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Eduardo Marques de Sa
sa@colibri.mat.uc.pt

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SOME SUBPOLYTOPES OF THE BIRKHOFF POLYTOPE*

EDUARDO MARQUES DE SÁ[†]

Abstract. Some special subsets of the set of uniformly tapered doubly stochastic matrices are considered. It is proved that each such subset is a convex polytope and its extreme points are determined. A minimality result for the whole set of uniformly tapered doubly stochastic matrices is also given. It is well known that if x and y are nonnegative vectors of \mathbb{R}^n and x is weakly majorized by y , there exists a doubly substochastic matrix S such that $x = Sy$. A special choice for such S is exhibited, as a product of doubly stochastic and diagonal substochastic matrices of a particularly simple structure.

Key words. Doubly-stochastic matrices, Inequalities, Polytopes, Majorization.

AMS subject classifications. 15A39, 52B11.

1. Introduction. A square, nonnegative matrix with row and column sums equal to 1 is called *doubly stochastic*. There is an extensive literature on Ω_n , the set of doubly stochastic matrices of order n . The name Birkhoff polytope given to Ω_n comes from a famous theorem of G. Birkhoff [1] who showed that Ω_n is a polytope whose vertices are the $n \times n$ permutation matrices.

For any *interval* F of $\{1, \dots, n\}$, of cardinality q , *i.e.*, a set of the form $F = \{r + 1, \dots, r + q\}$ (for some r , $0 \leq r < n$) let E_F be the $n \times n$ matrix

$$E_F := I_r \oplus J_q \oplus I_{n-r-q},$$

where J_q is the $q \times q$ matrix with all entries $= 1/q$. An *interval partition* of $\{1, \dots, n\}$, is a partition $\mathcal{P} = \{P_1, \dots, P_s\}$ of $\{1, \dots, n\}$ into disjoint, nonempty intervals P_i . For such \mathcal{P} , we let

$$E_{\mathcal{P}} := E_{P_1} E_{P_2} \cdots E_{P_s}. \quad (1.1)$$

The set \mathfrak{U}_n of the so-called *uniformly tapered doubly stochastic matrices* was introduced in [7, 11] by means of a set of linear inequalities. Theorem 1 of [9] asserts that \mathfrak{U}_n is the convex hull of all matrices $E_{\mathcal{P}}$. We shall prove that all $E_{\mathcal{P}}$ are vertices of \mathfrak{U}_n , and settle a minimality property of \mathfrak{U}_n . Note that $E_{\mathcal{P}}$ is the barycenter of the face of Ω_n consisting of all doubly stochastic matrices whose (i, j) -entry is 0 if the (i, j) -entry of $E_{\mathcal{P}}$ is 0. The facial structure of Ω_n has been thoroughly studied in [2, 3, 4, 5], however, the sub-polytopes of Ω_n we shall consider are not faces of Ω_n .

A *nested family* of intervals of $\{1, \dots, n\}$ is a set $\mathcal{F} = \{F_1, \dots, F_t\}$ of intervals of $\{1, \dots, n\}$, such that any two intervals in the family either have an empty intersection,

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[†]Departamento de Matemática, Universidade de Coimbra, 3000 Coimbra, Portugal (emsa@mat.uc.pt). Member of the research unit Centro de Matemática da Universidade de Coimbra/Fundação para a Ciência e a Tecnologia (CMUC/FCT).

or one of them is contained in the other. Note that, in these conditions, the matrices E_{F_1}, \dots, E_{F_t} commute. We define $\mathfrak{U}(\mathcal{F})$ as the set of all $n \times n$ matrices of the form

$$\prod_{i=1}^t [\alpha_i I + (1 - \alpha_i) E_{F_i}], \quad (1.2)$$

where $\alpha_1, \dots, \alpha_t$ run over $[0, 1]$, independently of each other. We shall prove that $\mathfrak{U}(\mathcal{F})$ is a subpolytope of \mathfrak{U}_n , and determine its vertices.

