2008

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Recommended Citation
DOI: https://doi.org/10.13001/1081-3810.1277

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ON A NEW CLASS OF STRUCTURED MATRICES RELATED TO THE DISCRETE SKEW-SELF-ADJOINT DIRAC SYSTEMS

B. FRITZSCHE†, B. KIRSTEIN†, AND A.L. SAKHNOVICH‡

Abstract. A new class of the structured matrices related to the discrete skew-self-adjoint Dirac systems is introduced. The corresponding matrix identities and inversion procedure are treated. Analogs of the Schur coefficients and of the Christoffel-Darboux formula are studied. It is shown that the structured matrices from this class are always positive-definite, and applications for an inverse problem for the discrete skew-self-adjoint Dirac system are obtained.

Key words. Structured matrices, Matrix identity, Schur coefficients, Christoffel-Darboux formula, Transfer matrix function, Discrete skew-self-adjoint Dirac system, Weyl function, Inverse problem.

AMS subject classifications. 15A09, 15A24, 39A12.

1. Introduction. It is well-known that Toeplitz and block Toeplitz matrices are closely related to a discrete system of equations, namely to Szegö recurrence. This connection have been actively studied during the last decades. See, for instance, [1]–[5], [12, 25] and numerous references therein. The connections between block Toeplitz matrices and Weyl theory for the self-adjoint discrete Dirac system were treated in [11]. (See [26] for the Weyl theory of the discrete analog of the Schrödinger equation.) The Weyl theory for the skew-self-adjoint discrete Dirac system

\[(1.1) \quad W_{k+1}(\lambda) - W_k(\lambda) = -\frac{i}{\lambda} C_k W_k(\lambda), \quad C_k = C_k^* = C_k^{-1}, \quad k = 0, 1, \ldots\]

was developed in [14, 18]. Here $C_k$ are $2p \times 2p$ matrix functions. When $p = 1$, system (1.1) is an auxiliary linear system for the isotropic Heisenberg magnet model. Explicit solutions of the inverse problem were constructed in [14]. A general procedure to construct the solutions of the inverse problem for system (1.1) was given in [18], using a new class of structured matrices $S$, which satisfy the matrix identity

\[(1.2) \quad AS - SA^* = i\Pi \Pi^*\].

*Received by the editors May 19, 2008. Accepted for publication September 11, 2008. Handling Editor: Harm Bart.
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Here, $S$ and $A$ are $(n+1)p \times (n+1)p$ matrices and $\Pi$ is an $(n+1)p \times 2p$ matrix. The block matrix $A$ has the form

$$A := A(n) = \left\{ a_{j-k} \right\}_{k,j=0}^n, \quad a_r = \begin{cases} 0 & \text{for } r > 0 \\ \frac{i}{2} I_p & \text{for } r = 0 \\ \frac{i}{2} I_p & \text{for } r < 0 \end{cases},$$

where $I_p$ is the $p \times p$ identity matrix. The matrix $\Pi = [\Phi_1 \ Ph \Phi_2]$ consists of two block columns of the form

$$\Phi_1 = \begin{bmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} \alpha_0 \\ \alpha_0 + \alpha_1 \\ \vdots \\ \alpha_0 + \alpha_1 + \cdots + \alpha_n \end{bmatrix}.$$

**Definition 1.1.** The class of the block matrices $S$ determined by the matrix identity (1.2) and formulas (1.3) and (1.4) is denoted by $\Omega_n$.

Notice that the blocks $\alpha_k$ in [18] are Taylor coefficients of the Weyl functions and that the matrices $C_n$ ($0 \leq n \leq l$) in (1.1) are easily recovered from the expressions $\Pi(n)^* S(n)^{-1} \Pi(n)$ ($0 \leq n \leq l$) (see Theorem 3.4 of [18]). In this way, the structure of the matrices $S$ determined by the matrix identity (1.2) and formulas (1.3) and (1.4), their inversion and conditions of invertibility prove essential. Recall that the self-adjoint block Toeplitz matrices satisfy [15]–[17] the identity $AS - SA^* = i \Pi J \Pi^*$ ($J = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}$), which is close to (1.2)–(1.4). We refer also to [20]–[24] and references therein for the general method of the operator identities. The analogs of various results on the Toeplitz matrices and $j$-theory from [6]–[11] can be obtained for the class $\Omega_n$, too.

