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Siegfried M. Rump
rump@tu-harburg.de

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VARIATIONAL CHARACTERIZATIONS OF THE SIGN-REAL AND THE SIGN-COMPLEX SPECTRAL RADIUS∗
SIEGFRIED M. RUMP†

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Abstract. The sign-real and the sign-complex spectral radius, also called the generalized spectral radius, proved to be an interesting generalization of the classical Perron-Frobenius theory (for nonnegative matrices) to general real and to general complex matrices, respectively. Especially the generalization of the well-known Collatz-Wielandt max-min characterization shows one of the many one-to-one correspondences to classical Perron-Frobenius theory. In this paper the corresponding inf-max characterization as well as variational characterizations of the generalized (real and complex) spectral radius are presented. Again those are almost identical to the corresponding results in classical Perron-Frobenius theory.

1. Introduction. Denote $\mathbb{R}_+ := \{ x \geq 0 : x \in \mathbb{R} \}$, and let $\mathbb{K} \in \{ \mathbb{R}_+, \mathbb{R}, \mathbb{C} \}$. The generalized spectral radius is defined [6] by

$$(1.1) \quad \rho^K(A) := \max \{ |\lambda| : \exists 0 \neq x \in \mathbb{K}^n, \exists \lambda \in \mathbb{K}, |Ax| = |\lambda x| \} \quad \text{for} \quad A \in M_n(\mathbb{K}).$$

Note that absolute value and comparison of matrices and vectors are always to be understood componentwise. For example, $A \preceq |C|$ for $A \in M_n(\mathbb{R})$, $C \in M_n(\mathbb{C})$ is equivalent to $A_{ij} \leq |C_{ij}|$ for all $i,j$.

For $\mathbb{K} = \mathbb{R}_+$ the quantity in (1.1) is the classical Perron root, for $\mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \}$ it is the sign-real or sign-complex spectral radius, respectively. Note that the quantities are only defined for matrices over the specific set $\mathbb{K}$, and also note that for $\rho^\mathbb{R}$ the maximum $|\lambda|$ is only taken over real $\lambda$ and real $x$. Vectors $0 \neq x \in \mathbb{K}^n$ and scalars $\lambda \in \mathbb{K}$ satisfying the nonlinear eigenequation $|Ax| = |\lambda x|$ are also called generalized eigenvectors and generalized eigenvalues, respectively.

Denote the set of signature matrices over $\mathbb{K}$ by $S(\mathbb{K})$, which are diagonal matrices $S$ with $|S_{ii}| = 1$ for all $i$. In short notation $S \in S(\mathbb{K}) : \iff S \in M_n(\mathbb{K})$ and $|S| = I$. For $\mathbb{K} = \mathbb{R}_+$ this is just the identity matrix $I$, for $\mathbb{K} = \mathbb{R}$ the set of $S = \text{diag}(\pm1)$ or diagonal orthogonal, and for $\mathbb{K} = \mathbb{C}$ the set of diagonal unitary matrices. Obviously, for $y \in \mathbb{K}^n$ there is $S \in S(\mathbb{K})$ with $Sy \succeq 0$. In case $|y| > 0$, this $S$ is uniquely determined. Note that $S^{-1} = S^* \in S(\mathbb{K})$ for all $S \in S(\mathbb{K})$.

By definition (1.1) there is $y \in \mathbb{K}^n$ with $|Ay| = |ry| = r|y|$ for $r := \rho^K(A)$, and therefore for $\mathbb{K} \in \{ \mathbb{R}_+, \mathbb{R}, \mathbb{C} \}$,

$$(1.2) \quad \exists S \in S(\mathbb{K}), \exists 0 \neq y \in \mathbb{K}^n : SAy = ry$$

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†Institut für Informatik III, Technical University Hamburg-Harburg, Schwarzenbergstr. 95, 21071 Hamburg, Germany (rump@tu-harburg.de).
and
\begin{equation}
\exists S_1, S_2 \in S(\mathbb{K}), \ \exists x \geq 0, x \neq 0 : S_1 AS_2 x = rx.
\end{equation}

Among the variational characterizations of the Perron root are
\begin{equation}
\max_{x \geq 0} \min_{x, \neq 0} \frac{(Ax)_i}{x_i} = \rho^{\mathbb{R}^+}(A) = \rho(A) = \inf_{x \geq 0} \max_{i} \frac{(Ax)_i}{x_i} \text{ for } A \geq 0
\end{equation}
and
\begin{equation}
\max_{y \geq 0} \min_{y, \neq 0} \frac{y^T Ax}{y^T x} = \rho(A) = \min_{y \geq 0} \max_{y, \neq 0} \frac{y^T Ax}{y^T x} \text{ for } A \geq 0.
\end{equation}

The purpose of this paper is to prove a generalization of both characterizations for the generalized spectral radius.

