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Ji Young Choi
Luz Maria DeAlba
luz.dealba@drake.edu
Leslie Hogben
Benard M. Kivunge
Sandra K. Nordstrom

See next page for additional authors

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THE NONNEGATIVE $P_0$-MATRIX COMPLETION PROBLEM

JI YOUNG CHOI†, LUZ MARIA DEALBA‡, LESLIE HOGBEN†, BENARD M. KIVUNGE†, SANDRA K. NORDSTROM†, AND MIKE SHEDENHELM†

Abstract. In this paper the nonnegative $P_0$-matrix completion problem is considered. It is shown that a pattern for $4 \times 4$ matrices that includes all diagonal positions has nonnegative $P_0$-completion if and only if its digraph is complete when it has a 4-cycle. It is also shown that any positionally symmetric pattern that includes all diagonal positions and whose graph is an $n$-cycle has nonnegative $P_0$-completion if and only if $n \neq 4$.

Key words. Matrix completion, $P_0$-matrix, Nonnegative $P_0$-matrix, L-digraph, n-cycle.

AMS subject classifications. 15A48.

1. Introduction. A partial matrix is a rectangular array of numbers in which some entries are specified while others are free to be chosen. A completion of a partial matrix is a specific choice of values for the unspecified entries. Let $\mathcal{N} = \{1, \ldots, n\}$. A pattern for $n \times n$ matrices is a subset of $\mathcal{N} \times \mathcal{N}$. A partial matrix specifies a pattern if its specified entries lie exactly in those positions listed in the pattern. For a particular class $\Pi$ of matrices, we say a pattern has $\Pi$-completion if every partial $\Pi$-matrix specifying the pattern can be completed to a $\Pi$-matrix. The $\Pi$-matrix completion problem for patterns is to determine which patterns have $\Pi$-completion. For example, the positive definite completion problem asks, “Which patterns have the property that any partial positive definite matrix specifying the pattern can be completed to a positive definite matrix?” The answer to this question is given in [4] through the use of graph theoretic methods. Matrix completion problems arise in applications whenever a full set of data is not available, but it is known that the full matrix of data must have certain properties. Such applications include seismic reconstruction problems and data transmission, coding, and image enhancement problems in electrical and computer engineering.

A positionally symmetric pattern is a pattern with the property that $(i, j)$ is in the pattern if and only if $(j, i)$ is also in the pattern. An asymmetric pattern is a pattern with the property that if $(i, j)$ is in the pattern, then $(j, i)$ is not in the pattern.

For $\alpha$ a subset of $\mathcal{N}$, the principal submatrix obtained from $A$ by deleting all rows and columns not in $\alpha$ is denoted by $A(\alpha)$. The principal subpattern $Q(\alpha)$ is obtained from the pattern $Q$ by deleting all positions whose first or second coordinate is not in $\alpha$. A principal minor is the determinant of a principal submatrix. A real $n \times n$ matrix is called a $P_0$-matrix if all of its principal minors are nonnegative. A $P_0$-
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A matrix is called a \( P_{0,1} \)-matrix if all diagonal entries are nonzero. A partial \( P_0 \)-matrix (\( P_{0,1} \)-matrix) is a partial matrix in which all fully specified principal submatrices are \( P_0 \)-matrices (\( P_{0,1} \)-matrices). A nonnegative \( P_0 \)-matrix (nonnegative \( P_{0,1} \)-matrix) is a \( P_0 \)-matrix (\( P_{0,1} \)-matrix) whose entries are nonnegative. A partial nonnegative \( P_0 \)-matrix (partial nonnegative \( P_{0,1} \)-matrix) is a partial \( P_0 \)-matrix (\( P_{0,1} \)-matrix) whose specified entries are nonnegative.

The following properties of nonnegative \( P_0 \)-matrices will be used: A nonnegative block triangular matrix, all of whose diagonal blocks are nonnegative \( P_0 \)-matrices, is a nonnegative \( P_0 \)-matrix. If \( A \) is a nonnegative \( P_0 \)-matrix and \( D \) is a positive diagonal matrix, then \( DA \) and \( D^{-1}AD \) are both nonnegative \( P_0 \)-matrices. If \( A \) is a nonnegative \( P_0 \)-matrix and \( P \) is a permutation matrix, then \( P^{-1}AP \) is a nonnegative \( P_0 \)-matrix.

In many situations we need to permute the entries of a partial matrix. We do this by defining a permutation of a pattern: If \( Q \) is a pattern for \( n \times n \) matrices and \( \pi \) is a permutation of \( N \), then \( \pi(Q) = \{(\pi(i), \pi(j)) : (i, j) \in Q\} \). For a partial matrix \( A \) specifying \( Q \), define the partial matrix \( \pi(A) \) specifying \( \pi(Q) \) by \( \pi(A)_{\pi(i)\pi(j)} = A_{ij} \) for \((i, j) \in Q \). Note that for a fully specified matrix \( A \), \( \pi(A) = P_{\pi}^{-1}AP_{\pi} \) with \( P_{\pi} = [c_{\pi(1)}, \ldots, c_{\pi(n)}]^T \). In completion problems we permute the entries of a given partial matrix \( A \) to obtain \( \pi(A) \), complete \( \pi(A) \) to \( \hat{\pi}(A) \), and use \( \pi^{-1}(\hat{\pi}(A)) \) to complete \( A \).

Throughout the paper we will denote the entries of a partial matrix as follows: \( d_i \) denotes a specified diagonal entry, \( a_{ij} \) a specified off-diagonal entry, and \( x_{ij} \) an unspecified entry, \( 1 \leq i, j \leq n \). In addition, \( c_{ij} \) may be used to denote the value assigned to the unspecified entry \( x_{ij} \) during the process of completing a partial matrix.

The next result is known [6]. We include the proof here, because the explicit values chosen in this proof will be used in subsequent proofs.

**Lemma 1.1.** A pattern for \( 3 \times 3 \) matrices that includes all diagonal positions has nonnegative \( P_0 \)-completion.