We denote by $\mathfrak{D}(n)$ the set of all $x \in \mathbb{R}^n$, such that $x_1 \geq \dots \geq x_n$, and $\mathfrak{D}_+(n)$ is the set of all nonnegative vectors of $\mathfrak{D}(n)$. We adopt the following *majorization symbols*: for $x, y \in \mathbb{R}^n$, we write $x \preceq_w y$ whenever

$$x'_1 + \dots + x'_k \leq y'_1 + \dots + y'_k, \quad \text{for all } k \in \{1, \dots, n\}, \quad (1.3)$$

where z'_1, \dots, z'_k denotes the non-increasing rearrangement of $z \in \mathbb{R}^n$; and we write $x \preceq y$ if (1.3) holds with equality for $k = n$. In [9], the reader may find the following refinement of a well-known theorem of Hardy, Littlewood and Pólya [8]: *if $x, y \in \mathfrak{D}(n)$ satisfy $x \preceq y$, there exists $R \in \mathfrak{U}_n$ such that $x = Ry$* , together with three proofs of this result. In section 2, we show that the third of these proofs, due to D.Z. Djokovic (see [9, p. 325]) may be conveniently adapted to give a little bit more than the referred refinement. Then we extend that result to the case of weak majorization.

2. Nested Families and Majorization. PROPOSITION 2.1. *For any \mathcal{F} , a nested family of intervals of $\{1, \dots, n\}$, $\mathfrak{U}(\mathcal{F})$ is a subset of \mathfrak{U}_n .*

Proof. Let us expand the polynomial

$$f(u_1, \dots, u_t) := \prod_{i=1}^t [\alpha_i + (1 - \alpha_i) u_i],$$

where the α_i are real numbers and the u_i are commutative variables, as a sum of monomials. The sum of all coefficients of f 's monomials is $f(1, \dots, 1)$, which obviously equals 1. So (1.2) is a convex combination of the products $E_{X_1} \cdots E_{X_s}$, for $0 \leq s \leq t$ and $X_1, \dots, X_s \in \mathcal{F}$. Note that, if $X \supseteq Y$, then $E_X E_Y = E_Y E_X = E_X$. Thus we only have to consider products $E_{X_1} \cdots E_{X_s}$ for pairwise disjoint sets X_1, \dots, X_s . Therefore (1.2) lies in \mathfrak{U}_n , and so $\mathfrak{U}(\mathcal{F}) \subseteq \mathfrak{U}_n$. \square

The proof of the following theorem is essentially due to D. Djokovic [9, p. 325].

THEOREM 2.2. *Let $x, y \in \mathfrak{D}(n)$ satisfy $x \preceq y$. There exists a nested family of intervals of $\{1, \dots, n\}$, and a matrix $R \in \mathfrak{U}(\mathcal{F})$, such that $x = Ry$.*

Proof. We consider the two cases of D.Z. Djokovic's proof [9, p. 325]. In Case 1, it is assumed there is $k < n$ such that $x_1 + \dots + x_k = y_1 + \dots + y_k$. By induction, there exist a nested family \mathcal{F}' of intervals of $\{1, \dots, k\}$, a nested family \mathcal{F}'' of intervals of $\{1, \dots, n - k\}$, and there exist $R' \in \mathfrak{U}(\mathcal{F}')$ and $R'' \in \mathfrak{U}(\mathcal{F}'')$ such that $x = Ry$, with $R := R' \oplus R''$. Define

$$\mathcal{F} := \mathcal{F}' \cup (\mathcal{F}'' + k),$$

where $\mathcal{F}'' + k$ is the family of all sets $\{i + k : i \in X\}$, for X running over \mathcal{F}'' . Clearly, \mathcal{F} is a nested family of intervals of $\{1, \dots, n\}$. On the other hand, it is also clear that $R' \oplus I_{n-k}$ and $I_k \oplus R''$ both lie in $\mathfrak{U}(\mathcal{F})$; therefore, R lies in $\mathfrak{U}(\mathcal{F})$ as well. So we are done with Case 1. In Case 2, D.Z. Djokovic proves that $x = R[\beta I + (1 - \beta)E_{\{1, \dots, n\}}]y$, where R is obtained as in Case 1. In our situation, this means R lies in $\mathfrak{U}(\mathcal{F})$ for some nested family \mathcal{F} of intervals. Note that $\mathcal{F} \cup \{\{1, \dots, n\}\}$ is also a nested family of intervals. So the theorem holds in this case as well. \square

Theorem 2.2 gives us a representation of matrix R as a product of type (1.2), of t doubly stochastic matrices of simple structure, where t is the cardinality of \mathcal{F} . On the other hand, the only sets $F_i \in \mathcal{F}$ which are relevant in (1.2) are those having cardinality at least 2. A straightforward argument, left to the reader, shows that *any maximal nested family of intervals of $\{1, \dots, n\}$ has precisely $n - 1$ elements of cardinality at least 2*. So, $n - 1$ is an upper bound to the number of relevant factors in R 's factorization (1.2).