We note that the only non-obvious property of the generalized spectral radius we use is [6, Corollary 2.4]
\begin{equation}
\rho^{\mathbb{K}}(A|\mu|) \leq \rho^{\mathbb{K}}(A) \text{ for } \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}, A \in M_n(\mathbb{K}) \text{ and } \mu \subseteq \{1, \ldots, n\}.
\end{equation}

2. Variational characterizations. For the following results we need three preparatory lemmata, the first showing that for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ there exists a generalized eigenvector in every orthant.

**Lemma 2.1.** Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $A \in M_n(\mathbb{K})$ be given. Then
\[ \forall S \in S(\mathbb{K}), \exists 0 \neq z \in \mathbb{K}^n, \exists \lambda \in \mathbb{R}^+_0 : S z \geq 0, |Az| = \lambda |z|. \]

**Remark 2.2.** The condition $S z \geq 0$ for $z \in \mathbb{K}^n$ means $S z \in \mathbb{R}^n$ and $S z \geq 0$, or shortly $S z \in \mathbb{R}^+_n$. Note that Lemma 2.1 is also true for $\mathbb{K} = \mathbb{R}^+_0$, in which case $S \in S(\mathbb{K})$ implies $S = I$.

**Proof of Lemma 2.1.** Let fixed $S \in S(\mathbb{K})$ be given and define $O := \{z \in \mathbb{K}^n : \|z\|_1 = 1, S z \geq 0\}$. The set $O$ is nonempty, compact and convex. If there exists some $z \in O$ with $A z = 0$ we are finished with $\lambda = 0$. Suppose $A z \neq 0$ for all $z \in O$ and define $\varphi(x) := \|A x\|_1^{-1} \cdot S^* |A x|$. Then $\varphi$ is well-defined and continuous on $O$, and $\varphi : O \to O$, such that by Brouwer’s theorem there exists a fixed point $z \in O$ with $\varphi(z) = \|A z\|_1^{-1} \cdot S^* |A z| = z$. Then $|Az| = \lambda S z = \lambda |z|$ with $\lambda = \|A z\|_1$.

The next lemma states a property of vectors out of the interior of a certain orthant.

**Lemma 2.3.** Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $A \in M_n(\mathbb{K})$ and define $r := \rho^{\mathbb{K}}(A)$. Then
\[ \forall S \in S(\mathbb{K}), \forall \varepsilon > 0, \exists z \in \mathbb{K}^n : S z > 0, |Az| \leq (r + \varepsilon) \cdot |z|. \]

**Proof.** We proceed by induction. For $n = 1$, it is $r = |A_{11}| \in \mathbb{R}^+_0$, and $z := \text{sign}(S_{11}) \in \mathbb{K}$ does the job. Suppose the lemma is proved for dimension less than $n$. For given $S \in S(\mathbb{K})$ there exists by Lemma 2.1 some $0 \neq z \in \mathbb{K}^n$ and $\lambda \in \mathbb{R}^+_0$ with $S z \geq 0$
and $|Az| = \lambda |z|$. Then $\lambda \leq r$ by definition (1.1). If $Sz > 0$ we are finished. Let 
$\mu := \{ j : z_j \neq 0 \}$ and let $\mathcal{P} := \{1, \ldots, n\} \setminus \mu$ such that 
\[
\begin{bmatrix} T & U \\ V & W \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 0 \end{bmatrix}
\]
with $T = A[\mu]$, $U = A[\mu, \mathcal{P}]$, $V = A[\mathcal{P}, \mu]$, $W = A[\mathcal{P}, \mathcal{P}]$, $z[\mu] = x$ and $z[\mathcal{P}] = 0.$

Then $|Tx| = \lambda |x|$, $Vx = 0$ and $|x| > 0$.

By the induction hypothesis there exists $y' \in K[\mathcal{P}]$ with $S[\mathcal{P}]y' > 0$ and 
$|Wy' | \leq (\rho^K(W) + \varepsilon)|y'| \leq (r + \varepsilon)|y'|$, 
where the latter inequality follows by (1.5). Define 
\[
\alpha := \begin{cases} 
\min_i \frac{|x_i|}{(Uy')_i} & \text{for } Uy' \neq 0 \\
1 & \text{otherwise,}
\end{cases}
\]
and set $y := \alpha y'$. Then $|y| > 0$ and 
\[
\begin{bmatrix} x \\ \varepsilon y \end{bmatrix} = \begin{bmatrix} Tx + \varepsilon Uy \\ \varepsilon Wy \end{bmatrix} \leq \begin{bmatrix} \lambda |x| + \varepsilon \alpha |Uy'| \\ \varepsilon \alpha (r + \varepsilon)|y'| \end{bmatrix} \leq (r + \varepsilon) \begin{bmatrix} |x| \\ \varepsilon |y'| \end{bmatrix}.
\]