**2. Structure of the matrices from $\Omega_n$.** Consider first the block matrix $S = \left\{ s_{kj} \right\}_{k,j=0}^n$ with the $p \times p$ entries $s_{kj}$, which satisfies the identity

$$AS - SA^* = iQ, \quad Q = \left\{ q_{kj} \right\}_{k,j=0}^n.$$

One can easily see that the equality

$$q_{kj} = s_{kj} + \sum_{r=0}^{k-1} s_{rj} + \sum_{r=0}^{j-1} s_{kr}$$

follows from (2.1). Sometimes we add comma between the indices and write $s_{k,j}$. Putting $s_{-1,j} = s_{k,-1} = q_{-1,j} = q_{k,-1} = 0$, from (2.2) we have

$$s_{k+1,j+1} - s_{kj} = q_{kj} + q_{k+1,j+1} - q_{k+1,j} - q_{k,j+1}, \quad -1 \leq k, j \leq n - 1.$$
Now, putting \( Q = i \Pi \Pi^* \) and taking into account (2.3), we get the structure of \( S \).

**Proposition 2.1.** Let \( S \in \Omega_n \). Then we have

\[
s_{k+1,j+1} - s_{kj} = \alpha_{k+1} \alpha_j^* \quad (-1 \leq k, j \leq n-1),
\]

excluding the case when \( k = -1 \) and \( j = -1 \) simultaneously. For that case, we have

\[
s_{00} = I_p + \alpha_0 \alpha_0^*.
\]

Notice that for the block Toeplitz matrix, the equalities \( s_{k+1,j+1} - s_{kj} = 0 \) \((0 \leq k, j \leq n-1)\) hold. Therefore, Toeplitz and block Toeplitz matrices can be used to study certain homogeneous processes and appear as a result of discretization of homogeneous equations. From this point of view, the matrix \( S \in \Omega_n \) is perturbed by the simplest inhomogeneity.

The authors are grateful to the referee for the next interesting remark.

**Remark 2.2.** From (1.2)–(1.4) we get another useful identity, namely,

\[
S - NSN^* = \hat{\Pi} \hat{\Pi}^*,
\]

where

\[
N = \{ \delta_{k-j-1} I_p \}_{k,j=0}^n = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
I_p & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
I_p & 0 & \cdots & 0
\end{bmatrix},
\]

\[
\hat{\Pi} = \begin{bmatrix}
I_p & \alpha_0 \\
0 & \alpha_1 \\
\vdots & \vdots \\
0 & \alpha_n
\end{bmatrix}.
\]

Indeed, it is easy to see that \((I_{(n+1)p} - N) A = \frac{i}{2} (I_{(n+1)p} + N)\). Hence, the identity

\[
i(S - NSN^*) = i(I_{(n+1)p} - N) \Pi \Pi^*(I_{(n+1)p} - N^*)
\]

follows from (1.2). By (2.7), we have \((I_{(n+1)p} - N) \Pi = \hat{\Pi}\), and so (2.6) is valid. Relations (2.4) and (2.5) are immediate from (2.6).

**Proposition 2.3.** Let \( S = \{ s_{kj} \}_{k,j=0}^n \in \Omega_n \). Then \( S \) is positive and, moreover, \( S \geq I_{(n+1)p}\). We have \( S > I_{(n+1)p}\) if and only if \( \det \alpha_0 \neq 0 \).

**Proof.** From (2.5) it follows that \( S(0) = s_{00} \geq I_p \) and that \( S(0) > I_p \), when \( \det \alpha_0 \neq 0 \). The necessity of \( \det \alpha_0 \neq 0 \) for the inequality \( S > I_{(n+1)p} \) to be true, follows from (2.5), too. We shall prove that \( S \geq I_{(n+1)p}\) and that \( S > I_{(n+1)p}\), when \( \det \alpha_0 \neq 0 \), by induction.

Suppose that \( S(r-1) = \{ s_{kj} \}_{k,j=0}^{r-1} \geq I_{rp} \) \((r \geq 1)\). According to (2.6), we can
present $S(r) = \{ s_{kj} \}_{k,j=0}^r$ in the form $S(r) = S_1 + S_2$, 

$$S_1 := \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \begin{bmatrix} \alpha_0^* & \alpha_1^* & \cdots & \alpha_r^* \end{bmatrix}, \quad S_2 := \begin{bmatrix} I_p & 0 \\ 0 & S(r-1) \end{bmatrix}. \quad (2.8)$$

By the assumption of induction, it is immediate that $S(r) \geq S_2 \geq I_{(r+1)p}$. Hence, we get $S = S(n) \geq I_{(n+1)p}$.

