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AN UPPER BOUND ON THE REACHABILITY INDEX FOR A SPECIAL CLASS OF POSITIVE 2-D SYSTEMS

ESTEBAN BAILO†, JOSEP GELONCH†, AND SERGIO ROMERO-VIVÓ‡

Abstract. The smallest number of steps needed to reach all nonnegative local states of a positive two-dimensional (2-D) system is the local reachability index of the system. The study of such a number is still an open problem which seems to be a hard task. In this paper, an expression depending on the dimension $n$ as well as an upper bound on the local reachability index of a special class of systems are derived. Moreover, this reachability index is greater than any other bound proposed in previous literature.

Key words. Hurwitz products, Influence digraph, Local reachability index, Nonnegative matrices, Positive two-dimensional (2-D) systems, Reachability.

AMS subject classifications. 93C55, 15A48, 93B03.

1. Introduction. Positive two-dimensional systems have received considerable attention in the last decade, as they naturally arise in different physical problems such as the pollutant diffusion in a river (see [16]), gas absorption and water stream heating (see [25]), among others. The structural properties of these 2-D state-space models have been recently analyzed in [18], [21] and [22].

One of the most frequent representations of positive 2-D systems is the Fornasini-Marchesini state-space model (see [17] and [18]) which is as follows:

\[ x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1} \]

with local states $x_{\cdot, \cdot} \in \mathbb{R}_+^n$, inputs $u_{\cdot, \cdot} \in \mathbb{R}_+^m$, state matrices $A_1, A_2 \in \mathbb{R}_+^{n \times n}$, input matrices $B_1, B_2 \in \mathbb{R}_+^{n \times m}$ and initial global state $\chi_0 := \{x_{h,k} : (h,k) \in C_0\}$ being the separation set $C_0 := \{(h,k) : h, k \in \mathbb{Z}, h + k = 0\}$. Let us denote this system by $(A_1, A_2, B_1, B_2)$.

If any nonnegative local state is achieved from the zero initial global state choosing a nonnegative input sequence, then the system is said to be locally reachable. Local
reachability is equivalent to the possibility, in a finite number of steps, of attaining each vector of the standard basis of \( \mathbb{R}^n \) or alternatively a positive multiple of the aforementioned vectors (see [18]).

The smallest number of steps needed to reach all local states of the system is the local reachability index of that system. This index plays an important role because it enables us to determine in a finite number of steps whether a system is locally reachable or not. Moreover, once this index is obtained, several new open problems could be addressed such as the study of reachability algorithms (see [4], [11] and [24]), of a canonical form by means of an appropriated positive similarity ([5] and [7]), of the definition of the pertaining complete sequence of invariants (see [5], [8] and [10]) as well as the analysis of this property under feedback ([6] and [20]) and so on.

The reachability index for positive 1-D systems has been studied in [5], [9], [14], [15] and [26] among others, and it is always bounded by \( n \) (see [12] and [13]). However, despite the numerous research efforts, an upper bound on such a number for a positive 2-D system is still an unanswered question.

With regard to this subject, different studies such as [18] and [23] have provided the first results, which have been revised in [2] and [3]. Lately, in [1], the local reachability index has been characterized for a particular class of positive 2-D systems, which are a generalization of the systems presented in [2] and [3], and an upper bound on this index has been derived which even turns to be \( \frac{(n+1)^2}{4} \) in suitable conditions. However, it is worth mentioning that the obtained upper bound is valid for this class of chosen systems but not in general.

This work has been organized as follows: Section II presents some notations and basic definitions. Finally, in Section III, a special class of positive 2-D systems is shown to be always (positively) locally reachable. Moreover, an expression depending on the dimension \( n \) for the corresponding indices is deduced as well as its associated tight upper bound.

2. Notations and Preliminary Definitions. We denote by \( \lfloor z \rfloor \) the lower integer-part of \( z \in \mathbb{R} \) and by \( \text{col}_j(A) \) the \( j \)th column of the matrix \( A \).