**Proof.** Let

\[
A = \begin{bmatrix}
d_1 & a_{12} & x_{13} \\
0 & d_2 & a_{23} \\
0 & 0 & d_3
\end{bmatrix}
\]

be a partial nonnegative \( P_0 \)-matrix and let \( Q \) be the pattern \( A \) specifies. We will look at two cases: 1) If \( a_{12}a_{23} = 0, a_{23}a_{31} = 0 \) or \( a_{12}a_{23}a_{31} \geq d_1d_2d_3 \), set \( x_{13} = 0 \) to complete \( A \). 2) If \( a_{12} > 0, a_{21} > 0, a_{23} > 0, a_{32} > 0, \) and \( a_{12}a_{23}a_{31} < d_1d_2d_3 \), set \( x_{13} = \frac{a_{12}a_{23}}{d_3} \) to complete \( A \). In both cases, this completes \( A \) to a nonnegative \( P_0 \)-matrix. Thus, the pattern \( Q \) that \( A \) specifies has nonnegative \( P_0 \)-completion.

This implies that any pattern \( R \) for a \( 3 \times 3 \) matrix with one unspecified off-diagonal entry has nonnegative \( P_0 \)-completion, since any partial matrix specifying \( R \) can be transformed via a permutation matrix in order to specify the above pattern \( Q \). Also, any matrix with more than one unspecified off-diagonal entry can be completed by first setting all except one unspecified off-diagonal entry to zero and then completing the resulting matrix as shown above.

A digraph \( G = (V_G, E_G) \) is a finite set \( V_G \) of positive integers, whose members are called vertices, and a set \( E_G \) of ordered pairs \((v, u)\) of distinct vertices, called arcs.
In the arc \((v, u)\), \(v\) is the tail and \(u\) is the head. The indegree \(d^-(u)\) of vertex \(u\) is the number of arcs with head \(u\). The outdegree \(d^+(v)\) of vertex \(v\) is the number of arcs with tail \(v\). The order of \(G\) is the number of vertices of \(G\). A subdigraph of the digraph \(G = (V_G, E_G)\) is a digraph \(H = (V_H, E_H)\), where \(V_H\) is a subset of \(V_G\) and \(E_H\) is a subset of \(E_G\) (note that \((v, u) \in E_H\) requires \(v, u \in V_H\), since \(H\) is a digraph).

If \(W\) is a subset of \(V_G\), the subdigraph induced by \(W\), \(<W>\), is the digraph \((W, E_W)\) with \(E_W\) the set of all arcs of \(G\) between the vertices in \(W\). A subdigraph induced by a subset of vertices is also called an induced subdigraph.

For a pattern \(Q\) for \(n \times n\) matrices that contains all diagonal positions, the digraph of \(Q\) (pattern-digraph) is the digraph having vertex set \(\mathcal{N}\) and, as arcs, the ordered pairs \((i, j) \in Q\) where \(i \neq j\). A partial matrix that specifies a pattern is also referred to as specifying the digraph of the pattern. Note that the use of a permutation on a pattern or partial matrix corresponds to renumbering the vertices of the pattern-digraph that the matrix specifies. Since nonnegative \(P_0\)-matrices are closed under permutation similarity, we are free to renumber digraph vertices as convenient.

When all diagonal entries in a matrix are nonzero or all diagonal positions are present in a pattern, digraphs can be used to study matrices (nonzero digraphs) and patterns (pattern-digraphs). When diagonal positions are omitted or diagonal entries of a matrix can be zero, it is sometimes necessary to use mardigraphs (cf. [6]) or digraphs that include loops (L-digraphs, defined below, cf. [8, Definition 6.2.11] and [1, p. 53]). In this paper we study only patterns that include all diagonal positions, so we use pattern-digraphs, but we will use L-digraphs to study matrices, since it is necessary to distinguish between zero and nonzero diagonal entries, both of which occur. Note that the term “digraph" is sometimes used to describe what is here called an L-digraph. We use our terminology because we need to distinguish L-digraphs from what we call digraphs.

An L-digraph is a digraph that is allowed to have loops, i.e., arcs \((v, v)\). The terms indegree, outdegree, order, sub-L-digraph, and induced sub-L-digraph are defined analogously to the corresponding terms for digraphs. For an L-digraph \(G\), let \(\text{Sub}(G)\) denote the set of all sub-L-digraphs of \(G\). The L-digraph \(G = (V_G, E_G)\) is isomorphic to the L-digraph \(H = (V_H, E_H)\) if there is a one-to-one map \(\phi\) from \(V_G\) onto \(V_H\) and \((v, v) \in E_G\) if and only if \((\phi(v), \phi(v)) \in E_H\).

Let \(A\) be a (fully specified) \(n \times n\) matrix. The nonzero-L-digraph of \(A\) is the L-digraph having vertex set \(\mathcal{N}\) and, as arcs, the ordered pairs \((i, j)\) where \(a_{ij} \neq 0\). If \(G\) is the nonzero-L-digraph of \(A\), then the nonzero-L-digraph of the principal submatrix \(A(\alpha)\) is isomorphic to \(<\alpha>\). We may abuse the notation and refer to \(<\alpha>\) as the nonzero-L-digraph of \(A(\alpha)\).

We use the term (L-)digraph to mean digraph or L-digraph. A path (respectively, semipath) in the (L-)digraph \(G = (V_G, E_G)\) is a sequence of vertices \(v_1, v_2, \ldots, v_{k-1}, v_k\) in \(V_G\) such that for \(i = 1, \ldots, k - 1\), the arc \((v_i, v_{i+1}) \in E_G\) (respectively, \((v_i, v_{i+1}) \in E_G\) or \((v_{i+1}, v_i) \in E_G\)) and all vertices are distinct except possibly \(v_1 = v_k\). Clearly, a path is a semipath, although the converse is false. The length of the (semipath) \(v_1, v_2, \ldots, v_{k-1}, v_k\) is \(k - 1\). A cycle is a path in which \(v_1 = v_k\). A cycle is even or odd according as its length is even or odd. A digraph whose vertex set consists of the \(n\) vertices \(v_1, \ldots, v_n\), and whose arc set consists of exactly the arcs in the two cycles
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$v_1, v_2, \ldots, v_n, v_1$ and $v_n, v_{n-1}, \ldots, v_1, v_n$ is a symmetric $n$-cycle. Two distinct vertices $v$ and $w$ are connected if there is a semipath $v = v_1, v_2, \ldots, v_k = w$. Any vertex $v$ is connected to itself, whether or not the loop $(v, v)$ is in $E_G$. The relationship of being connected is an equivalence relation on vertices of $G$. A sub-$(L)$-digraph induced by an equivalence class defined by this relation is called a component of $G$. The $(L)$-digraph $G$ is connected if it has only one component, i.e., if any two vertices of $G$ are connected. An $(L)$-digraph is strongly connected if for any two vertices $v$ and $w$ there is a path from $v$ to $w$.