Suppose that $\det \alpha_0 \neq 0$ and $S(r-1) > I_{(n+1)p}$. Let $S(r)f = f \ (f \in BC^{(r+1)p})$, i.e., let $f^*(S(r) - I_{(r+1)p})f = 0$. By (2.8), we have $S_1 \geq 0$, and by the assumption of induction, we have $S_2 - I_{(r+1)p} \geq 0$. So, it follows from $f^* (S(r) - I_{(r+1)p})f = 0$ that $f^*S_1f = 0$ and $f^* (S_2 - I_{(r+1)p})f = 0$. Hence, as $\alpha_0 \alpha_0^* > 0$ and $S(r-1) > I_{xp}$, we derive $f = 0$. In other words, $S(r)f = f$ implies $f = 0$, that is, $\det(S(r) - I_{(r+1)p}) \neq 0$.

From $\det(S(r) - I_{(r+1)p}) \neq 0$ and $S(r) \geq I_{(r+1)p}$, we get $S(r) > 0$. So, the condition $\det \alpha_0 \neq 0$ implies $S(n) > I_{(n+1)p}$ by induction. \[\square\]

**Remark 2.4.** Using formula (2.5) and representations $S(r) = S_1(r) + S_2(r)$ $(0 < r \leq n)$, where $S_1(r)$ and $S_2(r)$ are given by (2.8), one easily gets

$$S = I_{(n+1)p} + \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} \alpha_0^* & \alpha_1^* & \cdots & \alpha_n^* \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \alpha_0 \\ \alpha_{n-1} \end{bmatrix} \begin{bmatrix} 0 & \alpha_0^* & \cdots & \alpha_{n-1}^* \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & \alpha_0^* \end{bmatrix}$$

$$= I_{(n+1)p} + V_\alpha V_\alpha^*, \quad V_\alpha := \begin{bmatrix} \alpha_0 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_0 \end{bmatrix}. \quad (2.9)$$

Here, $V_\alpha$ is a triangular block Toeplitz matrix, and formula (2.9) is another way to prove Proposition 2.3. Further, we will be interested in a block triangular factorization of the matrix $S$ itself, namely, $S = V_-^{-1}(V_-^*)^{-1}$, where $V_-$ is a lower triangular matrix.

Similar to the block Toeplitz case (see [13] and references therein) the matrices $S \in \Omega_n$ admit the matrix identity of the form $A_1S - SA_1 = Q_1$, where $Q_1$ is of low
rank, $A_1 := \{\delta_{k-j+1} I_p\}_{k,j=0}^n = N^*$ and $N$ is given in (2.7). The next proposition follows easily from (2.4).

**Proposition 2.5.** Let $S \in \Omega_n$. Then we have

\[(2.10) A_1 S - SA_1 = y_1 y_2^* + y_3 y_4^* + y_5 y_6^*, \quad A_1^* S - SA_1^* = -y_2 y_1^* + y_4 y_3^* + y_6 y_5^*,\]

where

\[(2.11) y_1 = \begin{bmatrix} s_{10} \\ s_{20} \\ \vdots \\ s_{n0} \\ 0 \end{bmatrix}, \quad y_3 = -\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1_p \end{bmatrix}, \quad y_5 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad y_6 = \begin{bmatrix} 0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix},\]

\[(2.12) y_2^* = \begin{bmatrix} I_p \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \quad y_4^* = \begin{bmatrix} 0 \\ s_{n0} \\ s_{n1} \\ \cdots \\ s_{n,n-1} \end{bmatrix}.\]

Differently than the block Toeplitz matrix case, the rank of $A_1 S - SA_1$ is in general situation larger than the rank of $AS - SA^*$, where $A$ is given by (1.3). (To see this compare (1.2)–(1.4) and (2.10)–(2.12).)

