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GENERALIZED INVERSES OF BORDERED MATRICES

R. B. BAPAT† AND BING ZHENG‡

Abstract. Several authors have considered nonsingular borderings \( A = \begin{pmatrix} B & C \\ D & X \end{pmatrix} \) of \( B \) and investigated the properties of submatrices of \( A^{-1} \). Under specific conditions on the bordering, one can recover any \( g \)-inverse of \( B \) as a submatrix of \( A^{-1} \). Borderings \( A \) of \( B \) are considered, where \( A \) might be singular, or even rectangular. If \( A \) is \( m \times n \) and if \( B \) is an \( r \times s \) submatrix of \( A \), the consequences of the equality \( m + n - \text{rank}(A) = r + s - \text{rank}(B) \) with reference to the \( g \)-inverses of \( A \) are studied. It is shown that under this condition many properties enjoyed by nonsingular borderings have analogs for singular (or rectangular) borderings as well. We also consider \( g \)-inverses of the bordered matrix when certain rank additivity conditions are satisfied. It is shown that any \( g \)-inverse of \( B \) can be realized as a submatrix of a suitable \( g \)-inverse of \( A \), under certain conditions.

Key words. Generalized inverse, Moore-Penrose inverse, Bordered matrix, Rank additivity.

AMS subject classifications. 15A09, 15A03.

1. Introduction. Let \( A \) be an \( m \times n \) matrix over the complex field and let \( A^* \) denote the conjugate transpose of \( A \). We recall that a generalized inverse \( G \) of \( A \) is an \( n \times m \) matrix which satisfies the first of the four Penrose equations:

\[
\begin{align*}
(1) \quad AXA &= A & (2) \quad XAX &= X & (3) \quad (AX)^* &= AX & (4) \quad (XA)^* &= XA.
\end{align*}
\]

For a subset \( \{i,j,\ldots\} \) of the set \( \{1,2,3,4\} \), the set of \( n \times m \) matrices satisfying the equations indexed by \( \{i,j,\ldots\} \) is denoted by \( A\{i,j,\ldots\} \). A matrix in \( A\{i,j,\ldots\} \) is called an \( \{i,j,\ldots\} \)-inverse of \( A \) and is denoted by \( A^{\{i,j,\ldots\}} \). In particular, the matrix \( G \) is called a \( \{1\} \)-inverse or a \( g \)-inverse of \( A \) if it satisfies (1). As usual, a \( g \)-inverse of \( A \) is denoted by \( A^\dagger \). If \( G \) satisfies (1) and (2) then it is called a reflexive inverse or a \( \{1,2\} \)-inverse of \( A \). Similarly, \( G \) is called a \( \{1,2,3\} \)-inverse of \( A \) if it satisfies (1),(2) and (3). The Moore-Penrose inverse of \( A \) is the matrix \( G \) satisfying (1)-(4). Any matrix \( A \) admits a unique Moore-Penrose inverse, denoted \( A^\dagger \). If \( A \) is \( n \times n \) then \( G \) is called the group inverse of \( A \) if it satisfies (1), (2) and \( AG = GA \). The matrix \( A \) has group inverse, which is unique and denoted by \( A^\# \), if and only if \( \text{rank}(A) = \text{rank}(A^2) \).

We refer to [4], [6] for basic results on \( g \)-inverses.

Suppose

\[
A = \frac{p_1}{p_2} \begin{pmatrix} q_1 & q_2 \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} B & C \\ D & X \end{pmatrix}
\]

\[\tag{1.1}
\]

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is a partitioned matrix. We say that \( A \) is obtained by bordering \( B \). We will generally partition a \( g \)-inverse \( A^{-1} \) of \( A \) as

\[
A^{-1} = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix},
\]

which is in conformity with \( A^* \).

We say that the \( g \)-inverses of \( A \) have the “block independence property” if for any \( g \)-inverses

\[
A_i^{-1} = \begin{pmatrix} E_i & F_i \\ G_i & Y_i \end{pmatrix}, \quad i = 1, 2
\]

of \( A \), \( \begin{pmatrix} E_1 & F_1 \\ G_1 & Y_2 \end{pmatrix} \), \( \begin{pmatrix} E_1 & F_1 \\ G_2 & Y_1 \end{pmatrix} \) etc. are also \( g \)-inverses of \( A \).

If \( A \) is an \( m \times n \) matrix, then the following function will play an important role in this paper:

\[
\psi(A) = m + n - \text{rank}(A).
\]

An elementary result is given next. For completeness, we include a proof.

**Lemma 1.1.** If \( B \) is a submatrix of \( A \), then \( \psi(B) \leq \psi(A) \).

**Proof.** Let

\[
A = \begin{pmatrix} q_1 & q_2 \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} B & C \\ D & X \end{pmatrix}.
\]

Then

\[
\text{rank}(A) \leq \text{rank}(B) + \text{rank}(C) + p_2 \leq \text{rank}(B) + q_2 + p_2.
\]

From this inequality, we get \( \psi(B) \leq \psi(A) \). \( \square \)

Note that a matrix \( B \) with \( \text{rank}(B) = r \) can be completed to a nonsingular matrix \( A \) of order \( n \) if and only if \( \psi(B) \leq n \) [10, Theorem 5]. As another example of a result concerning \( \psi \), if

\[
A = \begin{pmatrix} q_1 & q_2 \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} B & C \\ D & O \end{pmatrix}
\]

is a nonsingular matrix of order \( n \), \( n = p_1 + p_2 = q_1 + q_2 \), then \( A^{-1} \) is of the form

\[
A^{-1} = \begin{pmatrix} q_1 & q_2 \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} E & F \\ G & O \end{pmatrix}
\]
if and only if $\psi(B) = \psi(A)$. This will follow from Theorem 3.1.