An $(L)$-digraph that contains all possible arcs between its vertices is called complete. A complete sub-$(L)$-digraph is called a clique. A cut-vertex is a vertex that if removed along with all incidental arcs, causes a component of the $(L)$-digraph to be separated into more than one component. If a connected $(L)$-digraph does not contain any cut-vertices, the $(L)$-digraph is nonseparable. A block is a maximal nonseparable sub-$(L)$-digraph, and a $(L)$-digraph where all blocks are cliques is called block-clique.

Let $S_n$ denote the group of permutations of $N$. Let $G_{\pi}$ denote the $L$-digraph of the permutation matrix $P_{\pi}$ for some $\pi \in S_n$; $G_{\pi}$ is called a permutation $L$-digraph (cf. [1, p. 291]).

The following two lemmas are obvious.

**Lemma 1.2.** Let $A$ be an $n \times n$ matrix and let $G$ be its nonzero-$L$-digraph. Then

$$\text{Det} A = \sum_{\pi \in S_n, G_{\pi} \in \text{Sub}(G)} (\text{sgn} \, \pi) a_{1\pi(1)} \cdots a_{n\pi(n)},$$

where the sum over the empty set is zero.

**Lemma 1.3.** An $L$-digraph $H$ is a permutation $L$-digraph if and only if $V_H = N$ and each component of $H$ is a single cycle.

**Corollary 1.4.** A nonnegative matrix whose nonzero $L$-digraph contains no even cycles is a $P_0$-matrix.

**Proof.** For every $\pi \in S_n$ such that $G_{\pi}$ is a sub-$L$-digraph of $G$, $G_{\pi}$ is composed of disjoint cycles, which by hypothesis must all be of odd length. Thus $\pi$ is the product of odd cycles and so $\text{sgn} \, \pi = 1$. Thus $\text{Det} A$ is the sum of positive terms, or zero if there are no permutation $L$-digraphs in $G$. Since any cycle in an induced sub-$L$-digraph of $G$ is also a cycle of $G$, every induced sub-$L$-digraph $< \alpha >$ inherits the property of containing no even cycles, and hence $\text{Det} A(\alpha)$ is nonnegative. \[\square\]

References [3, 6, 7] contain some results on nonnegative $P$- and nonnegative $P_0$-matrix completion problems. Of particular interest is Lemma 3.5 in [3]: Any partial positive $P$-matrix, the graph of whose specified entries is an $n$-cycle can be completed to a positive $P$-matrix. (For this result all diagonal entries are assumed specified, and “$n$-cycle” means what we call here a “symmetric $n$-cycle” because all patterns discussed in [3] are positionally symmetric.) In [6, Theorem 8.4], it is noted that the same method of proof applies to partial nonnegative $P_{0,1}$-matrices. In contrast, it is shown in [2] that the analogous statement for $P_0$-matrices is true if and only if $n \neq 4$. That is, a pattern for $n \times n$ matrices that includes all diagonal positions whose pattern-digraph is a symmetric $n$-cycle has $P_0$-completion if and only if $n \neq 4$. Note that the cases $n = 2$ and 3 are trivial, since the pattern includes all positions. The interesting
cases are $n = 4$ (no completion) and $n \geq 5$ (completion). In Section 3 it is shown that the same situation holds for the nonnegative $P_0$-matrix completion problem, that is, a pattern for $n \times n$ matrices that includes all diagonal positions whose pattern digraph is a symmetric $n$-cycle has nonnegative $P_0$-completion if and only if $n \neq 4$. Section 2 contains a classification of patterns for $4 \times 4$ matrices that include all diagonal positions as either having nonnegative $P_0$-completion or not having nonnegative $P_0$-completion, as [2] provided the analogous classification for $P_0$-completion. However, this classification has a more elegant description (cf. Theorem 2.6). In [2], it was established that asymmetric patterns have $P_0$-completion. This is not the case for nonnegative $P_0$-completion, as Lemma 2.1 makes clear.

2. Classification of Patterns for $4 \times 4$ Matrices. In this section we will classify all patterns for $4 \times 4$ matrices as either having nonnegative $P_0$-completion, or not. The patterns discussed here are assumed to include all diagonal positions. The digraphs are numbered following [5]; $q$ is the number of edges, $n$ is the diagram number. The classification is broken up into a series of lemmas.

Lemma 2.1. If the digraph of a pattern $Q$ contains a 4-cycle $\Gamma$ and the subdigraph induced by $\Gamma$ is not a clique, then $Q$ does not have nonnegative $P_0$-completion.

Proof. Without loss of generality, assume that $\Gamma$ is 1, 2, 3, 4, 1. Let $A$ be a partial nonnegative $P_0$-matrix specifying $Q$, with $a_{12} = 1, a_{23} = 1, a_{34} = 1, a_{41} = 1$, and all other specified entries equal to zero. Then every fully specified principal submatrix of $A$ is triangular, so $A$ is a partial nonnegative $P_0$-matrix. Suppose that $\tilde{A}$ is a nonnegative $P_0$-matrix that completes $A$, and let $\tilde{a}_{ij}$ denote the $ij$-entry of $\tilde{A}$, whether specified in $A$ or chosen for $\tilde{A}$. Since all of the diagonal entries of $\tilde{A}$ are zero, $\text{Det} \tilde{A}(\{i, j\}) = -\tilde{a}_{ij}\tilde{a}_{ji}$. But $\tilde{a}_{ij} \geq 0, \tilde{a}_{ji} \geq 0$, and $\text{Det} \tilde{A}(\{i, j\}) \geq 0$. Therefore, $\tilde{a}_{21} = 0, \tilde{a}_{32} = 0, \tilde{a}_{43} = 0$, and $\tilde{a}_{14} = 0$. In addition, $\tilde{a}_{13}\tilde{a}_{31} = 0$. Then $\text{Det} \tilde{A} = -1 + \tilde{a}_{13}\tilde{a}_{31}\tilde{a}_{24}\tilde{a}_{42} = -1$, a contradiction. Thus, $A$ cannot be completed to a nonnegative $P_0$-matrix. Therefore, $\langle \Gamma \rangle$ does not have nonnegative $P_0$-completion, so by [6, Lemma 3.1], neither does the pattern $Q$. \[\]

Lemma 2.2. The patterns for the digraphs listed below have nonnegative $P_0$-completion.

