and S. Hara. Interior-point methods for the monotone semidefinite linear complementarity problems. *SIAM Journal of Optimization*, 7:86–125, 1997.

[25] A. Kovascs. Trace preserving linear transformations on matrix algebras. *Linear and Multilinear Algebra*, 4:243–250, 1977.

[26] R. Loewy. On ranges of real Lyapunov transformations. *Linear Algebra and its Applications*, 13:79–89, 1976.

[27] R. Loewy and H. Schneider. Positive operators on the n-dimensional ice cream cone. *Journal of Mathematical Analysis and Applications*, 49:375–392, 1975.

[28] K.C. Sivakumar and M.J. Tsatsomeros. Semipositive matrices and their semipositive cones. *Positivity*, 22(1):379–398, 2018.

[29] R.J. Stern and H. Wolkowicz. Exponential nonnegativity on the ice cream cone. *SIAM Journal of Matrix Analysis and Applications*, 12(1):160–165, 1991.

[30] H. Schneider and M. Vidyasagar. Cross-positive matrices. *SIAM Journal on Numerical Analysis*, 7:508–519, 1970.

[31] B.S. Tam. On the structure of the cone of positive operators. *Linear Algebra and its Applications*, 167(1):65–85, 1992.

[32] H.D. Tuan, T.T.N. Trang, and D.T. Hieu. Positive definite preserving linear transformations on symmetric matrix spaces. Preprint, arXiv:1008.1347v1, 2010.

[33] M.J. Tsatsomeros. Geometric mapping properties of semipositive matrices. *Linear Algebra and its Applications*, 498:349–359, 2016.