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EXPLICIT POLAR DECOMPOSITION OF COMPANION MATRICES*

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Abstract. An explicit formula for the polar decomposition of an $n \times n$ nonsingular companion matrix is derived. The proof involves the largest and smallest singular values of the companion matrix.

Key words. companion matrices, polar decomposition, singular values

AMS(MOS) subject classification. 15A23, 15A18

1. Introduction.

Let

$$f(z) = z^n - a_{n-1}z^{n-1} - \cdots - a_1 z - a_0, \quad a_0 \neq 0,$$

be a complex polynomial and

$$C = \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ a_0 & a_1 & \dots & a_{n-1} \end{bmatrix} \tag{1}$$

be an $n \times n$ nonsingular companion matrix associated with $f(z)$. Let $C = PU$ be the left polar decomposition of C with positive-definite P and unitary U. The singular values of C, i.e., the eigenvalues of P, are well known ([1], [5], [6]). They yield bounds for zeros and for products of zeros of $f(z)$ [6], and they are used for the computation of robustness measures in systems theory [5]. In view of the wide range of applications, both of the polar decomposition and of companion matrices, an explicit formula for $C = PU$ is useful. It is the purpose of this note to derive explicit expressions for the factors P and U in terms of the coefficients a_ν of $f(z)$. As companion matrices have been included in collections of test matrices (see e.g., Table I of [3]) our formula adds yet one more possibility for testing computational algorithms in numerical linear algebra. Our formula also shows that companion matrices belong to the class

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of matrices for which the polar decomposition is finitely computable. Whether all complex square matrices have that property is an open problem, which has been studied in [2].

2. Polar decomposition formula. Our main result, Theorem 2.1, is the explicit formula for P and U in the left polar decomposition of a nonsingular companion matrix C where the coefficients of the polynomial $f(z)$ form the last row.

THEOREM 2.1. *Let the companion matrix C in* (1) *be partitioned as*

$$C = \begin{bmatrix} 0 & I_{n-1} \\ a_0 & d^* \end{bmatrix},$$

with $a_0 \neq 0$ and $d^* = (a_1, \ldots, a_{n-1})$. Define

$$w = \left[(|a_0|+1)^2 + |a_1|^2 + \cdots + |a_{n-1}|^2\right]^{\frac{1}{2}} = \left[(|a_0|+1)^2 + \|d\|^2\right]^{\frac{1}{2}} \tag{2}$$

and

$$v = \frac{a_0}{|a_0|w} \begin{bmatrix} -d \\ 1+|a_0| \end{bmatrix}. \tag{3}$$

Then

$$P = \frac{1}{w} \begin{bmatrix} wI_{n-1} - (w+|a_0|+1)^{-1}dd^* & d \\ d^* & w^2 - |a_0| - 1 \end{bmatrix} \tag{4}$$

is positive definite and $P^2 = CC^*$. Assume $P = (p_1, \ldots, p_n)$ and set $U = (v, p_1, \ldots, p_{n-1})$. Then U is unitary and $C = PU$ is the left polar decomposition of (1).

To prove Theorem 2.1 we first consider the singular values of C, i.e. the nonnegative square roots of the eigenvalues of

$$CC^* = \begin{bmatrix} I_{n-1} & d \\ d^* & s \end{bmatrix}, \tag{5}$$

where

$$s = \sum_{i=0}^{n-1} |a_i|^2 = |a_0|^2 + \|d\|^2.$$

Set $a_n = 1$, and define

$$F(z) = z^2 - \left(\sum_{i=0}^{n} |a_i|^2\right) z + |a_0|^2. \tag{6}$$

The following result is known (see, e.g., [1, pp. 224–225], [5], [6]). To make our note self-contained we include a simple proof.

LEMMA 2.2. *Let* $0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$ *be the singular values of* C. *Then* $\sigma_2 = \cdots = \sigma_{n-1} = 1$, *and* σ_1^2, σ_n^2 *are the zeros of* $F(z)$ *in* (6).

Proof. From (5) it follows that

$$(7)\quad \begin{aligned} \det(zI_n - CC^*) &= (z-s)\det[(z-1)I_{n-1}] - d^*\mathrm{adj}[(z-1)I_{n-1}]d \\ &= (z-1)^{n-2}\left[z^2 - (s+1)z + s - \|d\|^2\right] \\ &= (z-1)^{n-2}F(z). \end{aligned}$$

Thus CC^* has 1 as eigenvalue of multiplicity at least $(n-2)$. Since the eigenvalues of the principal submatrix I_{n-1} in (5) interlace those of CC^*, it follows that $\sigma_1^2 \leq 1 \leq \sigma_n^2$. □

Note that, as $F(z)$ in (6) is quadratic, the values of σ_1^2, σ_n^2 can be found explicitly in terms of $|a_0|^2$ and $\|d\|^2$, see [5, Th. 3.1]. Also $\sigma_1\sigma_n = |a_0|$ and $\sigma_1^2 + \sigma_n^2 = s + 1$. These relations give $\sigma_1 + \sigma_n = w$, and $\|d\|^2 = s - \sigma_1^2\sigma_n^2 = -\left(\sigma_n^2 - 1\right)\left(\sigma_1^2 - 1\right)$. From (2) follows

$$(8)\qquad \|d\|^2 = (w + |a_0| + 1)(w - |a_0| - 1).$$

For the computation of the square root of CC^* only a symmetric 2×2 matrix has to be considered. The following can easily be verified.