3. **Transfer matrix function and Weyl functions.** Introduce the $(r+1)p \times (n+1)p$ matrix

\[(3.1) P_k := \begin{bmatrix} I_{(r+1)p} & 0 \end{bmatrix}, \quad r \leq n.\]

It follows from (1.3) that $P_r A(n) = A(r) P_r$. Hence, using (1.2) we derive

\[(3.2) A(r) S(r) - S(r) A(r)^* = i \Pi(r) \Pi(r)^*, \quad \Pi(r) := P_r \Pi.\]

As $S > 0$, it admits a block triangular factorization

\[(3.3) S = V_+^{-1} (V_-^*)^{-1},\]

where $V_{\pm 1}$ are block lower triangular matrices. It is immediate from (3.3) that

\[(3.4) S(r) = V_-^{-1} (V_-^*)^{-1}, \quad V_-(r) := P_r V_- P_r^*.\]

Recall that $S$-node [21, 23, 24] is the triple $\left( A(r), S(r), \Pi(r) \right)$ that satisfies the matrix identity (3.2) (see also [21, 23, 24] for a more general definition of the $S$-node). Following [21, 23, 24], introduce the transfer matrix function corresponding to the $S$-node:

\[(3.5) w_A(r, \lambda) = I_{2p} - i \Pi(r)^* S(r)^{-1} (A(r) - \lambda I_{(r+1)p})^{-1} \Pi(r).\]
In particular, taking into account (3.4) and (3.5), we get

\[(3.6) \quad w_A(0, \lambda) = I_{2p} - \frac{2i}{\alpha - 2\lambda} \beta(0)^* \beta(0), \quad \beta(0) = V_- (0) \Pi(0).\]

By the factorization theorem 4 from [21] (see also [23, p. 188]), we have

\[(3.7) \quad w_A(r, \lambda) = \left( I_{2p} - i\Pi(r)^* S(r)^{-1} P^* (PA(r) P^* - \lambda I_p)^{-1} (PS(r)^{-1} P^*)^{-1} \times PS(r)^{-1} \Pi(r) \right) w_A(r - 1, \lambda), \quad P = [0 \cdots 0 I_p].\]

According to (1.3), we obtain

\[(3.8) \quad (PA(r) P^* - \lambda I_p)^{-1} = \left( \frac{i}{2} - \lambda \right)^{-1} I_p.\]

Using (3.4), we derive

\[(3.9) \quad PS(r)^{-1} P^* = (V_-(r))_{rr}^* (V_-(r))_{rr}, \quad PS(r)^{-1} \Pi(r) = (V_-(r))_{rr}^* PV_-(r) \Pi(r),\]

where \((V_-(r))_{rr}\) is the block entry of \(V_-(r)\) (the entry from the \(r\)-th block row and the \(r\)-th block column). In view of (3.8) and (3.9), we rewrite (3.7) in the form

\[(3.10) \quad w_A(r, \lambda) = \left( I_{2p} - \frac{2i}{\alpha - 2\lambda} \beta(r)^* \beta(r) \right) w_A(r - 1, \lambda), \quad \beta(r) = PV_-(r) \Pi(r) = (V_- \Pi)_r, \quad 0 < r \leq n.\]

Here, \((V_- \Pi)_r\) is the \(r\)-th \(p \times 2p\) block of the block column vector \(V_- \Pi\). Moreover, according to (3.9) and definitions (3.6), (3.11) of \(\beta\), we have

\[(3.12) \quad \left( PS(r)^{-1} P^* \right)^{-\frac{1}{2}} PS(r)^{-1} \Pi(r) = u(r) \beta(r), \quad u(r) := \left( PS(r)^{-1} P^* \right)^{-\frac{1}{2}} (V_-(r))_{rr}^*, \quad u(r)^* u(r) = I_p.\]

As \(u\) is unitary, the properties of \(\left( PS(r)^{-1} P^* \right)^{-\frac{1}{2}} PS(r)^{-1} \Pi(r)\) proved in [18, p. 2098] imply the next proposition.

**Proposition 3.1.** Let \(S \in \Omega_n\) and let \(\beta(k) \ (0 \leq k \leq n)\) be given by (3.3), (3.4), (3.6) and (3.11). Then we have

\[(3.13) \quad \begin{cases} \beta(k) \beta(k)^* = I_p & (0 \leq k \leq n), \\ \det \beta(k - 1) \beta(k)^* \neq 0 & (0 < k \leq n), \\ \det \beta_1(0) \neq 0, \end{cases}\]

where \(\beta_1(k), \beta_2(k)\) are \(p \times p\) blocks of \(\beta(k)\).
Remark 3.2. Notice that the lower triangular factor $V_-$ is not defined by $S$ uniquely. Hence, the matrices $\beta(k)$ are not defined uniquely, too. Nevertheless, in view of (3.12), the matrices $\beta(k)^* \beta(k)$ are uniquely defined, which suffices for our considerations.