$q = 0$
$q = 1$
$q = 2$  $n = 1 - 5$
$q = 3$  $n = 1 - 13$
$q = 4$  $n = 1 - 15, 17 - 27$
$q = 5$  $n = 1 - 6, 8 - 31, 33, 34, 36 - 38$
$q = 6$  $n = 1 - 3, 5, 6, 8 - 21, 23 - 27, 29, 32, 35, 36, 38 - 41, 43, 44, 46 - 48$
$q = 7$  $n = 1, 3 - 6, 9, 11, 14, 16, 19, 22, 24, 26, 28, 29, 31, 34, 36, 37$
$q = 8$  $n = 1, 10, 12, 18, 21, 27$
$q = 9$  $n = 8, 11$
$q = 12$

Proof. For each of these digraphs, the pattern of every nonseparable strongly connected induced subdigraph has nonnegative $P_0$-completion, thus by [6, Theorem
5.8] the patterns of these digraphs have nonnegative $P_0$-completion. 

**Lemma 2.3.** The patterns for the digraphs listed below have nonnegative $P_0$-completion.

$q = 5$    $n = 35$
$q = 6$    $n = 28, 30, 31$
$q = 7$    $n = 7$

**Proof.** Let $A$ be a partial nonnegative $P_0$-matrix specifying any of the patterns of the above digraphs. Then set all unspecified entries of the partial matrix $A$ to zero. It is straightforward to verify by computation that this completion results in a nonnegative $P_0$-matrix.

**Lemma 2.4.** The patterns for the digraphs $q = 8, n = 14$; $q = 8, n = 15$; $q = 7, n = 15$; $q = 7, n = 17$; $q = 7, n = 21$; and $q = 7, n = 23$ all have nonnegative $P_0$-completion.

**Proof.** Let

$$A = \begin{bmatrix} d_1 & a_{12} & x_{13} & a_{14} \\ a_{21} & d_2 & a_{23} & a_{24} \\ x_{31} & a_{32} & d_3 & a_{34} \\ x_{41} & a_{42} & x_{43} & d_4 \end{bmatrix}$$

be a partial nonnegative $P_0$-matrix specifying the pattern of the digraph $q = 8, n = 14$ with the vertices labeled as in Figure 2.1.

We will consider two cases: 1) $a_{12}a_{21} = 0$ or $a_{23}a_{32} = 0$ or $a_{24}a_{42} = 0$, and 2) $a_{12}, a_{21}, a_{23}, a_{32}, a_{24},$ and $a_{42}$ are all nonzero. Notice that the submatrices $A(\{1, 2\})$, $A(\{2, 3\})$, and $A(\{2, 4\})$ are fully specified, and thus their determinants are given to be nonnegative in all cases.

**Case 1:** $a_{12}a_{21} = 0$ or $a_{23}a_{32} = 0$ or $a_{24}a_{42} = 0$. We prove the case $a_{12}a_{21} = 0$; the other two cases are similar. Consider the two subcases: i) $a_{12} = 0$ and ii) $a_{21} = 0$. For both subcases set $x_{31} = 0$, $x_{41} = 0$, and $x_{13} = 0$. Next, complete the submatrix $A(\{2, 3, 4\})$ to a nonnegative $P_0$-matrix by using Lemma 1.1. This will determine $x_{43}$, and we will say $x_{43} = c_{43}$. Thus $A$ is completed to $\hat{A}$. 

**Fig. 2.1.** Digraphs having $P_0$-completion.
For subcase i), the principal minors of $\hat{A}$ are shown in Table 2.1.

The first three principal minors are known to be nonnegative because $A$ is a partial nonnegative $P_0$-matrix. $\text{Det}(A(3, 4))$ and $\text{Det}(A(2, 3, 4))$ are nonnegative due to the selection of $c_{43}$. $\text{Det}(\hat{A}(2, 3)) = d_1 \cdot \text{Det}(\hat{A}(2, 3)) \geq 0$, $\text{Det}(\hat{A}(2, 3, 4)) = a_{14}a_{21}a_{42} + d_1 \cdot \text{Det}(\hat{A}(2, 4)) \geq 0$, and $\text{Det}(\hat{A}(1, 3, 4)) = d_1 \cdot \text{Det}(\hat{A}(3, 4)) \geq 0$.

Recall that the completion of $A(2, 3, 4)$ sets $c_{43}$ to be either 0 or $\frac{a_{21}a_{42}}{d_2}$.

If $c_{43} = 0$, then $a_{24}a_{42} = 0$, $a_{23}a_{42} = 0$, or $a_{23}a_{43}a_{42} \geq d_2 d_3 d_4$, and $\text{Det}(\hat{A}) = d_1 d_2 d_3 d_4 + a_{23}a_{43}a_{42} d_1 + a_{14}a_{21}a_{42} d_3 - a_{24}a_{42} d_3 - a_{23}a_{42} d_4$. If $a_{24}a_{42} = 0$, then $\text{Det}(\hat{A}) = a_{23}a_{43}a_{42} d_1 + a_{14}a_{21}a_{42} d_3 + d_1 d_4 \cdot \text{Det}(\hat{A}(2, 3)) \geq 0$. If $a_{23}a_{43}a_{42} \geq d_2 d_3 d_4$, then $\text{Det}(\hat{A}) \geq d_1 d_2 d_3 d_4 + d_1 d_2 d_3 d_4 + a_{14}a_{21}a_{42} d_3 - a_{24}a_{42} d_3 - a_{23}a_{42} d_4 = a_{14}a_{21}a_{42} d_3 + d_1 d_4 \cdot \text{Det}(\hat{A}(2, 4)) + d_1 d_4 \cdot \text{Det}(\hat{A}(2, 3)) \geq 0$.