It is well known [10, p. 27] that if $x, y \in \mathfrak{D}_+(n)$ satisfy $x \preceq_w y$, then $x = Sy$ for some doubly sub-stochastic matrix S . In the following theorem we give a factorization for a special choice of S , in the spirit of Theorem 2.2.

We shall use the following notation: for each $p \in \{1, \dots, n\}$, Δ_p is the $n \times n$ diagonal matrix

$$\Delta_p := \text{Diag}(\underbrace{1, 1, \dots, 1}_p, 0, 0, \dots, 0).$$

THEOREM 2.3. *Let $x \in \mathfrak{D}_+(n)$ be a vector whose distinct coordinates are $\chi_1 > \dots > \chi_s$. Suppose m_i is the number of times χ_i occurs in x . If $y \in \mathfrak{D}_+(n)$ satisfies $x \preceq_w y$, then the following conditions hold:*

(I) *There exist real numbers $\theta_1, \dots, \theta_s$ in the interval $[0, 1]$, a nested family \mathcal{F} on $\{1, \dots, n\}$ and a matrix R in $\mathfrak{U}(\mathcal{F})$, such that $x = DRy$, where D is the diagonal matrix*

$$D := \prod_{i=1}^s [\theta_i I + (1 - \theta_i) \Delta_{m_1 + \dots + m_i}]. \quad (2.1)$$

(II) *The following entities exist: a positive integer p , real numbers $\sigma_1, \dots, \sigma_p$ in the interval $[0, 1]$, nested families, $\mathcal{F}_1, \dots, \mathcal{F}_p$, of intervals of $\{1, \dots, n\}$, and matrices $R_1 \in \mathfrak{U}(\mathcal{F}_1), \dots, R_p \in \mathfrak{U}(\mathcal{F}_p)$, such that $x = [D_p R_p \cdots D_2 R_2 D_1 R_1]y$, where*

$$D_i := \sigma_i I + (1 - \sigma_i) \Delta_{n - m_s}, \text{ for } i = 1, \dots, p. \quad (2.2)$$

Proof. For each $z \in \mathbb{R}^n$ let $\Sigma(z) := z_1 + \dots + z_n$. For each $t \in \mathbb{R}$ let $x(t) \in \mathfrak{D}(n)$ be the vector with i -th entry $\max\{x_i, t\}$. Clearly $x(t) \geq x$ for all t , with equality iff $t \leq x_n$. $\Sigma(x(t))$ is a continuous function, and it is strictly increasing with t , for $t \geq x_n$. As $x \preceq_w y$, we have $\Sigma(x) = \Sigma(x(x_n)) \leq \Sigma(y) \leq \Sigma(x(y_1))$. So there is a

unique $\tau \geq x_n$ such that $\Sigma(x(\tau)) = \Sigma(y)$. We prove

$$\sum_{i=1}^k [y_i - x(\tau)_i] \geq 0, \quad (2.3)$$

for $k = 1, \dots, n-1$. If $\tau \geq x_1$, then $x(\tau) = (\tau, \dots, \tau)$ and (2.3) is obvious. Now assume $\tau < x_1$, and let $v := \sup\{i : x_i > \tau\}$. Note that $1 \leq v < n$. As $x_i(\tau) = x_i$ for $i \in \{1, \dots, v\}$, (2.3) is true for $k \in \{1, \dots, v\}$. So we are left with the case $v < k < n$. Clearly

$$\sum_{i=1}^k [y_i - x(\tau)_i] = \sum_{i=k+1}^n (\tau - y_i). \quad (2.4)$$

On the other hand, as $x \preceq_w y$ and $(y_i - \tau)_{i=1}^n$ is non-increasing, we have

$$\begin{aligned} 0 = \Sigma(y) - \Sigma(x(\tau)) &= \sum_{i=1}^v (y_i - x_i) + \sum_{i=v+1}^n (y_i - \tau) \\ &\geq \sum_{i=v+1}^n (y_i - \tau) \geq \frac{n-v}{n-k} \sum_{i=k+1}^n (y_i - \tau). \end{aligned} \quad (2.5)$$