Several authors ([4], [5], [8], [10], [11], [12]) have considered nonsingular borderings $A$ of $B$ and investigated the properties of submatrices of $A^{-1}$. Under specific conditions on the bordering, one can recover a special g-inverse of $B$ as a submatrix of $A^{-1}$. It turns out that in all such cases the condition $\psi(B) = \psi(A)$ holds. The main theme of the present paper is to investigate borderings $A$ of $B$, where $A$ might be singular, or even rectangular. We show that if $\psi(A) = \psi(B)$ is satisfied then many properties enjoyed by nonsingular borderings have analogs for singular (or rectangular) borderings as well. For example, any g-inverse of $B$ can be obtained as a submatrix of $A^{-1}$ where $A$ is a bordering of $B$ with $\psi(A) = \psi(B)$. This will be shown in Section 4. In Section 5 we show how to obtain the Moore-Penrose inverse and the group inverse by a general, not necessarily nonsingular, bordering. In the next two sections we consider general borderings $A$ of $B$ and obtain some results concerning $A^{-1}$.

We say that rank additivity holds in the matrix equation $A = A_1 + \cdots + A_k$ if $\text{rank}(A) = \text{rank}(A_1) + \cdots + \text{rank}(A_k)$. Let $R(A)$ and $N(A)$ denote the range space of $A$ and the null space of $A$ respectively. We will need the following well-known result.

**Lemma 1.2.** [2] Let $A, B$ be $m \times n$ matrices. Then the following conditions are equivalent:

(i) $\text{rank}(B) = \text{rank}(A) + \text{rank}(B - A)$.

(ii) Every $B$ is a g-inverse of $A$.

(iii) $AB^{-1}(B - A) = O$, $(B - A)B^{-1}A = O$ for any $B$.

(iv) There exists a $B$ that is a g-inverse of both $A$ and $B - A$.

It follows from the proof of Lemma 1.1 that if $\psi(B) = \psi(A)$ then rank additivity holds in

\[
\begin{pmatrix}
B & C \\
D & X
\end{pmatrix} = \begin{pmatrix}
B & O \\
O & O
\end{pmatrix} + \begin{pmatrix}
O & C \\
O & O
\end{pmatrix} + \begin{pmatrix}
O & O \\
D & X
\end{pmatrix}
\]

and in

\[
\begin{pmatrix}
B & C \\
D & X
\end{pmatrix} = \begin{pmatrix}
B & O \\
O & O
\end{pmatrix} + \begin{pmatrix}
O & O \\
D & O
\end{pmatrix} + \begin{pmatrix}
O & C \\
O & X
\end{pmatrix}.
\]

In Section 2 we discuss necessary and sufficient conditions for the block matrix

\[
\begin{pmatrix}
E & F \\
G & Y
\end{pmatrix}
\]

to be a g-inverse of \(\begin{pmatrix}
B & C \\
D & X
\end{pmatrix}\) under the assumption of rank additivity in (1.3) and (1.4). In section 3, necessary and sufficient conditions for the block matrix

\[
\begin{pmatrix}
E & F \\
G & Y
\end{pmatrix}
\]

to be a g-inverse of \(\begin{pmatrix}
B & C \\
D & X
\end{pmatrix}\) are considered under the assumption $\psi(A) = \psi(B)$. Certain related results are also proved. Some additional references on g-inverses of bordered matrices as well as generalizations of Cramer’s rule are [1], [14], [16], [17].

**2. G-inverses of a bordered matrix.** Let $A = \begin{pmatrix}
B & C \\
D & X
\end{pmatrix}$ be a block matrix which is a bordering of $B$. In this section we will study some necessary and sufficient...
conditions for a partitioned matrix \( \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \), conformal with \( A^* \), to be a g-inverse of \( A \).

**Theorem 2.1.** Let \( A = \begin{pmatrix} B & C \\ D & X \end{pmatrix} \). Then rank additivity holds in (1.3) and (1.4) and \( H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \) is a g-inverse of \( A \) if and only if the following conditions hold.

(i) \( BEB = B, CGC = C, DFD = D, XGC = DFX = -DEC, X = XYX - DEC \).

(ii) \( CYD, BFX, CYX, XGB, XYD, BEC, DEB, CGB, BFD \) are null matrices. Furthermore, if \( EBE = E \), then \( X = XYX \).