**Definition 2.1.** The Hurwitz products of the \( n \times n \) matrices \( A_1 \) and \( A_2 \) are defined as follows:

- \( A_1^{i} \perp_{ij}^{j} A_2 = 0 \), when either \( i \) or \( j \) is negative,
- \( A_1^{i} \perp_{ij}^{j} A_2 = A_1^{i} \), if \( i \geq 0 \), \( A_1^{0} \perp_{ij}^{j} A_2 = A_2^{j} \), if \( j \geq 0 \),
- \( A_1^{i} \perp_{ij}^{j} A_2 = A_1(A_1^{i-1} \perp_{ij}^{j} A_2) + A_2(A_1^{i} \perp_{ij}^{j-1} A_2) \), if \( i, j > 0 \).

Note that \( \sum_{i+j=\ell} A_1^{i} \perp_{ij}^{j} A_2 = (A_1 + A_2)^{\ell} \).
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Definition 2.2. (see [18]) A 2-D space-state model (1.1) is (positively) locally reachable if, upon assuming $\chi_0 = 0$, for every state $x^* \in \mathbb{R}^n$, there exists $(h, k) \in \mathbb{Z} \times \mathbb{Z}$, $h + k > 0$, and a nonnegative input sequence $u_\cdot$ such that $x_{h,k} = x^*$. When so, the state is said to be (positively) reachable in $h + k$ steps. The smallest number of steps allowing to reach all nonnegative local states represents the local reachability index $I_{LR}$ of such a system.

Characterizations of the local reachability of the positive 2-D systems (1.1) can be established in terms of the reachability matrix of the quadruple $(A_1, A_2, B_1, B_2)$. The reachability matrix in $k$-steps is given by

$$\mathcal{R}_k = [B_1 B_2 A_1 B_1 A_1 B_2 + A_2 B_1 A_2 B_2 A_2^2 B_1 \cdots A_2^{k-1} B_2]$$

$$= [(A_1^{i-1} 1,2 \cdots A_2)B_1 + (A_1^{i-1} 1,2 \cdots A_2)B_2]_{i,j \geq 0, 0 < i + j \leq k}$$

where $k$ belongs to $\mathbb{N}$. It is known (see [18]) that the local reachability property holds if and only if there are $n$ pairs $(h_i, k_i) \in \mathbb{N} \times \mathbb{N}$, $i = 1, \ldots, n$, and $n$ indices $j = j(i) \in \{1, 2, \ldots, m\}$ such that $(A_1^{h_i-1} 1, 2 \cdots A_2)col_j(B_1) + (A_1^{h_i-1} 1, 2 \cdots A_2)col_j(B_2)$ is a positive $i$th monomial vector, that is, there exists $k \in \mathbb{N}$ such that $\mathcal{R}_k$ contains an $n \times n$ monomial matrix. We recall that a positive $i$th monomial vector (or simply $i$-monomial vector throughout this paper) is a positive multiple of the $i$th unit vector of $\mathbb{R}^n$. In the same way, a monomial matrix is a nonsingular matrix having a unique positive entry in each row and column.

To study the properties of the local reachability index we use digraph theory. Namely, we consider a family of coloured digraphs constructed from the matrices of the system $(A_1, A_2, B_1, B_2)$ as follows:

Definition 2.3. (see [18]) Associated with system (1.1), a directed digraph called 2-D influence digraph is defined. It is denoted by $D^{(2)}(A_1, A_2, B_1, B_2)$ and it is given by $(S, V, A_1, A_2, B_1, B_2)$, where $S = \{s_1, s_2, \ldots, s_m\}$ is the set of sources, $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices, $A_1$ and $A_2$ are subsets of $V \times V$ whose elements are named $A_1$-arcs and $A_2$-arcs (or simply 1-arcs and 2-arcs) respectively, while $B_1$ and $B_2$ are subsets of $S \times V$ whose elements are tagged $B_1$-arcs and $B_2$-arcs (or simply 1-arcs and 2-arcs) respectively. There is an $A_1$-arc ($A_2$-arc) from $v_j$ to $v_i$ if and only if the $(i, j)$th entry of $A_1$ ($A_2$) is nonzero. There is a $B_1$-arc ($B_2$-arc) from $s_t$ to $v_i$ if and only if the $(i, t)$th entry of $B_1$ ($B_2$) is nonzero.