So (2.4) is nonnegative. This proves (2.3). Therefore $x \leq x(\tau) \preceq y$. By Theorem 2.2 we know that

$$x(\tau) = Ry, \quad (2.6)$$

where $R \in \mathfrak{U}(\mathcal{F})$ for some nested family of intervals, \mathcal{F} . From now on we assume that x and y lie in $\mathfrak{D}_+(n)$.

Proof of (I). If $x = x(\tau)$, then (I) holds with $D := I$, i.e. with $\theta_i := 1$ for $i = 1, \dots, s$. Now assume $x \neq x(\tau)$. Let $u := \min\{i : x_i < \tau\}$. Then define $\theta_i := 1$ for $i = 1, \dots, u-1$, $\theta_u := \chi_u/\tau$ and $\theta_j := \chi_j/\chi_{j-1}$ for $j = u+1, \dots, s$. We clearly have $x = Dx(\tau)$, for D as given in (2.1). So (I) holds.

Proof of (II). The proof is easy when $s = 1$, i.e. when all entries of x are equal. For, we define $p := 1$, $\sigma_1 := x_n/\tau$ if $\tau > 0$ and $\sigma_1 := 0$ if $\tau = 0$ (note that in this case $x = x(\tau)$). Then put $R_1 := R$, the matrix of (2.6). With these definitions (II) holds. We now work out the case $s \geq 2$. For any $z \in \mathbb{R}^n$, let $\kappa(z)$ be the smallest integer greater than $[\Sigma(z) - \Sigma(x)]/(m_s \chi_{s-1})$. In particular

$$\kappa(z) m_s \chi_{s-1} \geq \Sigma(z) - \Sigma(x). \quad (2.7)$$

The proof goes by induction on $\kappa(y)$. Note that $\kappa(y) = \kappa(x(\tau))$. We have two cases.

CASE 1: when $m_s \tau \geq \Sigma(y) - \Sigma(x)$. Define $p := 2$,

$$\sigma_1 := \frac{m_s \tau - \Sigma(y) + \Sigma(x)}{m_s \tau},$$

$\sigma_2 := 0$ and $R_1 := R$, the matrix of (2.6). Moreover, let D_i be as given in (2.2) and let $y' := D_1 x(\tau)$. As $\Sigma(y') = \Sigma(x(\tau)) - m_s \tau(1 - \sigma_1)$, some easy computations show $\Sigma(y') = \Sigma(x)$. This identity may be written as:

$$\sum_{i=1}^{n-m_s} x(\tau)_i + m_s \tau \sigma_1 = \sum_{i=1}^{n-m_s} x_i + m_s \tau. \quad (2.8)$$

As $\sigma_1 \leq 1$, this implies, for each $k \in \{1, \dots, m_s\}$:

$$\sum_{i=1}^{n-m_s} x(\tau)_i + k \tau \sigma_1 \geq \sum_{i=1}^{n-m_s} x_i + k \tau. \quad (2.9)$$

Taking into account that $x(\tau) \geq x$, (2.8)-(2.9) show that $x \preceq_w y'$. So, for some nested family of intervals \mathcal{F}_2 , there exists $R_2 \in \mathcal{U}(\mathcal{F}_2)$ such that $x = R_2 y'$. Therefore $x = [D_2 R_2 D_1 R_1] y$ and (II) holds. CASE 2: when $m_s \tau < \Sigma(y) - \Sigma(x)$. Here, we let $\sigma_1 := 0$ and D_1 be as in (2.2). The vector $y' := D_1 x(\tau)$ clearly satisfies $\Sigma(y') = \Sigma(y) - m_s \tau > \Sigma(x)$. It is now easy to show that

$$x \preceq_w y'. \quad (2.10)$$

On the other hand,

$$\begin{aligned} 0 < \Sigma(x(\tau)) - \Sigma(x) - m_s \tau &= \sum_{i=1}^s m_i \cdot \max\{0, \tau - \chi_i\} - m_s \tau \\ &\leq n \cdot \max\{0, \tau - \chi_{s-1}\}. \end{aligned}$$