**Proof.** “Only if” part: Assume rank additivity in (1.3) and (1.4) and that \( H \) is a g-inverse of \( A \). Then by (ii) of Lemma 1.2, \( H \) is also a g-inverse of each summand matrix in (1.3) and (1.4). Using the definition of g-inverse, we easily get \( BEB = B, CGC = C, DFD = D, XYD = O, CYX = O \), and

\[
DFX + XYX = X, \quad XGC + XYX = X.
\]

On the other hand, by (iii) of Lemma 1.2, we have

\[
\begin{pmatrix} B & O \\ O & O \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & C \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow BEC = O,
\]

\[
\begin{pmatrix} O & C \\ O & O \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} B & O \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow CGB = O,
\]

\[
\begin{pmatrix} B & O \\ O & O \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & D & O \\ O & D & X \end{pmatrix} = \begin{pmatrix} O & O & O \\ O & O & O \end{pmatrix} \Rightarrow BFD = O, \quad BFX = O,
\]

\[
\begin{pmatrix} O & C \\ O & O \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & O & O \\ O & O & O \end{pmatrix} = \begin{pmatrix} O & O & O \\ O & O & O \end{pmatrix} \Rightarrow CYD = O, \quad CYX = O,
\]

\[
\begin{pmatrix} O & C \\ O & X \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & O & O \\ O & O & O \end{pmatrix} = \begin{pmatrix} O & O & O \\ O & O & O \end{pmatrix} \Rightarrow CYD = O, \quad XYD = O.
\]

\[
\begin{pmatrix} O & D \\ X & G \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & O & O \\ O & O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \\ O & O \end{pmatrix} \Rightarrow XGB = O, \quad DEB = O,
\]

\[
\begin{pmatrix} O & X \\ G & Y \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} B & O \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \\ O & O \end{pmatrix} \Rightarrow XGB = O, \quad DEB = O,
\]

\[
\begin{pmatrix} O & O \\ D & O \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & C \\ O & X \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \\ O & O \end{pmatrix} \Rightarrow XGC = DFX = -DEC.
\]
Also, (2.1) and (2.2) imply $X = X Y X - D E C$.

"If" part: If the conditions (i) and (ii) hold, then it is easy to verify that $H$ is a $g$-inverse of each summand matrix in (1.3) and (1.4). By (iv) in Lemma 1.2, rank additivity holds in (1.3) and (1.4). It is also easily verified that $H$ is a $g$-inverse of $A$.

If $E B E = E$, then $D E C = 0$ and so $X = X Y X$.

We note certain consequences of Theorem 2.1.

**Corollary 2.2.** Let $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$. Then rank additivity holds in (1.3) and (1.4) and the matrix $H = \begin{pmatrix} E & F \\ G & O \end{pmatrix}$ is a $g$-inverse of $A$ if and only if the following conditions hold.

(i) $B E B = B$, $C G C = C$, $D F D = D$, $D E C = -X$.

(ii) $B E C, D E B, C G B, B F D$ are null matrices.

Furthermore if $E B E = E$, then $X = O$.

**Corollary 2.3.** Let $A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$. Then $R(B) \cap R(C) = \{0\}$, $R(B^*) \cap R(D^*) = \{0\}$ and $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ is a $g$-inverse of $A$ if and only if the following conditions hold.

(i) $B E B = B$, $C G C = C$, $D F D = D$.

(ii) $C Y D, D E C, B E C, D E B, C G B, B F D$ are null matrices.

In this case, the $g$-inverses of $A$ have the block independence property.

**Remark 2.4.** As the conditions $R(B) \cap R(C) = \{0\}$, $R(B^*) \cap R(D^*) = \{0\}$ together with $X = O$ imply rank additivity in (1.3) and (1.4), Corollary 2.3 is a direct consequence of Theorem 2.1. In particular, conditions (i) and (ii) indicate that the block matrices in $\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ can be independently chosen if it is a $g$-inverse of $A$. In other words, the $g$-inverses of $A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$ have the block independence property. Thus Corollary 2.3 complements the known result (see Theorem 3.1 in [15] and Lemma 5(1.2e) in [7]) that the $g$-inverses of $A$ have the block independence property if and only if

$$\text{rank}(A) = \text{rank}(B) + \text{rank}(C) + \text{rank}(D).$$

The next result can also be viewed as a generalization of Corollary 2.3. This type of rank additivity has been considered, for example, in [13].

**Theorem 2.5.** Let $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ and suppose

$$\text{rank}(A) = \text{rank}(B) + \text{rank}(C) + \text{rank}(D) + \text{rank}(X).$$
Then $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ is a $g$-inverse of $A$ if and only if the following conditions hold.

(i) $BEB = B$, $CGC = C$, $DFD = D$, $XYX = X$.

(ii) $BFX, CYD, CYX, DFX, XGB, XGC, XYD, BEC, BFD, CGB, DEB, DEC$ are null matrices.

Proof. Note that the condition $\text{rank}(A) = \text{rank}(B) + \text{rank}(C) + \text{rank}(D) + \text{rank}(X)$ implies rank additivity in

$$A = \begin{pmatrix} B & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & C \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ D & O \end{pmatrix} + \begin{pmatrix} O & O \\ O & X \end{pmatrix}.$$

Now the proof is similar to that of Theorem 2.1. \[ \square \]

A generalization of Theorem 2.5 is stated next; the proof is omitted.

**Theorem 2.6.** Let $A = (A_{i,j})$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$ be an $m \times n$ block matrix. If $\text{rank}(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} \text{rank}(A_{i,j})$, then $G = (G_{l,s})$, $l = 1, 2, \cdots, n$, $s = 1, 2, \cdots, m$ is a $g$-inverse of $A$ if and only if the following equations hold.

$A_{i,j}G_{j,l}A_{l,s} = \begin{cases} A_{i,j} & (i,j) = (l,s) \\ O & (i,j) \neq (l,s) \end{cases}$.