The above lemma is obviously not true when replacing $r + \varepsilon$ by $r$, as the example 
$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ with $\rho^K(A) = 1$ for $K \in \{\mathbb{R}, \mathbb{R}, \mathbb{C}\}$ shows. It is, at least for $K = \mathbb{R}$, 
also not valid for irreducible $|A|$. Consider 
\[
A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.
\]

It has been shown in [5, Lemma 5.6] that $\rho^\mathbb{R}(A) = 1$. We show that $|Au| \leq u$ is not 
possible for $u > 0$. Set $u := (x, y, z)^T$, then $|Au| \leq u$ is equivalent to 
\[
-x \leq y + z \leq x \\
-y \leq -x + z \leq y \\
-z \leq -x - y \leq z.
\]

The second and third row imply that 
\[
x \leq y + z \quad \text{and} \quad y \leq -x + z,
\]
and by the first and second row, 
\[
x = y + z \quad \text{and} \quad y = -x + z
\]
so that $y = x - z = -x + z$ and therefore $y = 0$, which means $u$ cannot be positive.
Variational Characterizations of the Generalized Spectral Radius

Third, we need a generalization of a theorem by Collatz [3, Section 2] to the complex case.

**Lemma 2.4.** Let \( A \in M_n(\mathbb{C}) \), \( A^* z = \lambda z \) for \( 0 \neq z \in \mathbb{R}^n \), \( \lambda \in \mathbb{C} \). Then for all \( x \in \mathbb{R}^n \) with \( |x| > 0 \) and \( x_i z_i \geq 0 \) for all \( i \) the following estimations hold true:

\[
\begin{align*}
\min \Re \mu_i & \leq \Re \lambda \leq \max \Re \mu_i, \\
\min \Im \mu_i & \leq \Im \lambda \leq \max \Im \mu_i,
\end{align*}
\]

where \( \mu_i := (Ax)_i / x_i \), for \( 1 \leq i \leq n \).

**Remark 2.5.** Note that \( x \) and the left eigenvector \( z \) of \( A \) are assumed to be real.

**Proof of Lemma 2.4.** Similar to Collatz’s original proof for the case \( A \geq 0 \) we note that

\[
\sum (\lambda - \mu_i) x_i z_i = \sum x_i (A^* z)_i - \sum (Ax)_i z_i = x^* A^* z - z^* Ax = 0,
\]

the latter because \( x \) and \( z \) are real. Now \( x_i z_i \) are real nonnegative for all \( i \), and by \( |x| > 0 \) not all products \( x_i z_i \) can be zero. The assertion follows.

With these preparations we can prove the first two-sided characterization of \( \rho^K \).

**Theorem 2.6.** Let \( K \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\} \) and \( A \in M_n(K) \). Then

\[
(2.1) \quad \max_{S \in S(K)} \max_{x \in K^n, \ x \neq 0} \min_{x_i \geq 0} \left| \frac{(Ax)_i}{x_i} \right| = \rho^K(A) = \max_{S \in S(K)} \inf_{x \in K^n} \max_{x_i \geq 0} \left| \frac{(Ax)_i}{x_i} \right|.
\]

**Remark 2.7.** The characterization is almost identical to the classical Perron-Frobenius characterization (1.4). The difference is that for nonnegative \( A \) the nonnegative orthant is the generic one, and vectors \( x \) can be restricted to this generic orthant. For general real or complex matrices, there is no longer a generic orthant, and therefore the max-min and inf-max characterization is maximized over all orthants. Note that in the left hand side the two maximums can be replaced by \( \max_{x \in K^n} \), but are separated for didactic purposes.

**Proof of Theorem 2.6.** The result is well-known for \( K = \mathbb{R}_+ \), and the left equality was shown in [5, Theorem 3.1] for \( K = \mathbb{R} \), and for \( K = \mathbb{C} \) it was shown in a different context in [4] and [2]; see also [6, Theorem 2.3]. We need to prove the right equality for \( K \in \{\mathbb{R}, \mathbb{C}\} \). Let \( S \in S(K) \) be fixed but arbitrary and denote \( r := \rho^K(A) \). By Lemma 2.3, for every \( \varepsilon > 0 \) there exists some \( x \in K^n \) with \( Sx > 0 \) and \( |Ax| \leq (r + \varepsilon)|x| \), so that \( r \) is larger than or equal to the r.h.s. of (2.1). We will show \( r \) is less than or equal to the r.h.s. of (2.1) to finish the proof. By (1.3) and \( \rho^K(A^*) = \rho^K(A) \) there is \( S_1, S_2 \in S(K) \) and \( 0 \neq z \in \mathbb{R}^n \) with \( z \geq 0 \) and \( S_1 A^* S_2 z = rz \). Then for any \( x \in K^n \) with \( S_1 x > 0 \), Lemma 2.4 implies that

\[
\max_i \left| \frac{(Ax)_i}{x_i} \right| = \max_i \left| \frac{(S^*_2 A S^*_1) \cdot S_1 x)_i}{(S_1 x)_i} \right| \geq \Re r = r.
\]