If $c_{43} = \frac{a_{21}a_{42}}{d_2}$, then

$$\text{Det}(\hat{A}) = \left( \frac{a_{14}a_{21}a_{42}a_{23}d_3}{d_2} - \frac{a_{21}a_{42}a_{23}a_{42}d_1}{d_2} \right) + \frac{a_{21}a_{42}a_{23}a_{42}d_1}{d_2} - \frac{a_{21}a_{42}a_{23}a_{42}d_1}{d_2} \geq 0.$$
of $A$ by a positive diagonal matrix, we may assume without loss of generality that $d_1 = d_2 = d_3 = d_4 = 1$. By use of a diagonal similarity we may also assume without loss of generality that $a_{21} = a_{32} = a_{42} = 1$. Thus,

$$A = \begin{bmatrix} 1 & a_{12} & x_{13} & a_{14} \\ 1 & 1 & a_{23} & a_{24} \\ x_{31} & 1 & 1 & a_{34} \\ x_{41} & 1 & x_{43} & 1 \end{bmatrix}.$$ 

The submatrices in Table 2.2 are fully specified and thus their determinants are nonnegative. We will look at three subcases:

i) $a_{14} \geq 1$ and $a_{34} \geq 1$,   
ii) $a_{14} < 1$, and 
iii) $a_{34} < 1$.

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<thead>
<tr>
<th>Principal Submatrix</th>
<th>Determinant</th>
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<tr>
<td>$A((1, 2))$</td>
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</tr>
<tr>
<td>$A((2, 3))$</td>
<td>$1 - a_{23}$</td>
</tr>
<tr>
<td>$A((2, 4))$</td>
<td>$1 - a_{24}$</td>
</tr>
</tbody>
</table>

For subcase i), set $x_{31} = 0$, $x_{41} = 0$, $x_{43} = 0$ and $x_{13} = a_{23}$. The principal minors of $\hat{A}$ are shown in Table 2.3.

<table>
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<tr>
<td>$A((1, 4))$</td>
<td>1</td>
</tr>
<tr>
<td>$A((3, 4))$</td>
<td>1</td>
</tr>
<tr>
<td>$A((1, 2, 3))$</td>
<td>$1 - a_{12}$</td>
</tr>
<tr>
<td>$A((1, 2, 4))$</td>
<td>$1 - a_{12} + a_{14} - a_{24}$</td>
</tr>
<tr>
<td>$A((1, 3, 4))$</td>
<td>1</td>
</tr>
<tr>
<td>$A((2, 3, 4))$</td>
<td>$1 - a_{23} - a_{24} + a_{23}a_{34}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$1 - a_{12} + a_{14} - a_{24}$</td>
</tr>
</tbody>
</table>

$\text{Det}\hat{A}((1, 2, 3)) = 1 - a_{12} = \text{Det}\hat{A}((1, 2)) \geq 0$. Since $a_{14} \geq 1$, $\text{Det}\hat{A}((1, 2, 4)) \geq 1 - a_{12} + 1 - a_{24} = \text{Det}\hat{A}((1, 2)) + \text{Det}\hat{A}((2, 4)) \geq 0$. $\text{Det}\hat{A}((2, 3, 4)) = \text{Det}A((2, 4)) + a_{23}(a_{34} - 1) \geq 0$, since $a_{34} \geq 1$. And

$$\text{Det}A = \text{Det}\hat{A}((1, 2, 4)) \geq 0.$$ 

For subcase ii), specify the unspecified entries in the following order. First, set $x_{41} = 1$ and $x_{31} = 1$. Then, complete the submatrix $A((2, 3, 4))$ to a nonnegative $P_0$-matrix.
Determinant

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by using Lemma 1.1. This will determine \( x_{43} \), which we will call \( c_{43} \). Next, complete \( A(\{1,3,4\}) \) to a nonnegative \( P_0 \)-matrix by Lemma 1.1, and say \( x_{13} = c_{13} \). The resulting matrix is

\[
\hat{A} = \begin{bmatrix}
1 & a_{12} & c_{13} & a_{14} \\
1 & 1 & a_{23} & a_{24} \\
1 & 1 & 1 & a_{34} \\
1 & 1 & c_{43} & 1 \\
\end{bmatrix}.
\]

The principal minors of \( \hat{A} \) are shown in Table 2.4.

<table>
<thead>
<tr>
<th>Principal Submatrix</th>
<th>Determinant</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A({3,4}) )</td>
<td>( 1 - a_{34} c_{43} )</td>
</tr>
<tr>
<td>( A({2,3,4}) )</td>
<td>( 1 - a_{23} - a_{24} + a_{23} a_{34} + a_{24} c_{43} - a_{34} c_{43} )</td>
</tr>
<tr>
<td>( A({1,3}) )</td>
<td>( 1 - c_{13} )</td>
</tr>
<tr>
<td>( A({1,3,4}) )</td>
<td>( 1 - a_{14} - c_{13} + a_{34} c_{13} + a_{14} c_{43} - a_{34} c_{43} )</td>
</tr>
<tr>
<td>( A({1,4}) )</td>
<td>( 1 - a_{14} )</td>
</tr>
<tr>
<td>( A({1,2,3}) )</td>
<td>( 1 - a_{12} + a_{34} a_{23} )</td>
</tr>
<tr>
<td>( A({1,2,4}) )</td>
<td>( 1 - a_{12} - a_{23} + a_{12} a_{23} )</td>
</tr>
<tr>
<td>( A )</td>
<td>( 1 - a_{12} - a_{23} + a_{12} a_{23} - a_{24} + a_{12} a_{42} + a_{23} a_{34} - a_{12} a_{23} a_{34} + a_{24} c_{43} - ) ( a_{12} a_{24} c_{43} - a_{34} c_{43} + a_{12} a_{34} c_{43} )</td>
</tr>
</tbody>
</table>