### 3. $G$-inverses of a block matrix $A$ with $\psi(A) = \psi(B)$

Let $A$ and $H$ be matrices of order $m \times n$ and $n \times m$ respectively, partitioned as follows:

$$A = \begin{pmatrix} q_1 & q_2 \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} B & C \\ D & X \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} q_1 & q_2 \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix},$$

where $p_1 + p_2 = m$ and $q_1 + q_2 = n$. By $\eta(A)$ we denote the row nullity of $A$, which by definition is the number of rows minus the rank of $A$. If $m = n$, $A$ is nonsingular, $H = A^{-1}$ and if $A$ and $H$ are partitioned as in (3.1) then it was proved by Fiedler and Markham [10], and independently by Gustafson [9], that

(3.2) $\eta(B) = \eta(Y)$.

The following result, proved in [3], will be used in the sequel. We include an alternative simple proof for completeness.

**Lemma 3.1.** Let $A$ and $H$ be matrices of order $m \times n$ and $n \times m$ respectively, partitioned as in (3.1). Assume $\text{rank}(A) = r$ and $\text{rank}(H) = k$. Then the following assertions are true.

(i) If $AHA = A$, then

$$-(m - r) \leq \eta(Y) - \eta(B) \leq n - r.$$

(ii) If $HAH = H$, then

$$-(n - k) \leq \eta(B) - \eta(Y) \leq m - k.$$
Proof. (i) According to a result on bordered matrices and g-inverses [11, Theorem 1], there exist matrices $P$, $Q$ and $Z$ of order $m \times (m-r)$, $(n-r) \times n$ and $(n-r) \times (m-r)$ respectively, such that the matrix

$$S = \begin{pmatrix} A & P \\ Q & Z \end{pmatrix}$$

is nonsingular and the submatrix formed by the first $n$ rows and the first $m$ columns of $T = S^{-1}$ is $W$. Thus we may write

$$S = \begin{pmatrix} q_1 & q_2 & m-r \\ p_1 & p_2 & n-r \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} q_1 & p_1 & p_2 & n-r \\ q_2 & m-r & V_1 & V_2 \end{pmatrix}.$$

Since $S$ is nonsingular, we have, using (3.2),

$$\eta(B) = \eta\left(\begin{pmatrix} Y & U_2 \\ V_2 & W \end{pmatrix}\right) = q_2 + m - r - \text{rank} \left(\begin{pmatrix} Y & U_2 \\ V_2 & W \end{pmatrix}\right).$$

Now by Lemma 1.1

$$\text{rank}(Y) \leq \text{rank} \left(\begin{pmatrix} Y & U_2 \\ V_2 & W \end{pmatrix}\right) \leq \text{rank}(Y) + m + n - 2r,$$

and hence

$$-(m-r) \leq \eta(Y) - \eta(B) \leq n - r.$$

The result (ii) follows from (i). \[\square\]

The following result, proved using Lemma 3.1, will be used in the sequel.

**Theorem 3.2.** Let $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ with $\psi(A) = \psi(B)$. Then for any $g$-inverse $A^{-} = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ of $A$, $Y = O$.

**Proof.** Assume the sizes of the block matrices in $A$ to be as in (3.1). By Lemma 3.1 we have

$$-(m-r) \leq \eta(Y) - \eta(B) \leq n - r.$$

It follows that

$$-m + r \leq q_2 - \text{rank}(Y) - p_1 + \text{rank}(B).$$

Using $\psi(A) = \psi(B)$ and the inequality above, $\text{rank}(Y) = 0$ and hence $Y = O$. \[\square\]

**Theorem 3.3.** Let $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$. Then $\psi(A) = \psi(B)$ and $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ is a $g$-inverse of $A$ if and only if the following equations hold.
Generalized Inverses

(i) $Y = O$, $BEB = B$, $GC = I$, $DF = I$.
(ii) $DEC = -X$.
(iii) $BEC = O$, $DEB = O$, $BF = O$, $GB = O$.

Furthermore, if $EBE = E$, then $X = O$.

Proof. If $H$ is a g-inverse of $A$ with $\psi(A) = \psi(B)$, then by Theorem 3.2, we know $Y = O$. From the proof of Lemma 1.1, the condition $\psi(A) = \psi(B)$ also indicates rank additivity in (1.3) and (1.4). Note that $C$ and $D$ are also of full column rank and of full row rank respectively under the condition $\psi(A) = \psi(B)$. Then the proof of the theorem is similar to that of Theorem 2.1. □

The proof of the following result is also similar and is omitted.

**Theorem 3.4.** Let $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$, $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ and consider the statements:

(i) $Y = O$, $BEB = B$, $GC = I$, $DF = I$, $BF = O$, $GB = O$.
(ii) $EB + FD$ is hermitian.
(iii) $BE + CG$ is hermitian.
(iv) $EBE + FDE = E$ (v) $EBE + ECG = E$.

Then

(a) $\psi(A) = \psi(B)$ and $H \in A[1, 2, 3]$ if and only if (i), (ii), (iv) hold, $DEC = -X$, $EC = FDEC$ and $DEB = O$.
(b) $\psi(A) = \psi(B)$ and $H \in A[1, 2, 4]$ if and only if (i), (iii), (v) hold, $DEC = -X$, $DE = DCG$ and $BEC = O$.
(c) $\psi(A) = \psi(B)$ and $H = A^T$ if and only if (i)-(v) hold, $DE + XG = O$ and $EC + FX = O$.