Therefore $\tau > \chi_{s-1}$. Taking (2.7) into account we obtain:

$$\begin{aligned} \Sigma(y') - \Sigma(x) &= \Sigma(y) - \Sigma(x) - m_s \tau \\ &\leq \kappa(y) m_s \chi_{s-1} - m_s \tau < [\kappa(y) - 1] m_s \chi_{s-1}. \end{aligned}$$

This yields $\kappa(y') \leq \kappa(y) - 1$, and this, taken together with (2.10), allows us to use induction: there exist nested families of intervals, $\mathcal{F}'_1, \dots, \mathcal{F}'_q$, matrices $R'_1 \in \mathcal{U}(\mathcal{F}'_1), \dots, R'_q \in \mathcal{U}(\mathcal{F}'_q)$ and diagonal matrices, D'_1, \dots, D'_q , of the type of (2.2), such that $x = [D'_q R'_q \cdots D'_1 R'_1] y'$. Therefore

$$x = [D'_q R'_q \cdots D'_1 R'_1 D_1 R] y$$

and the proof is done. \square

Incidentally, in the course of proof, we showed the existence of a z such that $x \leq z \preceq_w y$. This is a result of [6] (see also [10, p. 123] and references therein). However, we got a little bit more: that we may choose z of the form $x(\tau)$. We point out that our inductive proof of Theorem 2.3(II) also yields an upper bound for the number, p , of factors $D_i R_i$, namely $p \leq \kappa(y) + 1$. This gives an indication on the complexity of the procedure given by the proof.

3. Extreme Points. There exist 2^{n-1} distinct interval partitions of $\{1, \dots, n\}$, and so this is the cardinality of the set $\{E_{\mathcal{P}}\}$ of all matrices defined in (1.1). Theorem 1 of [9] says that $\{E_{\mathcal{P}}\}$ contains the set of all extreme points of \mathfrak{U}_n . Our aim now is to prove that any $E_{\mathcal{P}}$ is an extreme point of \mathfrak{U}_n .

LEMMA 3.1. *Let $w \in \mathbb{R}^n$ be a vector satisfying $w_1 > \dots > w_n$, R an element of \mathfrak{U}_n and \mathcal{G} an interval partition of $\{1, \dots, n\}$. The identity $Rw = E_{\mathcal{G}}w$ implies $R = E_{\mathcal{G}}$.*

Proof. By Theorem 1 of [9], R is a convex combination of the $E_{\mathcal{P}}$, for all partitions \mathcal{P} , i.e., $R = \sum \lambda_{\mathcal{P}} E_{\mathcal{P}}$, for some nonnegative coefficients $\lambda_{\mathcal{P}}$ which sum up 1. As $Rw = E_{\mathcal{G}}w$,

$$E_{\mathcal{G}}w = \sum \lambda_{\mathcal{P}} E_{\mathcal{P}}w. \quad (3.1)$$

The second proof of Theorem 2 of [9] shows that the 2^{n-1} vectors $E_{\mathcal{P}}w$ are pairwise distinct, and are the extreme points of $\{x \in \mathfrak{D}(n) : x \preceq w\}$. Therefore (3.1) implies that all $\lambda_{\mathcal{P}}$ are 0, except $\lambda_{\mathcal{G}}$ that equals 1. Thus $R = E_{\mathcal{G}}$ as required. \square

THEOREM 3.2. *For any interval partition \mathcal{G} , $E_{\mathcal{G}}$ is an extreme point of \mathfrak{U}_n .*

Proof. Pick any $E_{\mathcal{G}}$ and write it as a convex combination of the $E_{\mathcal{P}}$. Then an equation like (3.1) arises. The argument under (3.1) now proves that $E_{\mathcal{G}}$ is not a convex combination of the *other* generators $E_{\mathcal{P}}$ of \mathfrak{U}_n . This means $E_{\mathcal{G}}$ is an extreme point of \mathfrak{U}_n . \square