By the choice of \( c_{43} \), \( \det \hat{A}(\{3,4\}) \) and \( \det \hat{A}(\{2,3,4\}) \) are nonnegative. And by the choice of \( c_{13} \), \( \det \hat{A}(\{1,3\}) \) and \( \det \hat{A}(\{1,3,4\}) \) are nonnegative. Since \( a_{14} < 1 \), \( \det \hat{A}(\{1,4\}) \geq 0 \). Also, \( \det \hat{A}(\{1,2,3\}) = \det \hat{A}(\{1,2\}) \cdot \det \hat{A}(\{2,3\}) \geq 0 \) and \( \det \hat{A}(\{1,2,4\}) = \det \hat{A}(\{1,2\}) \cdot \det \hat{A}(\{2,4\}) \geq 0 \). Lastly,

\[
\det \hat{A} = \det \hat{A}(\{1,2\}) \cdot \det \hat{A}(\{2,3,4\}) \geq 0.
\]

Subcase iii) is similar to subcase ii). Therefore, the pattern for \( q = 8 \), \( n = 14 \) has nonnegative \( P_0 \)-completion.

A partial nonnegative \( P_0 \)-matrix \( B \) specifying the pattern of the digraph \( q = 7 \), \( n = 15 \) or \( q = 7 \), \( n = 21 \) with the vertices labelled as in Figure 2.1 can be extended to a partial nonnegative \( P_0 \)-matrix specifying \( q = 8 \), \( n = 14 \) by setting the unspecified entry \( x_{12} \) or \( x_{24} \), respectively, of \( B \) to zero. Then the resulting matrix can be completed to a nonnegative \( P_0 \)-matrix as above. Also, notice that a matrix specifying the pattern of \( q = 8 \), \( n = 15 \) is the transpose of a matrix specifying \( q = 8 \), \( n = 14 \). Therefore, any partial nonnegative \( P_0 \)-matrix specifying \( q = 8 \), \( n = 15 \) can be completed to a nonnegative \( P_0 \)-matrix by taking its transpose, completing it as above, and taking its transpose again. In addition, a partial nonnegative \( P_0 \)-matrix \( C \) specifying \( q = 7 \), \( n = 17 \) or \( q = 7 \), \( n = 23 \) with the vertices labeled as in Figure 2.1 can be
extended to a partial nonnegative $P_0$-matrix specifying $q = 8$, $n = 15$ by setting the unspecified entry $x_{42}$ or $x_{21}$, respectively, of $C$ to zero. Then $C$ can be completed to a nonnegative $P_0$-matrix as above. So the patterns for $q = 7$, $n = 15$; $q = 7$, $n = 21$; $q = 8$, $n = 15$; $q = 7$, $n = 17$; and $q = 7$, $n = 23$ have nonnegative $P_0$-completion.

**Lemma 2.5.** The pattern for the digraph $q = 8$, $n = 6$ (Figure 2.2) has nonnegative $P_0$-completion.

**Theorem 2.6.** A pattern for $4 \times 4$ matrices, that includes all diagonal positions, has nonnegative $P_0$-completion if and only if its digraph is complete when it has a 4-cycle, that is, if and only if its digraph is not one of the following.

$$
\begin{align*}
q &= 4 & n &= 16 \\
q &= 5 & n &= 7, 32 \\
q &= 6 & n &= 4, 7, 22, 33, 34, 37, 42, 45 \\
q &= 7 & n &= 2, 8, 10, 12, 13, 18, 20, 25, 27, 30, 32, 33, 35, 38 \\
q &= 8 & n &= 2 - 5, 7 - 9, 11, 13, 16, 17, 19, 20, 22 - 26 \\
q &= 9 & n &= 1 - 7, 9, 10, 12, 13 \\
q &= 10 & n &= 1 - 5 \\
q &= 11 & 
\end{align*}
$$

**Proof.** Each of the listed digraphs contains a 4-cycle whose induced subdigraph is not a clique. By Lemma 2.1, the patterns of these digraphs do not have nonnegative $P_0$-completion. Lemmas 2.2, 2.3, 2.4, and 2.5 demonstrate that the patterns for all remaining digraphs have nonnegative $P_0$-completion.

One question emerging from Theorem 2.6 is whether either the theorem or its obvious generalization is true for larger digraphs. That is, whether

1. a pattern has nonnegative $P_0$-completion if and only if its digraph has the property that the induced subdigraph of any 4-cycle is a clique, or
2. a pattern has nonnegative $P_0$-completion if and only if its digraph has the property that the induced subdigraph of any even cycle is a clique.

Neither of these statements is true. Example 2.7 contains a counterexample to item 1. That is, we give a partial nonnegative $P_0$-matrix that cannot be completed to a partial nonnegative $P_0$-matrix and that specifies a digraph that does not contain any 4-cycles. Furthermore, Theorem 3.2 in the next section shows that any partial nonnegative $P_0$-matrix that specifies a symmetric 6-cycle can be completed to a nonnegative $P_0$-matrix, and thus contradicts item 2.
Fig. 2.3. Digraph not containing any 4-cycles and not having \( P_0 \)-completion.