When $p = 1$ and $C_k \neq \pm I_2$, the matrices $C_k = C_k^* = C_k^{-1}$ (i.e., the potential of the system (1.1)) can be presented in the form $C_k = I_2 - 2\beta(k)^* \beta(k)$, where $\beta(k)^* \beta(k) = 1$. Therefore, it is assumed in [18] for the system (1.1) on the interval $0 \leq k \leq n$, that

\begin{equation}
C_k = I_{2p} - 2\beta(k)^* \beta(k),
\end{equation}

where $\beta(k)$ are $p \times 2p$ matrices and (3.13) holds. Relation (3.14) implies $C_k^* = C_k^{-1}$ follow. Consider the fundamental solution $W_r(\lambda)$ of the system (1.1) normalized by $W_0(\lambda) = I_{2p}$. Using (3.6) and (3.10), one easily derives

\begin{equation}
W_{r+1}(\lambda) = \left(\frac{\lambda - i}{\lambda}\right)^{r+1} w_A\left(r, \frac{\lambda}{2}\right), \quad 0 \leq r \leq n.
\end{equation}

Similar to the continuous case, the Weyl functions of the system (1.1) are defined via Möbius (linear-fractional) transformation

\begin{equation}
\varphi(\lambda) = (W_{11}(\lambda) R(\lambda) + W_{12}(\lambda) Q(\lambda))(W_{21}(\lambda) R(\lambda) + W_{22}(\lambda) Q(\lambda))^{-1},
\end{equation}

where $W_{ij}$ are $p \times p$ blocks of $W$ and

\begin{equation}
\mathcal{W}(\lambda) = \{W_{ij}(\lambda)\}_{i,j=1}^2 := W_{n+1}(\lambda)^*.
\end{equation}

Here, $R$ and $Q$ are any $p \times p$ matrix functions analytic in the neighborhood of $\lambda = i$ and such that

\begin{equation}
\det \left(W_{21}(i) R(i) + W_{22}(i) Q(i)\right) \neq 0.
\end{equation}

One can easily verify that such pairs always exist (see [18, p. 2090]). A matrix function $\varphi(\lambda)$ of order $p$, analytic at $\lambda = i$, generates a matrix $S \in \Omega_n$ via the Taylor coefficients

\begin{equation}
\varphi\left(i \frac{1+z}{1-z}\right) = -(\alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n) + O(z^{n+1}) \quad (z \to 0)
\end{equation}

and identity (1.2). By Theorem 3.7 in [18], such $\varphi$ is a Weyl function of some system (1.1) if and only if $S$ is invertible. Now, from Proposition 2.3 it follows that $S > 0$, and the next proposition is immediate.
Proposition 3.3. Any $p \times p$ matrix function $\varphi$, which is analytic at $\lambda = i$, is a Weyl function of some system (1.1) on the interval $0 \leq k \leq n$, such that (3.13) and (3.14) hold.

Moreover, from the proof of the statement (ii) of Theorem 3.7 in [18], the Corollary 3.6 in [18] and our Proposition 3.3, we get:

Proposition 3.4. Let the $p \times p$ matrix function $\varphi$ be analytic at $\lambda = i$ and admit expansion (3.19). Then $\varphi$ is a Weyl function of the system (1.1) $(0 \leq k \leq n)$, where $C_k$ are defined by the formulas (1.2)–(1.4), $\Pi = [\Phi_1 \Phi_2]$, (3.3), (3.11) and (3.14). Moreover, any Weyl function of this system admits expansion (3.19).

4. Schur coefficients and Christoffel-Darboux formula. The sequence $\{\alpha_k\}_{k=0}^n$ uniquely determines via formulas (1.2)–(1.4) or (1.3), (1.4), (2.4) and (2.5) the $S$-nodes $(A, S, \Pi)$. Then, using (3.3), (3.11) and (3.14), we uniquely recover the system (1.1) $(0 \leq k \leq n)$, or equivalently, we recover the sequence $\{\beta_k^*\beta_k\}_{k=0}^n$, such that (3.13) holds. By Proposition 3.4, one can use Weyl functions of this system to obtain the sequence $\{\alpha_k\}_{k=0}^n$.