The two previous results will be used in the proof of the next result.

**Theorem 3.5.** Let $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$. Then the following conditions are equivalent:

1. $\psi(A) = \psi(B)$ and $\begin{pmatrix} B & C \\ D & X \end{pmatrix}^T = \begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$.
2. $\psi(A) = \psi(B)$ and $\begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$ is a g-inverse of $A$.
3. $X = O$, $C^\dagger C = I$, $DD^\dagger = I$, $BD^\dagger = O$, $C^\dagger B = O$.
4. $X = O$, $C^\dagger C = I$, $DD^\dagger = I$, $BD^\dagger = O$, $C^\dagger B = O$.
5. $\psi(A) = \psi(B)$ and $\begin{pmatrix} E & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$ is a g-inverse of $A$ for some $E \in B^{[1, 2]}$.
6. $\psi(A) = \psi(B)$ and $\begin{pmatrix} E & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$ is a g-inverse of $A$ for some $E$.
7. $\psi(A) = \psi(B)$ and $\begin{pmatrix} B & C \\ D & X \end{pmatrix}^\dagger = \begin{pmatrix} B^\dagger & F \\ G & Y \end{pmatrix}$ for some matrices $F, G, Y$.
8. $\psi(A) = \psi(B)$ and $\begin{pmatrix} B^\dagger & F \\ C^\dagger & Y \end{pmatrix}$ is a $[1, 2, 3]$-inverse of $A$ for some $F, Y$.
9. $\psi(A) = \psi(B)$ and $\begin{pmatrix} B^\dagger & D^\dagger \\ G & Y \end{pmatrix}$ is a $[1, 2, 4]$-inverse of $A$ for some $G, Y$.

Proof. Clearly, (1) ⇒ (2).
(2 ⇒ (3): This follows from Theorem 3.3.)
(3) $\Leftrightarrow$ (4): Since $BD^\dagger = O$ and $C^\dagger B = O$ are equivalent to $BD^* = O$ and $C^* B = O$ respectively, we have this implication.

(3) $\Rightarrow$ (1): Note that $BD^\dagger = O$ and $C^\dagger B = O$ imply $DB^\dagger = O$ and $B^\dagger C = O$. Then it is easy to verify that $\begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$ is $A^\dagger$ thus (1) holds.

Clearly, (1) $\Rightarrow$ (5) $\Rightarrow$ (6).

(5) $\Rightarrow$ (3): By Theorem 3.3, if $\begin{pmatrix} E & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$ is a $g$-inverse of $A$ for some matrix $E$, then we have $X^\dagger = O$, $C^\dagger C = I$, $DD^\dagger = I$, $BD^\dagger = O$ and $C^\dagger B = O$. Note that $X^\dagger = O \Leftrightarrow X = O$, thus (3) holds.

(6) $\Rightarrow$ (1): This follows from (6) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1).

Obviously, (1) $\Rightarrow$ (7), (1) $\Rightarrow$ (8) and (1) $\Rightarrow$ (9).

(7) $\Rightarrow$ (1): By Theorem 3.3, we have $X = O$, $Y = O$, $GC = I$, $DF = I$, $BF = O$ and $GB = O$. Clearly, $G \in C\{1,2,4\}$ and $F \in D\{1,2,3\}$. Using the hermitian property of the matrices $\begin{pmatrix} B & C \\ D & X \end{pmatrix}$, $\begin{pmatrix} B^\dagger & F \\ G & Y \end{pmatrix}$, $\begin{pmatrix} B & C \\ D & X \end{pmatrix}$, $BB^\dagger$ and $B^\dagger B$, it is easy to conclude that $CG$ and $FD$ are also hermitian. Thus $F = D^\dagger$ and $G = C^\dagger$. Note that $Y = X^\dagger = O$ and (1) is proved.

Similarly, using Theorem 3.4 we can show (8) $\Rightarrow$ (1) and (9) $\Rightarrow$ (1) and the proof is complete. \[ \square \]

4. Obtaining any $g$-inverse by bordering. By Theorem 3.3 if $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ with $\psi(A) = \psi(B)$ and if $H = \begin{pmatrix} E & F \\ G & O \end{pmatrix}$ is a $g$-inverse of $A$, then $E$ is a $g$-inverse of $B$ which also satisfies $DEC = -X$, $BEC = O$ and $DEB = O$. Such an $E$, hereafter, will be denoted by $E_{(C,D,X)}$. Note that $E_{(C,D,X)}$ is not uniquely determined by $C,D,X$, since $A^-$ is not unique. In this section we will investigate the converse problem, that is: for a given $g$-inverse $E$ of $B$, how to construct $C$, $D$ and $X$ so that $H = \begin{pmatrix} E & F \\ G & O \end{pmatrix}$ is a $g$-inverse of $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ with $\psi(A) = \psi(B)$ for some matrices of proper sizes.

We first state some well-known lemmas to be used later; see, for example, \[ [4],[6]. \]

Lemma 4.1. The following three statements are equivalent: (i) $E$ is a $g$-inverse of $B$, (ii) $BE$ is an idempotent matrix and $\text{rank}(BE) = \text{rank}(B)$, and (iii) $EB$ is an idempotent matrix and $\text{rank}(EB) = \text{rank}(B)$.