Example 2.7. The partial nonnegative \( P_0 \)-matrix

\[
A = \begin{bmatrix}
0 & 1 & 0.01 & x_{14} & 0.01 & x_{16} \\
x_{21} & 0 & 1 & 0.01 & x_{25} & 0.01 \\
x_{31} & x_{32} & 0 & 1 & x_{35} & x_{36} \\
x_{41} & x_{42} & x_{43} & 0 & 1 & x_{46} \\
x_{51} & x_{52} & 0.01 & x_{54} & 0 & 1 \\
1 & x_{62} & x_{63} & x_{64} & x_{65} & 0
\end{bmatrix}
\]

specifies the digraph \( D \), shown in Figure 2.3. Note that \( D \) does not contain any 4-cycles. But \( A \) cannot be completed to a nonnegative \( P_0 \)-matrix: Examination of the \( 2 \times 2 \) principal minors shows that if a completion \( \hat{A} \) of \( A \) is a nonnegative \( P_0 \)-matrix, then \( x_{21}, x_{32}, x_{43}, x_{54}, x_{65}, x_{16}, x_{31}, x_{51}, x_{42}, x_{62}, x_{35}, x_{41} \), and \( x_{35} \) must be zero. With these choices, \( \text{Det} \, \hat{A}(\{1, 3, 5, 6\}) = -0.0001x_{36} \) and \( \text{Det} \, \hat{A}(\{1, 2, 3, 4\}) = -x_{41} \), so \( x_{36} \) and \( x_{41} \) must be zero. Then, \( \text{Det} \, \hat{A}(\{1, 2, 4, 6\}) = -0.01x_{46} \) and \( \text{Det} \, \hat{A}(\{3, 4, 5, 6\}) = -x_{63} \), so \( x_{36} = x_{63} = 0 \). With these choices, \( \text{Det} \, \hat{A} = -0.99999 - 0.0001x_{52} \), so it is impossible for \( \hat{A} \) to be a nonnegative \( P_0 \)-matrix.

3. Symmetric \( n \)-cycle. If a positionally symmetric pattern has nonnegative \( P_0 \)-completion, then each principal subpattern associated with a component of the digraph either includes all diagonal positions or omits all diagonal positions [7]. Any pattern that omits all diagonal positions has nonnegative \( P_0 \)-completion [6, Theorem 4.7]. Thus, to determine which positionally symmetric patterns have nonnegative \( P_0 \)-completion, we need to discuss only patterns that include all diagonal positions.

In this section we prove that a pattern which includes all diagonal positions and whose digraph is a symmetric \( n \)-cycle has nonnegative \( P_0 \)-completion if and only if \( n \neq 4 \).

Lemma 3.1. Let \( G \) be the digraph of the symmetric 5-cycle 1, 2, 3, 4, 5, 1. Any partial nonnegative \( P_0 \)-matrix specifying \( G \) that has at least one diagonal entry equal to zero can be completed to a nonnegative \( P_0 \)-matrix.

Proof. Let \( A \) be a partial nonnegative \( P_0 \)-matrix specifying \( G \). By use of a permutation similarity, we may assume that \( d_1 \) is zero. By examination of \( \text{Det} \, A(\{1, 2\}) \) and \( \text{Det} \, A(\{1, 5\}) \), either \( a_{21} = 0 \) or \( a_{12} = 0 \), and either \( a_{51} = 0 \) or \( a_{15} = 0 \). There are now two cases.
Case 1: $a_{15} = a_{12} = 0$ or $a_{21} = a_{51} = 0$. The digraph of the pattern specified by $A\{2, 3, 4, 5\}$ is block-clique (see Figure 3.1 with $n = 5$), so $A\{2, 3, 4, 5\}$ can be completed to a nonnegative $P_0$-matrix [3, Theorem 4.1]. Set the remaining unspecified entries to zero. This completes $A$ to a block-triangular matrix whose diagonal blocks are nonnegative $P_0$-matrices.

Case 2: $a_{15} \neq 0$ and $a_{21} \neq 0$, or $a_{51} \neq 0$ and $a_{12} \neq 0$. Without loss of generality assume $a_{15} \neq 0$ and $a_{21} \neq 0$ and by use of a diagonal similarity, $a_{15} = a_{21} = 1$ (and necessarily $a_{51} = a_{12} = 0$). Then $A$ can be completed to a nonnegative $P_0$-matrix by setting

$$x_{24} = \begin{cases} \frac{d_4 a_{43} d_3}{a_{45} a_{32}} & \text{if } a_{43} a_{32} \neq 0 \\ 0 & \text{if } a_{43} a_{32} = 0 \end{cases} \quad \text{and} \quad x_{35} = \begin{cases} \frac{d_4 a_{43} d_3}{a_{54} a_{43}} & \text{if } a_{54} a_{43} \neq 0 \\ 0 & \text{if } a_{54} a_{43} = 0 \end{cases}$$

and all other unspecified entries equal to zero. The fact that this process yields a $P_0$-matrix can be verified by computing all the principal minors, most of which are clearly nonnegative. Observe

$$\text{Det}A\{2, 3, 4\} = d_2 d_3 d_4 + x_{24} a_{43} a_{32} - d_4 a_{23} a_{32} = 24 \cdot \text{Det}A\{2, 3\} \geq 0. \quad \text{If } a_{32} = 0,$$

$$\text{Det}A\{2, 3, 4\} = d_2 d_3 d_4 - d_4 a_{34} a_{43} a_{32} = d_2 \cdot \text{Det}A\{3, 4\} \geq 0. \quad \text{If } a_{43} a_{32} \neq 0,$$

$$\text{Det}A\{2, 3, 4\} = 2d_2 d_3 d_4 - d_4 a_{23} a_{32} - d_4 a_{34} a_{43} = \text{Det}A\{3, 4\} + \text{Det}A\{2, 3\} \geq 0. \quad \text{The computation of } \text{Det}A\{3, 4, 5\}.$$

If $a_{43} = 0$, we have $d_2 d_3 a_{34} a_{43} = 0$, $x_{24} = 0$, $x_{35} = 0$, and $\text{Det}A\{2, 3, 4, 5\} = \text{Det}A\{2, 3\} \cdot \text{Det}A\{4, 5\}$. If $a_{32} = 0$ and $a_{54} = 0$, then $d_4 d_5 a_{32} a_{32} = 0$ and $d_2 d_1 a_{45} a_{54} = 0$, $x_{24} = 0$, $x_{35} = 0$, and $\text{Det}A\{2, 3, 4, 5\} = d_2 d_5 \cdot \text{Det}A\{3, 4\}$. If $a_{43} \neq 0$ and $a_{32} \neq 0$, then $x_{24} = \frac{d_2 d_1 d_3}{a_{45} a_{54}}$ and $\text{Det}A\{2, 3, 4, 5\} = a_{23} a_{34} a_{43} a_{54} - a_{15} a_{45} d_2 d_3 - a_{23} a_{34} d_2 d_5 - a_{23} a_{32} d_4 d_5 + d_2 d_3 d_4 d_5 + a_{32} a_{43} d_2 x_{24} + a_{43} a_{54} d_2 x_{35} \geq 0$. The case $a_{43} \neq 0$ and $a_{54} \neq 0$ is similar to the case $a_{43} \neq 0$ and $a_{32} \neq 0$.