Remark 4.1. Thus, there are one to one correspondences between the sequences $\{\alpha_k\}_{k=0}^n$, the $S$-nodes $(A, S, \Pi)$ satisfying (1.2), the systems (1.1) $(0 \leq k \leq n)$ with $C_k$ of the form (3.14) and the sequences $\{\beta_k^*\beta_k\}_{k=0}^n$, such that (3.13) holds.

Next, we consider a correspondence between $\{\beta_k^*\beta_k\}_{k=0}^n$ and some $p \times p$ matrices $\{\rho_k\}_{k=0}^n$ ($\|\rho_k\| \leq 1$). Notice that $0 \leq \beta_1(k)\beta_1(k)^* \leq I_p$, and suppose that these inequalities are strict:

$$0 < \beta_1(k)\beta_1(k)^* < I_p \quad (0 \leq k \leq n).$$

In view of the first relation in (3.13) and inequalities (4.1), we have det $\beta_1(k) \neq 0$ and det $\beta_2(k) \neq 0$. So, we can put

$$\rho_k := (\beta_2(k)^*\beta_2(k))^{-\frac{1}{2}}\beta_2(k)^*\beta_1(k).$$

It follows from (4.2) that

$$\rho_k^*\rho_k = (\beta_2(k)^*\beta_2(k))^{-\frac{1}{2}}\beta_2(k)^*(I_p - \beta_2(k)\beta_2(k)^*)\beta_2(k)(\beta_2(k)^*\beta_2(k))^{-\frac{1}{2}}$$

$$= I_p - \beta_2(k)^*\beta_2(k).$$

By (4.2) and (4.3), we obtain

$$[\rho_k (I_p - \rho_k^*\rho_k)^{\frac{1}{2}}] = u_k\beta(k), \quad \|\rho_k\| < 1,$$

where

$$u_k := (\beta_2(k)^*\beta_2(k))^{-\frac{1}{2}}\beta_2(k)^*, \quad u_ku_k^* = I_p,$$
i.e., \( u_k \) is unitary.

**Remark 4.2.** Under condition (4.1), according to (4.4) and (4.5), the sequence \( \{\beta_k^* \beta_k\}_{k=0}^n \) is uniquely recovered from the sequence \( \{\rho_k\}_{k=0}^n (\|\rho_k\| \leq 1) \):

\[
\beta_k^* \beta_k = \begin{bmatrix} \rho_k^* & (I_p - \rho_k \rho_k^*)^{1/2} \end{bmatrix} \begin{bmatrix} \rho_k & (I_p - \rho_k \rho_k^*)^{1/2} \end{bmatrix}.
\]

By Remark 4.1 this means that the \( S \)-node can be recovered from the sequence \( \{\rho_k\}_{k=0}^n \).

Therefore, similar to the Toeplitz case, we call \( \rho_k \) the Schur coefficients of the \( S \)-node \((A, S, \Pi)\).

Besides Schur coefficients, we obtain an analog of the Christoffel-Darboux formula.

**Proposition 4.3.** Let \( S \in \Omega_n \), let \( w_A(r, \lambda) \) be introduced by (3.5) for \( r \geq 0 \) and put \( w_A(-1, \lambda) = I_{2p} \). Then we have

\[
\sum_{k=-1}^{n-1} w_A(k, \mu)^* \beta(k+1)^* \beta(k+1) w_A(k, \lambda) = \frac{(2\lambda - i)(2\pi i + i)}{4i(\pi - \lambda)} \left( w_A(n, \mu)^* w_A(n, \lambda) - I_{2p} \right).
\]

**Proof.** From (3.10) it follows that

\[
w_A(k + 1, \mu)^* w_A(k + 1, \lambda) - w_A(k, \mu)^* w_A(k, \lambda) = w_A(k, \mu)^* \left( \left( I_{2p} - \frac{2i}{2\pi i + i} \beta(k + 1)^* \beta(k + 1) \right) \right. \\
\times \left. \left( I_{2p} + \frac{2i}{2\lambda - i} \beta(k + 1)^* \beta(k + 1) \right) - I_{2p} \right) w_A(k, \lambda).
\]

Using \( \beta(k)\beta(k)^* = I_p \), we rewrite (4.8) in the form

\[
w_A(k + 1, \mu)^* w_A(k + 1, \lambda) - w_A(k, \mu)^* w_A(k, \lambda) = \frac{4i(\pi - \lambda)}{(2\lambda - i)(2\pi i + i)} w_A(k, \mu)^* \beta(k + 1)^* \beta(k + 1) w_A(k, \lambda).
\]