Finally we give a second two-sided characterization of the generalized spectral radius.
Theorem 2.8. Let $K \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$ and $A \in M_n(K)$. Then
\[
\max_{S_1, S_2 \in S(K)} \min_{x \in K^n} \frac{|y^* A x|}{|y^*| |x|} = \rho^K(A) = \max_{S_1, S_2 \in S(K)} \min_{x \in K^n} \frac{|y^* A x|}{|y^*| |x|}.
\]

Proof. Let, according to (1.2), $SAx = rx$ for $S \in S(K)$, $0 \neq x \in K^n$ and $r = \rho^K(A)$. Define $S_1$ such that $S_1 x \geq 0$ and set $S_2 = S_1 S$. Then for every $y \in K^n$ with $S_2 y \geq 0$ and $|y^*| |x| \neq 0$, it is $S_1 x = |x|$, $S_2 y = |y|$, $S_2 S_1 S = I$ and
\[
y^* A x = y^* S_1^* S_1 S A x = r y^* S_1^* S_1 x = r y^* |x|,
\]

That means for the specific choice of $S_1$, $S_2$ and $x$, the ratio $|y^* A x|/(|y^*| |x|)$ is equal to $r$ independent of the choice of $y$ provided $S_2 y \geq 0$. Therefore, both the left and the right hand side of (2.2) are greater than or equal to $r = \rho^K(A)$. This proves also that the extrema are actually achieved.

On the other hand, let $S_1, S_2 \in S(K)$ and $x \in K^n$, $S_1 x \geq 0$ be fixed but arbitrarily given. Denote $\mu := \{j: x_j \neq 0\}$, $k := |\mu|$, and $\overline{\tau} := \{1, \ldots, n\} \setminus \mu$. By Lemma 2.1, there exists $\tilde{y} \in K^k$ with $\tilde{y} \neq 0$, $S_2 |\mu| \tilde{y} \geq 0$ and $|A^*| |\mu| \cdot |\tilde{y}| = \lambda |\tilde{y}|$ for $\lambda \geq 0$. Therefore $\rho^K(A^*|\mu|) = \rho^K(A|\mu|)$. Define $y \in K^n$ by $y|\mu| := \tilde{y}$ and $y|\overline{\tau}| := 0$. Then $|y^*| |x| = |y| |\mu| |x| |\mu| \neq 0$ and $x|\overline{\tau}| = 0$ imply that
\[
|y^* A x| = |y| |\mu| A |\mu| |x| |\mu| \leq |y| |\mu| A |\mu| |x| |\mu| = \lambda |y| |\mu| |x| |\mu| = \lambda |y^*| |x|.
\]

By (1.5),
\[
\frac{|y^* A x|}{|y^*| |x|} \leq \lambda \leq \rho^K(A).
\]

Therefore, for that choice of $y$ (depending on $S_1$, $S_2$ and $x$) the left hand side of (2.2) is less than or equal to $\rho^K(A)$. It remains to prove that the right hand side of (2.2) is less than or equal to $\rho^K(A)$. Let $S_1, S_2$ be given, fixed but arbitrary. By Lemma 2.1, there exists $0 \neq y \in K^n$ with $S_2 y \geq 0$ and $|A^* y| = \lambda |y|$ for $\lambda \in \mathbb{R}_+$. Then for all $x \in K^n$,
\[
|y^* A x| \leq |y^* A| |x| = \lambda |y^*| |x|,
\]

such that for that choice of $y$ (depending on $S_1, S_2$) the ratio $|y^* A x|/(|y^*| |x|)$ is less than or equal to $\lambda$ for all $x \in K^n$ with $|y^*| |x| \neq 0$. It follows that the right hand side of (2.2) is less than or equal to $\lambda \leq \rho^K(A^*) = \rho^K(A)$, and the proof is finished.

We note that Theorem 2.8 and its proof cover the case $K = \mathbb{R}_+$, where in this case $S(\mathbb{R}_+)$ consists only of the identity matrix. That means for general $A \geq 0$,
\[
\max_{x \geq 0} \min_{y \geq 0} \frac{y^T A x}{y^T x} = \rho(A) = \min_{y \geq 0} \max_{x \geq 0} \frac{y^T A x}{y^T x}.
\]
Finally we note that for the classical Perron-Frobenius theory this characterization is mentioned without proof in the classical book by Varga [7] for irreducible matrices. As in other textbooks, the result is referenced as if it were included in [1], where in turn we only found a reference to an internal report.

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