LEMMA 2.3. *Let*

$$H = \begin{bmatrix} 1 & \|d\| \\ \|d\| & |a_0|^2 + \|d\|^2 \end{bmatrix}.$$

Then, $\det(zI - H) = F(z) = \left(z - \sigma_1^2\right)\left(z - \sigma_n^2\right)$, *and*

$$H^{\frac{1}{2}} = w^{-1}\begin{bmatrix} 1 + |a_0| & \|d\| \\ \|d\| & w^2 - |a_0| - 1 \end{bmatrix}.$$

Proof of Theorem 2.1. The case with $\sigma_1 = 1$ or $\sigma_n = 1$ is equivalent to $F(1) = 0$, or because of (7), equivalent to $d = 0$. In this case (5) implies

$$P = (CC^*)^{\frac{1}{2}} = \mathrm{diag}(1, \ldots, 1, |a_0|).$$

Furthermore $C = PU$ with P as above and

$$U = \begin{bmatrix} 0 & I_{n-1} \\ \frac{a_0}{|a_0|} & 0 \end{bmatrix},$$

agreeing with (4) and (3).

In the case $\sigma_1 < 1 < \sigma_n$, that is $d \neq 0$, we define vectors

$$v_1 = \frac{1}{\|d\|}\begin{bmatrix} d \\ 0 \end{bmatrix}, \quad v_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then (5), and $CC^*v_1 = v_1 + \|d\|v_n$ and $CC^*v_n = \|d\|v_1 + sv_n$, imply

$$CC^*(v_1, v_n) = (v_1, v_n)\, H.$$

Now consider the eigenvalue 1 of CC^* and let $y_2, \ldots, y_{n-1}$ be an orthonormal set of eigenvectors of CC^* satisfying $CC^*y_i = y_i$, for $i = 2, \ldots, n-1$. Note that for each y_i we have $y_i^* = (x_i^*, 0)$ and $d^*x_i = 0$. Then $V = (y_2, \ldots, y_{n-1}, v_1, v_n)$ is a unitary matrix, and

$$V^*CC^*V = \begin{bmatrix} I_{n-2} & 0 \\ 0 & H \end{bmatrix}.$$

Hence

$$P = (CC^*)^{\frac{1}{2}} = V\begin{bmatrix} I_{n-2} & 0 \\ 0 & H^{\frac{1}{2}} \end{bmatrix}V^*,$$

where $H^{\frac{1}{2}}$ is given in Lemma 2.3. Thus

$$P = I_n + (v_1, v_n)(H^{\frac{1}{2}} - I_2)\begin{bmatrix} v_1^* \\ v_n^* \end{bmatrix}.$$

From (8) it follows that

$$H^{\frac{1}{2}} - I_2 = w^{-1}\begin{bmatrix} -\|d\|^2(w + |a_0| + 1)^{-1} & \|d\| \\ \|d\| & w^2 - |a_0| - 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

On multiplication, the above expression for P yields (4).

For a nonsingular companion matrix C given by (1) it is well known that

$$C^{-1} = \begin{bmatrix} \frac{-d^*}{a_0} & \frac{1}{a_0} \\ I_{n-1} & 0 \end{bmatrix}$$

For the unitary factor of $C = PU$, we have $U = P(C^{-1})^*$. Hence

$$U = (p_1, \ldots, p_{n-1}, p_n)\begin{bmatrix} \frac{-d}{\bar{a}_0} & I_{n-1} \\ \frac{1}{\bar{a}_0} & 0 \end{bmatrix} = (v, p_1, \ldots, p_{n-1}),$$

with

$$v = \frac{1}{\bar{a}_0} P \begin{bmatrix} -d \\ 1 \end{bmatrix} = \frac{1}{\bar{a}_0 w} \begin{bmatrix} [-w + \|d\|^2 (w + |a_0| + 1)^{-1} + 1] d \\ -\|d\|^2 + w^2 - |a_0| - 1 \end{bmatrix}.$$

Using (8) yields (3) and completes the proof. □

It is well known (see, e.g., [4] or [7]) that for a given nonsingular matrix the unitary factors in the left and in the right polar decomposition are equal. Now define

$$\gamma = 1 - \frac{1}{w + |a_0| + 1}$$

and set

$$Q = \frac{1}{w} \begin{bmatrix} |a_0| + |a_0|^2 & \bar{a}_0 d^* \\ a_0 d & w I_{n-1} + \gamma d d^* \end{bmatrix}.$$

It is not difficult to verify that Q is positive definite and $Q^2 = C^* C$. Hence if C is nonsingular and U is given as in Theorem 2.1, then $C = UQ$ is the right polar decomposition of (1).

Let C_{lr}, C_{lc}, C_{fr}, C_{fc} be the companion matrices where the coefficients of the polynomial $f(z)$ form the last row, last column, first row, first column, respectively. So far in our note we have considered $C = C_{lr}$. Using the $n \times n$ permutation matrix (the reverse unit matrix) $K = (k_{ij})$ where $k_{i,n-i+1} = 1$, and 0 elsewhere, we note that

$$C_{lr} = C^T, \quad C_{fr} = KCK, \quad C_{fc} = KC^T K.$$

Hence the polar decompositions of the preceding three types of companion matrices are products that involve the matrices U, K, and P or Q. For any real nonsingular 2×2 matrix the right polar decomposition in closed form is given in [8].

There is a relation between the singular values σ_1 and σ_n of C and the zeros λ of the polynomial $f(z)$, namely $\sigma_1 \leq |\lambda| \leq \sigma_n$. Is it possible that the eigenvalues $e^{i\varphi_\nu}$, $\nu = 1, \ldots, n$, of the unitary factor U also provide information on the geometry of the zeros of $f(z)$?

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