Definition 2.4. A path in $D^{(2)}(A_1, A_2, B_1, B_2)$ from $v_{i_1}$ to $v_{i_p}$ is an alternating sequence of vertices and arcs $(v_{i_1}, (v_{i_1}, v_{i_2}), v_{i_2}, \ldots, (v_{i_{p-1}}, v_{i_p}), v_{i_p})$ such that $(v_{i_k}, v_{i_{k+1}}) \in A_1 \cup A_2 \cup B_1 \cup B_2$ for all $k = 1, 2, \ldots, p - 1$. A path is termed closed if the initial and final vertices coincide. In accordance with reference [18], an $s_t$-path is a path where $v_{i_1} = s_t$. The path length is defined to be equal to the number of arcs it contains, that is, $p + q$ being $p$ the number of 1-arcs (2-arcs) occurring in
the path $P$. Furthermore, the pair $(p, q)$ is called the composition of $P$. Finally, for abbreviation, a circuit is a closed path and if each vertex appears exactly once as the first vertex of an arc, then the circuit is said to be a cycle.

**Remark 2.5.** Let $P$ and $Q$ be two paths $\{v_{i_1}, (v_{i_1}, v_{i_2}), v_{i_2}, \ldots, (v_{i_p-1}, v_{i_p}), v_{i_p}\}$ and $\{v_{j_1}, (v_{j_1}, v_{j_2}), v_{j_2}, \ldots, (v_{j_q-1}, v_{j_q}), v_{j_q}\}$ such that $v_{i_p} = v_{j_1}$. To shorten, we denote the path from $v_{i_1}$ to $v_{j_q}$ consisting of the suitable adjacent joint of $P$ and $Q$ briefly by $P \cup Q$. Besides that, if $C$ is a cycle, from now on, $\eta C$ stands for the circuit resulting of doing $\eta$ laps around the cycle, $\eta$ being a positive integer.

**Remark 2.6.** In the sequel, to avoid ambiguities, we will write an 1-arc (2-arc) connecting two consecutive vertices $v_k$ and $v_{k+1}$ simply as $v_k \rightarrow v_{k+1} (v_k \rightarrow v_{k+1})$, that is, using continuous arrows (dashed arrows).

**Definition 2.7.** (see [18] and [19]) If there exists an $s_\ell$-path in $D^{(2)}(A_1, A_2, B_1, B_2)$ from the source $s_\ell$ to the vertex $v_i$, then $v_i$ is said to be reachable from $s_\ell$. Besides that, if for any $\ell \in \{1, \ldots, m\}$, all $s_\ell$-paths of composition $(p, q)$ end in the same vertex $v \in V$, then $v$ is said to be deterministically reachable from the source $s_\ell$ with composition $(p, q)$.

The shortest length of the $s_\ell$-paths deterministically reaching $v$ is called the $\ell$-index of $v$, i.e.,

$$I_\ell(v) = \min \{p + q | (p, q)\ell \text{ is the composition of an } s_\ell - \text{path deterministically reaching } v\}.$$ 

The determination index of $v$ is $I_D(v) = \min_{\ell=1,\ldots,m} \{I_\ell(v)\}$.