Equality (4.7) follows from (4.9). \( \Box \)

5. **Inversion of** \( S \in \Omega_n \). To recover the system (1.1) from \( \{\alpha_k\}_{k=0}^n \), it is convenient to use formula (3.11). The matrices \( V_-(r) \) \((r \geq 0)\) in this formula can be constructed recursively.
Proposition 5.1. Let \( S = V_-(r+1)^{-1} \in \Omega_n \). Then \( V_-(r+1) \) (\( 0 \leq r < n \)) can be constructed by the formula
\[
V_-(r+1) = \begin{bmatrix}
V_-(r) & 0 \\
t(r)S_{21}(r)V_-(r)^*V_-(r) & t(r)
\end{bmatrix},
\]
where \( S_{21}(r) = [s_{r+1,0} \quad s_{r+1,1} \quad \ldots \quad s_{r+1,n}] \),
\[
t(r) = \left(s_{r+1,r+1} - S_{21}(r)V_-(r)^*V_-(r)S_{21}(r)^*\right)^{-\frac{1}{2}}.
\]

Proof. To prove the proposition it suffices to assume that \( V_-(r) \) satisfies (3.4) and prove \( S(r+1) = V_-(r+1)^{-1}(V_-(r+1)^*)^{-1} \). In view of Proposition 2.3 and (3.4), we have \( s_{r+1,r+1} - S_{21}(r)V_-(r)^*V_-(r)S_{21}(r)^* > 0 \), i.e., formula (5.2) is well defined.

Now, it is easily checked that \( S(r+1)^{-1} = V_-(r+1)^*V_-(r+1) \) (see formula (2.7) in [17]).

Put \( T = \{t_{kj}\}_{k,j=0}^n = S^{-1}, \)
\[
Q = \{\hat{q}_{kj}\}_{k,j=0}^n = T\Pi_1\Pi_1^*T, \quad X = T\Phi_1, \quad Y = T\Phi_2,
\]
where \( t_{kj} \) and \( \hat{q}_{kj} \) are \( p \times p \) blocks of \( T \) and \( Q \), respectively. Similar to [15, 16, 20, 22] and references therein, we get the next proposition.

Proposition 5.2. Let \( S \in \Omega_n \). Then \( T = S^{-1} \) is recovered from \( X \) and \( Y \) by the formula
\[
t_{kj} = \hat{q}_{kj} + \hat{q}_{k+1,j+1} - \hat{q}_{k+1,j} - \hat{q}_{k,j+1} + t_{k+1,j+1},
\]
or, equivalently, by the formula
\[
t_{kj} = \hat{q}_{kj} + 2\sum_{r=1}^{n-k} \hat{q}_{k+r,j+r} - \sum_{r=1}^{n-k} \hat{q}_{k+r,j+r-1} - \sum_{r=1}^{n-k+1} \hat{q}_{k+r-1,j+r},
\]
where we fix \( t_{kj} = 0 \) and \( \hat{q}_{kj} = 0 \) for \( k > n \) or \( j > n \), and
\[
Q = XX^* + YY^*.
\]
The block vectors \( X \) and \( Y \) are connected by the relations
\[
\sum_{r=0}^{n}(X_r - X_r^*) = 0, \quad \sum_{r=0}^{n-k} X_{n-r} = \sum_{r=0}^{n-k} \hat{q}_{k+r,r} \quad (k \geq 0),
\]
\[
\sum_{r=0}^{n-k} X_{n-r}^* = \sum_{r=0}^{n-k} \hat{q}_{r,k+r} \quad (k > 0).
\]
Proof. From the identity (1.2) and formula (5.3), it follows that

\[(A^* - \lambda I_{(n+1)p})^{-1}T - T(A - \lambda I_{(n+1)p})^{-1}\]

(5.8)

where \(\hat{Q}\) satisfies (5.6). The identity \(TA - A^*T = i\hat{Q}\) yields (5.4), which, in its turn, implies (5.5).