Lemma 4.2. $E$ is a $\{1,2\}$-inverse of $B$ if and only if $E$ is a $g$-inverse of $B$ and $\text{rank}(E) = \text{rank}(B)$.

Lemma 4.3. Let $H = UV$ be a rank factorization of a square matrix. Then the following three statements are equivalent: (i) $H$ is an idempotent matrix, (ii) $I - H$ is an idempotent matrix, and (iii) $VU = I$.

Theorem 4.4. (i) Let $E$ be a $g$-inverse of the $p_1 \times q_1$ matrix $B$ with $\text{rank}(B) = r$. Then there exist $C,D,$ and $X$ such that $E = E_{(C,D,X)}$, where $\text{rank}(C) \leq p_1 - r$ and $\text{rank}(D) \leq q_1 - r$. 
(ii) If \( E = E(C,D,X) \), then there exist matrices \( U, V, \bar{U} \) and \( \bar{V} \) such that

\[
\begin{pmatrix}
  G \\
  V
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
  F \\
  \bar{U} \\
  \bar{V}
\end{pmatrix}
\]

are the rank factorizations of \( I - BE \) and \( I - EB \) respectively.

(iii) \( \text{rank}(E(C,D,X)) = \text{rank}(B) + \text{rank}(R) \), where

\[
R = \begin{pmatrix}
  -X & DEU \\
  VEC & VEU
\end{pmatrix}
\]

for some matrices \( U \) and \( \bar{V} \) as in (ii).

Proof. For a given g-inverse \( E \) of \( B \), we use rank factorizations of \( I - BE \) and \( I - EB \), by which there exist \( C, D, X, F, G, U, \bar{U}, V, \bar{V} \) satisfying the following identities

\[
\begin{align*}
  I - BE &= (C \ U) \begin{pmatrix}
  G \\
  V
\end{pmatrix}, \\
  I - EB &= (F \ \bar{U}) \begin{pmatrix}
  D \\
  \bar{V}
\end{pmatrix},
\end{align*}
\]

\[ X = -DEC. \]

To prove (i), we only need to show that these \( C, D, X, F \) along with \( Y = O \) satisfy the conditions (i),(ii) and (iii) in Theorem 3.3. In fact, from (4.2) and (4.3), we have, in view of Lemma 4.3, that

\[
\begin{pmatrix}
  G \\
  V
\end{pmatrix} (C \ U) = I \quad \text{and} \quad \begin{pmatrix}
  D \\
  \bar{V}
\end{pmatrix} (F \ \bar{U}) = I,
\]

implying

\[
GC = I \quad \text{and} \quad DF = I.
\]

Again from (4.2) and (4.3), we have, by \( (I - BE)B = O \) and \( B(I - EB) = O \),

\[
\begin{pmatrix}
  G \\
  V
\end{pmatrix} B = O \quad \text{and} \quad B \begin{pmatrix}
  F \\
  \bar{U}
\end{pmatrix} = O,
\]

and by \( BE(I - BE) = O \) and \( (I - EB)EB = O \),

\[
BE(C \ U) = O \quad \text{and} \quad \begin{pmatrix}
  D \\
  \bar{V}
\end{pmatrix} EB = O.
\]

Now by (4.4), \( GB = O \) and \( BF = O \), and by (4.5), \( BEC = O \) and \( DEB = O \).

(ii) Let \( E = E(C,D,X) \). By Theorem 3.3, \( BEC = O \), which means \( R(C) \subseteq N(BE) = R(I - BE) \). Note that \( C \) is of full column rank under the condition \( \psi(A) = \psi(B) \). Thus there exists a matrix \( U \) so that \( R((C \ U)) = R(I - BE) \)
and the matrix \((C \ U)\) is of full column rank. Hence, there exists a matrix of full row rank which can be partitioned as \(\begin{pmatrix} G \\ V \end{pmatrix}\) such that

\[
I - BE = (C \ U) \begin{pmatrix} G \\ V \end{pmatrix}.
\]

On the other hand, \(DEB = O\) implies \(N(I - EB) = R(EB) \subseteq N(D)\). So there exists a matrix \(\tilde{V}\) such that \(\begin{pmatrix} D \\ \tilde{V} \end{pmatrix}\) is of full row rank and

\[
N(I - EB) = N\left(\begin{pmatrix} D \\ \tilde{V} \end{pmatrix}\right).
\]

From this we conclude that there exists a matrix of full column rank which can be partitioned as \((F \ \bar{U})\) such that

\[
I - EB = (F \ \bar{U}) \begin{pmatrix} D \\ \tilde{V} \end{pmatrix}.
\]

Now we prove (iii). If \(E = E_{(C,D,X)}\), then from the proof of (ii) there exist matrices \(U, V, \bar{U}\) and \(\tilde{V}\) such that (4.2) and (4.3) hold. Hence \(BE (C \ U) = O\) and \(\begin{pmatrix} D \\ \tilde{V} \end{pmatrix} \bar{E} = O\). Therefore we have

\[
\begin{pmatrix} B \\ D \\ \tilde{V} \end{pmatrix} E(B \ C \ U) = \begin{pmatrix} B & O & O \\ O & DE & DU \\ O & V E & V EU \end{pmatrix} = \begin{pmatrix} B & O \\ O & R \end{pmatrix},
\]

where \(R = \begin{pmatrix} D \\ \tilde{V} \end{pmatrix} E(C \ U)\).