![Fig. 3.1. Block-clique](image)

**Theorem 3.2.** A pattern that includes all diagonal positions and whose digraph is a symmetric $n$-cycle has nonnegative $P_0$-completion for $n \geq 5$.

**Proof.** The proof is by induction on $n$. Let $A$ be a partial nonnegative $P_0$-matrix specifying a 5-cycle. If all the diagonal entries are positive, then $A$ is a partial nonnegative $P_{0,1}$-matrix and can be completed to a nonnegative $P_{0,1}$-matrix by [6, Theorem 8.4]. If at least one diagonal entry is zero, apply Lemma 3.1.

Assume true for $n - 1$. Let $A$ be an $n \times n$ partial nonnegative $P_0$-matrix specifying the pattern whose digraph is the symmetric $n$-cycle $1, 2, \ldots, n, 1$. By multiplication
by a positive diagonal matrix we may assume that each diagonal entry of $A$ is either 1 or 0. (Note that subscript numbering is modulo $n$.) We now consider two cases.

**Case 1:** There exists an index $i$ such that $d_i = d_{i+1} = 1$ and at least one of $a_{i,i+1}$ and $a_{i+1,i}$ is nonzero. Rename so that $d_1 = d_2 = 1$ and $a_{12} \neq 0$. Then we can use the completion of an appropriate $(n-1) \times (n-1)$ principal submatrix specifying an $(n-1)$-cycle to complete $A$ to a nonnegative $P_0$-matrix as in [3, Lemma 3.5] (see also [6, Theorem 8.4] and [2]). This case uses the induction hypothesis.

**Case 2:** The matrix does not satisfy the conditions of Case 1 and there exists an index $i$ such that $d_i = d_{i+1} = 1$. Necessarily $a_{i,i+1} = a_{i+1,i} = 0$. Rename so that $d_1 = d_2 = 1$ (and $a_{12} = a_{23} = 0$). At least one of $a_{21}$ and $a_{13}$ must be zero, because: if $d_0 = 0$, then $\det A(\{1,n\}) = -a_{12}a_{21}$; if $d_0 = 1$, then $a_{12}$ and $a_{21}$ must both be zero, since Case 1 does not apply. The digraph of the pattern specified by $A(\{2,\ldots,n\})$ is block-clique (see Figure 3.1), so $A(\{2,\ldots,n\})$ can be completed to a nonnegative $P_0$-matrix. Set the remaining unspecified entries to zero, thus obtaining a nonnegative block-triangular matrix $\hat{A}$ with diagonal blocks, $[d_1] = [1]$ and the completion of $A(\{2,\ldots,n\})$, both of which are $P_0$-matrices. So $\hat{A}$ is a nonnegative $P_0$-matrix.

**Case 3:** There does not exist an index $i$ such that $d_i = d_{i+1} = 1$. Then for each $i$, $\det A(i,i+1) = -a_{i,i+1}a_{i+1,i}$, so $a_{i,i+1} = 0$ or $a_{i+1,i} = 0$. There are now two subcases.

Subcase i): Whenever $n$ is odd, or if $a_{k,k+1} = 0$ for some $k \leq n$ and $a_{j+1,j} = 0$, for some $j \leq n$, we can set all unspecified entries to zero to get $\hat{A}$. The nonzero $L$-digraph of $\hat{A}$ does not contain any 2-cycles, since for all $i$, $a_{i,i+1} = 0$ or $a_{i+1,i} = 0$. If at least one $a_{k,k+1} = 0$ and at least one $a_{j+1,j} = 0$, there is no cycle of length greater than 1. Thus in either case the nonzero $L$-digraph of $\hat{A}$ does not contain any even cycle, and so by Corollary 1.4, $\hat{A}$ is a nonnegative $P_0$-matrix.

Subcase ii): Let $n$ be even and, for all $i = 1,2,\ldots,n, a_{i,i+1} \neq 0$ or for all $i = 1,\ldots,n, a_{i+1,i} \neq 0$. Without loss of generality $a_{i,i+1} \neq 0$ for all $i$. Complete $A$ to $\hat{A}$ by choosing $x_{2n} = a_{23}$ and $x_{n-1,3} = a_{n-1,n}$, and set all other unspecified entries to zero. The nonzero $L$-digraph $\hat{G}$ of $\hat{A}$ contains the $n$-cycle $1,2,\ldots,n,1$; the 3-cycle $1,2,n,1$; the $(n-3)$-cycle $3,4,\ldots,n-2,n-1,3$ and possibly some loops (see Figure 3.2). Thus there are exactly two permutation $L$-digraphs in $\hat{G}$, one having arc set the $n$-cycle and one having arc set the 3-cycle and the $(n-3)$-cycle. The two permutations have opposite signs and the products of the entries of $\hat{A}$ corresponding to these two permutation $L$-digraphs are equal, so $\det A = 0$ by Lemma 1.2. The nonzero $L$-digraph of any principal submatrix which is not the whole matrix cannot contain the $n$-cycle and thus has no even cycles. Therefore $\hat{A}$ is a nonnegative $P_0$-matrix.

**Corollary 3.3.** A pattern that includes all diagonal positions and whose digraph is a symmetric $n$-cycle has nonnegative $P_0$-completion for $n \neq 4$.

**Proof.** The cases $n = 2$ and $n = 3$ are trivial because the pattern includes all positions. The case $n = 4$ was established in Theorem 2.6, and Theorem 3.2 covers the case $n \geq 5$. The pattern that includes all diagonal positions and whose digraph is an $n$-cycle
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has nonnegative $P_0$-completion if and only if $n \neq 4$, because each entry in the partial matrix corresponding to the reverse of each arc in the cycle can be set to zero to obtain a partial nonnegative $P_0$-matrix specifying a symmetric $n$-cycle.

REFERENCES