



## THE COMMON INVARIANT SUBSPACE PROBLEM AND TARSKI'S THEOREM\*

GRZEGORZ PASTUSZAK<sup>†</sup>

**Abstract.** This article presents a computable criterion for the existence of a common invariant subspace of  $n \times n$  complex matrices  $A_1, \dots, A_s$  of a fixed dimension  $1 \leq d \leq n$ . The approach taken in the paper is model-theoretic. Namely, the criterion is based on a constructive proof of the renowned Tarski's theorem on quantifier elimination in the theory ACF of algebraically closed fields. This means that for an arbitrary formula  $\varphi$  of the language of fields, a quantifier-free formula  $\varphi'$  such that  $\varphi \leftrightarrow \varphi'$  in ACF is given explicitly. The construction of  $\varphi'$  is elementary and based on the effective Nullstellensatz. The existence of a common invariant subspace of  $A_1, \dots, A_s$  of dimension  $d$  can be expressed in the first-order language of fields, and hence, the constructive version of Tarski's theorem yields the criterion. In addition, some applications of this criterion in quantum information theory are discussed.

**Key words.** Common invariant subspaces, Common eigenvectors, Quantifier elimination, Effective Nullstellensatz, Quantum information theory.

**AMS subject classifications.** 15A18, 03C10, 03C98, 81P05, 81P45.

---

\*Received by the editors on November 14, 2016. Accepted for publication on August 12, 2017. Handling Editor: Michael Tsatsomeris.

<sup>†</sup>Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Toruń, Poland (past@mat.uni.torun.pl). Supported by grant no. DEC-2011/02/A/ST1/00208 of National Science Center of Poland.