On the other hand,

\[
(E \ F \ \bar{U}) \begin{pmatrix} B & O \\ O & R \end{pmatrix} \begin{pmatrix} E \\ G \\ V \end{pmatrix} = EBE + (F \ \bar{U}) R \begin{pmatrix} G \\ V \end{pmatrix} = EBE + (I - EB) E(I - BE) = E.
\]

Thus we have \(\text{rank}(E_{(C,D,R)}) = \text{rank}(B) + \text{rank}(R)\).}

Theorem 4.4(i) and its proof show that for a given matrix \(B\) and its g-inverse \(E\) we can find matrices \(C\) of full column rank with \(R(C) \subseteq N(EB)\) and \(D\) of full row rank with \(R(EB) \subseteq N(D)\), as well as \(X = -DE\), \(F\) and \(G\) such that matrix
\[
\begin{pmatrix}
  E & F \\
  G & O
\end{pmatrix}
\] is a \(g\)-inverse of
\[
\begin{pmatrix}
  B & C \\
  D & O
\end{pmatrix}
\] with \(\psi(A) = \psi(B)\). Furthermore, we have the following.

**Corollary 4.5.** Let \(B\) and its \(g\)-inverse \(E\) be given. Then the matrix \(A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}\) satisfies \(\psi(A) = \psi(B)\) and has a \(g\)-inverse of the form \(\begin{pmatrix} E & F \\ G & O \end{pmatrix}\) if and only if \(C\) is of full column rank with \(R(C) \subseteq N(BE)\) and \(D\) of full row rank with \(R(EB) \subseteq N(D)\). In this case, \(X = -DEC\), \(F \in D\{1,3\}\), \(G \in C\{1,4\}\), \(BF = O\) and \(GB = O\).

**Proof.** Necessity: This follows from Theorem 3.3.
Sufficiency: The proof of sufficiency is similar to that of Theorem 4.4(i), (ii).

As a special case we recover the following known result.

**Corollary 4.6.** [11, Theorem 1] Let \(E\) be a \(g\)-inverse of \(B\). Then for any matrix \(C\) of full column rank with \(R(C) = N(BE)\) and any matrix \(D\) of full row rank with \(N(D) = R(EB)\), the matrix
\[
A = \begin{pmatrix}
  B & C \\
  D & -DEC
\end{pmatrix}
\]
is nonsingular and
\[
A^{-1} = \begin{pmatrix}
  E & F \\
  G & O
\end{pmatrix},
\]
where \(F \in D\{1,3\}\), \(BF = O\), \(G \in C\{1,4\}\) and \(GB = O\).

5. **Moore-Penrose inverse and group inverse by bordering.** For a given \(g\)-inverse \(E\) of \(B\), Corollary 4.5 shows that \(C\) and \(D\) can be chosen with the conditions \(R(C) \subseteq N(BE)\) and \(R(D^*) \subseteq N((EB)^*)\) so that \(A = \begin{pmatrix} B & C \\ D & -DEC \end{pmatrix}\) satisfies \(\psi(A) = \psi(B)\) and has a \(g\)-inverse of the form \(\begin{pmatrix} E & F \\ G & O \end{pmatrix}\). Further, Corollary 4.6 provides an approach to border the matrix \(B\) into a nonsingular matrix such that in its inverse, the block matrix on the upper left corner is \(E\). We now show how to border the matrix if \(E\) is the Moore-Penrose inverse or the group inverse of \(B\).

**Theorem 5.1.** Let \(B\) be given. Then the matrix \(A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}\) satisfies \(\psi(A) = \psi(B)\) and has a \(g\)-inverse of the form \(\begin{pmatrix} B^\dagger & F \\ G & O \end{pmatrix}\) if and only if \(C\) has full column rank with \(R(C) \subseteq N(B^*)\) and \(D\) has full row rank with \(R(D^*) \subseteq N(B)\). In this case, \(X = -DB^\dagger C = O\) and
\[
A^\dagger = \begin{pmatrix}
  B^\dagger & C^\dagger \\
  D^\dagger & O
\end{pmatrix}.
\]

**Proof.** Note that \(N(BB^\dagger) = N(B^*)\) and \(N((EB)^*) = N(B^\dagger) = N(B)\), and the necessity and sufficiency follow from Corollary 4.5.
It is easy to verify that \((B^\dagger D^\dagger C^\dagger O)\) is a \(g\)-inverse of \(A\). Thus by Corollary 3.5(2), we have

\[
A^\dagger = \begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & O \end{pmatrix},
\]

where \(X = -DB^\dagger C = O\). 

Combining Corollary 4.6 with Theorem 5.1, we have

**Corollary 5.2.** \([5]\)

Let \(B\) be a \(p_1 \times q_1\) matrix with \(\text{rank}(B) = r\). Suppose the columns of \(C \in \mathbb{C}^{p_1 \times (p_1 - r)}\) are a basis of \(N(B^*)\) and the columns of \(D^* \in \mathbb{C}^{q_1 \times (q_1 - r)}\) are a basis for \(N(B)\). Then the matrix

\[
A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}
\]

is nonsingular and its inverse is

\[
A^{-1} = \begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & O \end{pmatrix}.
\]

If \(B\) is square and has group inverse, we can get a bordering \(\begin{pmatrix} B & * \\ * & O \end{pmatrix}\) of \(B\) such that it has a \(g\)-inverse in the form \(\begin{pmatrix} B^\sharp & * \\ * & O \end{pmatrix}\). Part (ii) of the following result is known. We generalize it to any bordering, not necessarily nonsingular, in part (i).