To derive (5.7), we rewrite (5.8) in the form

\[(A^* - \lambda I_{(n+1)p})^{-1}X - X^*(A - \lambda I_{(n+1)p})^{-1}\Phi_1\]

(5.9)

\[= i\Phi_1^*(A^* - \lambda I_{(n+1)p})^{-1}\hat{Q}(A - \lambda I_{(n+1)p})^{-1}\Phi_1.\]

(5.10)

It is easily checked (see formula (1.10) in [17]) that

\[(A - \lambda I_{(n+1)p})^{-1}\Phi_1 = \left(\frac{i}{2} - \lambda\right)^{-1}\text{col}[I_p \ 0 \ \cdots \ 0],\]

(5.11)

\[\Phi_1^*(A^* - \lambda I_{(n+1)p})^{-1}X = i\left(\frac{i}{2} + \lambda\right)^{-1}[I_p \ 0 \ \cdots \ 0].\]

(5.12)

Notice that we have

\[\Phi_1^*T\Phi_1 = \Phi_1^*X = X^*\Phi_1,\]

(5.13)

which implies the first equality in (5.7) . Multiply both sides of (5.10) by \(\lambda^2 + \frac{1}{4}\) and use (5.11), (5.12) and the first equality in (5.7) to rewrite the result in the form

\[\frac{i}{\lambda - 1}\left(0 \ (\zeta - 1)I_p \ (\zeta^2 - 1)I_p \ \cdots \ (\zeta^n - 1)I_p\right)X\]

\[+X^*\text{col}[0 \ \zeta^{-1}(\zeta - 1)I_p \ \cdots \ \zeta^{-n}(\zeta^n - 1)I_p]
\]

(5.14)

\[= i[I_p \ 0 \ \cdots \ 0] \zeta^p I_p \ zeta^{-1}I_p \ \cdots \ zeta^{-n}I_p].\]

The equalities for the coefficients corresponding to the same degrees of \(\zeta\) on the left-hand side and on the right-hand side of (5.14) imply the second and the third relations in (5.7) .
6. Factorization and similarity conditions. The block matrix

\[
K = \begin{bmatrix}
K_0 \\
K_1 \\
\vdots \\
K_n
\end{bmatrix},
\]

(6.1)

where \(K_j\) are \(p \times (n+1)p\) matrices of the form

\[
K_j = i\beta(j)[\beta(0)^* \beta(1)^* \cdots \beta(j-1)^* \beta(j)^*/2 0 \cdots 0],
\]

(6.2)

plays an essential role in [18]. From the proof of Theorem 3.4 in [18] the following result is immediate.

**Proposition 6.1.** Let a \((n+1)p \times (n+1)p\) matrix \(K\) be given by formulas (6.1) and (6.2), and let conditions (3.13) hold. Then \(K\) is similar to \(A\):

\[
K = V_-AV_-^1,
\]

(6.3)

where \(V_-^\pm\) are block lower triangular matrices.

Proposition 6.1 is a discrete analog of the theorem on similarity to the integration operator [19].

**Remark 6.2.** Note that \(V_-^1\) can be chosen so that

\[
V_-^1 \begin{bmatrix}
\beta_1(0) \\
\vdots \\
\beta_1(n)
\end{bmatrix} = \Phi_1.
\]

(6.4)

Moreover, \(V_-^1\) is a factor of \(S\), i.e., \(S = V_-^1(V_*^1)^{-1} \in \Omega_n\). Any matrix \(S \in \Omega_n\) can be obtained in this way.

An analogue of Proposition 6.1 for the self-adjoint discrete Dirac system and block Toeplitz matrices \(S\) follows from the proof of Theorem 5.2 in [11].

**Proposition 6.3.** Let a \((n+1)p \times (n+1)p\) matrix \(K\) be given by formulas (6.1) and

\[
K_j = i\beta(j)J[\beta(0)^* \cdots \beta(j-1)^* \beta(j)^*/2 0 \cdots 0], \quad J = \begin{bmatrix}
0 & I_p \\
I_p & 0
\end{bmatrix},
\]

(6.5)

where \(\beta(k)\) are \(p \times 2p\) matrices. Let conditions \(\beta(k)J\beta(k)^* = I_p\ (0 \leq k \leq n)\) hold. Then \(K\) is similar to \(A\): \(K = V_-AV_-^1\), where \(V_-^\pm\) are block lower triangular matrices. Moreover, \(V_-\) can be chosen so that \(S = V_-^1(V_*^1)^{-1}\) is a block Toeplitz matrix.
Acknowledgment. The authors are grateful to Professor W. Schempp and GALA project for the opportunity to meet in Grossbothen and discuss this paper.

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