THEOREM 3.3. *\mathfrak{U}_n is minimal among all sets \mathfrak{M} of $n \times n$ matrices satisfying the conditions: \mathfrak{M} is convex, and, if $x, y \in \mathfrak{D}(n)$ satisfy $x \preceq y$, there exists $M \in \mathfrak{M}$ such that $x = My$.*

Proof. Assume $\mathfrak{M} \subseteq \mathfrak{U}_n$ satisfies the given conditions. With w as in Lemma 3.1 we have, for any interval partition \mathcal{P} : $E_{\mathcal{P}}w \in \mathfrak{D}(n)$ and $E_{\mathcal{P}}w \preceq w$. So $E_{\mathcal{P}}w = M_{\mathcal{P}}w$, for some $M_{\mathcal{P}} \in \mathfrak{M}$. Lemma 3.1 implies $E_{\mathcal{P}} = M_{\mathcal{P}}$, and so $E_{\mathcal{P}} \in \mathfrak{M}$. Therefore $\mathfrak{M} = \mathfrak{U}_n$. \square

We now prove the convexity of the set $\mathfrak{U}(\mathcal{F})$, whose members are matrix products as (1.2), and determine the set of its extreme points.

THEOREM 3.4. *Given a nested family \mathcal{F} of intervals of $\{1, \dots, n\}$, the set $\mathfrak{U}(\mathcal{F})$ is convex, and $\{E_{\mathcal{X}} : \mathcal{X} \subseteq \mathcal{F}\}$ is the set of $\mathfrak{U}(\mathcal{F})$'s extreme points.*

Proof. By Theorem 3.2 we only need to prove that $\mathfrak{U}(\mathcal{F})$ is the convex hull of the $E_{\mathcal{X}}$, for $\mathcal{X} \subseteq \mathcal{F}$. We argue by induction on $t = |\mathcal{F}|$. Let M_1, \dots, M_r be the elements of \mathcal{F} which are maximal for inclusion. Without loss of generality, assume $M_1 = F_1, \dots, M_r = F_r$. Define $\mathcal{F}_i := \{X \in \mathcal{F} : X \subseteq F_i\}$, for $i = 1, \dots, r$. Clearly, $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_r$, and this union is disjoint. In the first place suppose $r = 1$, that is $F_1 \supseteq [F_1 \cup \dots \cup F_t]$. By induction, $\mathfrak{U}(\{F_2, \dots, F_t\}) = \text{conv}\{E_{\mathcal{X}} : \mathcal{X} \subseteq \{F_2, \dots, F_t\}\}$.

We therefore have

$$\begin{aligned}
 \mathfrak{U}(\mathcal{F}) &= \bigcup_{\alpha \in [0,1]} [\alpha I + (1 - \alpha)E_{F_1}] \cdot \mathfrak{U}(\{F_2, \dots, F_t\}) \\
 &= \bigcup_{\alpha \in [0,1]} [\alpha \mathfrak{U}(\{F_2, \dots, F_t\}) + (1 - \alpha)E_{F_1}] \\
 &= \text{conv} \left(\{E_{F_1}\} \cup \{E_{\mathcal{X}} : \mathcal{X} \subseteq \{F_2, \dots, F_t\}\} \right) \\
 &= \text{conv}\{E_{\mathcal{Y}} : \mathcal{Y} \subseteq \mathcal{F}\}.
 \end{aligned}$$

This settles the case $r = 1$. We now assume $r \geq 2$. By induction, $\mathfrak{U}(\mathcal{F}_i) = \text{conv}\{E_{\mathcal{X}_i} : \mathcal{X}_i \subseteq \mathcal{F}_i\}$. The proof is finished in the following two lines:

$$\begin{aligned}
 \mathfrak{U}(\mathcal{F}) &= \bigoplus_{i=1}^r \mathfrak{U}(\mathcal{F}_i) = \bigoplus_{i=1}^r \text{conv}\{E_{\mathcal{X}_i} : \mathcal{X}_i \subseteq \mathcal{F}_i\} \\
 &= \text{conv} \bigoplus_{i=1}^r \{E_{\mathcal{X}_i} : \mathcal{X}_i \subseteq \mathcal{F}_i\} = \text{conv}\{E_{\mathcal{X}} : \mathcal{X} \subseteq \mathcal{F}\}. \quad \square
 \end{aligned}$$

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