**Theorem 5.3.** Let \(B\) be \(n \times n\) and with index 1. Then

(i) there exist matrices \(C\) of full column rank with \(R(C) \subseteq N(B)\) and \(D\) of full row rank with \(R(B) \subseteq N(D)\) which satisfy \(DC = I\) such that \(\begin{pmatrix} B^\sharp & C \\ D & O \end{pmatrix}\) is a \(g\)-inverse of \(\begin{pmatrix} B & C \\ D & O \end{pmatrix}\) with \(\psi(A) = \psi(B)\);

(ii) ([8], [14], [17]) for any matrix \(C\) of full column rank with \(R(C) = N(B)\) and any matrix \(D\) of full row rank with \(R(B) = N(D)\), the matrix

\[
A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}
\]

is nonsingular and

\[
A^{-1} = \begin{pmatrix} B^\sharp & C(DC)^{-1} \\ (DC)^{-1}D & O \end{pmatrix}.
\]

**Proof.** (i): Consider the rank factorization of \(I - BB^\sharp\) given by

\[
I - BB^\sharp = (C \ U) \begin{pmatrix} D \\ V \end{pmatrix}.
\]
Note that $BB^t = B^2B$, and we have
\[
I - B^2B = \begin{pmatrix} C & U \end{pmatrix} \begin{pmatrix} D \\ V \end{pmatrix}.
\]

Obviously $R(C) \subseteq N(A)$ and $R(B) \subseteq N(D)$. As in the proof of Theorem 4.4(i), we conclude that \( \begin{pmatrix} B^2 & C \\ D & O \end{pmatrix} \) is a $g$-inverse of \( \begin{pmatrix} B & C \\ D & O \end{pmatrix} \) with $\psi(A) = \psi(B)$, since $X = -DB^tC = O$.

(ii): By Corollary 4.6, the nonsingularity of the matrix \( \begin{pmatrix} B^2 & C \\ D & O \end{pmatrix} \) under the conditions $R(C) = N(A)$ and $R(B) = N(D)$ can be easily seen. We now prove that for any matrix $C$ of full column rank with $R(C) = N(B)$ and any matrix $D$ of full row rank with $R(B) = N(D)$, $DC$ is nonsingular.

In fact, if $DCx = O$, then $Cx \in R(C)$ and $Cx \in N(D)$. Since $R(C) = N(B)$, $N(D) = R(B)$ and $R(B) \cap N(B) = \{0\}$, we have $Cx = O$ and therefore $x = 0$. Thus $DC$ is nonsingular.

By Lemma 4.3, $C(DC)^{-1}D$ is an idempotent matrix and
\[
I - BB^t = I - B^2B = C(DC)^{-1}D
\]
is a rank factorization. From Corollary 4.6, we know that
\[
\begin{pmatrix} B & C(DC)^{-1} \\ D & O \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & C \\ (DC)^{-1}D & O \end{pmatrix}
\]
are nonsingular and in fact
\[
\begin{pmatrix} B & C(DC)^{-1} \\ D & O \end{pmatrix}^{-1} = \begin{pmatrix} B^2 & C(DC)^{-1} \\ D & O \end{pmatrix}.
\]

Note that
\[
\begin{pmatrix} B & C(DC)^{-1} \\ D & O \end{pmatrix} = \begin{pmatrix} B & C \\ D & O \end{pmatrix} \begin{pmatrix} I & O \\ O & (DC)^{-1} \end{pmatrix}.
\]

The result follows immediately from the two equations preceding the one above. \(\square\)

Remark 5.4. Theorem 5.3(ii) can be used to compute the group inverse of the matrix $(I - T)^2$ which plays an important role in the theory of Markov chains, where $T$ is the transition matrix of a finite Markov chain. For an $n$-state ergodic chain, it is well-known that the transition matrix $T$ is irreducible and that $\text{rank}(I - T) = n - 1$ [6, Theorem 8.2.1]. Hence by Theorem 5.3(ii) we can compute the group inverse $(I - T)^2$ of $I - T$ by a bordered matrix.

Let $c$ be a right eigenvector of $T$ and $d$ a left eigenvector, that is $c$ and $d$ satisfy $Tc = c$ and $d^*T = d^*$, respectively. Then the bordered matrix $\begin{pmatrix} I - T & c \\ d^* & 0 \end{pmatrix}$ is nonsingular and
\[
\begin{pmatrix} I - T & c \\ d^* & 0 \end{pmatrix}^{-1} = \begin{pmatrix} (I - T)^2 & d^* \\ \frac{d^*c}{d^*} & 0 \end{pmatrix}.
\]
Thus the group inverse \((I - T)\dagger\) can be obtained by computing the inverse of a nonsingular matrix.

REFERENCES

[7] Chen Yonglin and Zhou Bingjun. On \(g\)-inverses and nonsingularity of a bordered matrix \[
\begin{pmatrix}
A & B \\
C & O
\end{pmatrix}