We observe that $I_D(v)$ stands for the minimum number of steps where a positive multiple of the $v$th vector of the standard basis of $\mathbb{R}^n$ is deterministically reachable. Therefore, taking into account that the system is locally reachable when each one of the $v$th vectors for any $v \in V$ is deterministically reachable (see [18]), then, obviously, the local reachability index $I_{LR}$ of a positive 2-D system is the maximum of the determination indices $v$ for all $v \in V$, that is, $I_{LR} = \max_{v \in V} \{I_D(v)\}$.

### 3. Local Reachability Index for a new family of systems.

In [3], the authors showed with the following example that the local reachability index can be greater than $\frac{(n+1)^2}{4}$.

**Example 3.1.** Let $(A_1, A_2, B_1, B_2)$ be a positive 2-D system given by

$$(A_1, A_2, B_1, B_2) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right), \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right), \left(\begin{array}{ccc} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right), \left(\begin{array}{ccc} 0 \\ 0 \\ 0 \end{array}\right)$$
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with a 2-D influence digraph corresponding to Fig. 3.1 where continuous arrows (dashed arrows) represent 1-arcs (2-arcs). Thus, the system is locally reachable with $I_{LR} = 6$, while $\frac{(n+1)^2}{4} = 4$ for $n = 3$.

\[ \begin{array}{ccc}
& \bullet & \\
\circ & \rightarrow & v_1 \\
& \downarrow & \rightarrow & v_2 \\
& & \rightarrow & v_3 \\
\end{array} \]

**Fig. 3.1. Digraph for the system in example 3.1**

On the basis of this example, we have obtained a new family of systems whose local reachability index largely exceeds the prior upper bound.

Let us consider an $n$th order positive 2-D system $(A_1, A_2, B_1, B_2)$ with $n \geq n_1 \geq 3$, $B_2 = O$, where $O$ is the zero matrix of an appropriate size, $B_1 = [\alpha e_1]$ with $\alpha > 0$ (that is, a 1-monomial column vector) and 2-D influence digraph given in Fig. 3.2 where the vertices $v_1, v_2, \ldots, v_n$ have been relabeled as the subindices $1, 2, \ldots, n$ to simplify. The preceding replacement is also carried out throughout this paper.

Therefore, the matrices of the system $(A_1, A_2, B_1, B_2)$ are given as follows:

\[(3.1) A_1 = \begin{bmatrix} \hat{A}'_{11} & \hat{A}'_{12} \\ O & O \end{bmatrix}, \quad A_2 = \begin{bmatrix} \hat{A}''_{11} & O \\ \hat{A}''_{21} & \hat{A}''_{22} \end{bmatrix}, \quad B_1 = [\alpha e_1] \quad \text{and} \quad B_2 = [O] \]

where

\[
\hat{A}'_{11} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & + & + & 0 \\
0 & 0 & \cdots & 0 & 0 & +
\end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}, \quad \hat{A}'_{12} = \begin{bmatrix}
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & + \\
\vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0
\end{bmatrix} \in \mathbb{R}^{n_1 \times (n-n_1)},
\]

\[
\hat{A}''_{11} = \begin{bmatrix}
0 & + & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & +
\end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}, \quad \hat{A}''_{21} = \begin{bmatrix}
+ & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix} \in \mathbb{R}^{(n-n_1) \times n_1},
\]
and

\[
\hat{A}_{22}' = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
+ & 0 & \cdots & 0 & 0 \\
0 & + & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & + & 0
\end{bmatrix} \in \mathbb{R}^{(n-n_1) \times (n-n_1)},
\]

+ denoting a strictly positive entry.

Notice that we may consider for the sake of simplicity all strictly positive entries equal to one. Furthermore, it is worth pointing out that we may assume that the digraph for this family of systems is reduced to be the digraph given in Fig. 3.3 in the case \( n = n_1 \). Then, we observe that the matrices of the system given in example 3.1 are structured as those seen above for \( n = 3 \) and \( n_1 = 3 \). It is convenient to establish now two preliminary technical lemmas in order to get an expression of the local reachability index for this class of systems.

Preliminarily, to simplify notation, we define the paths (see Fig. 3.2) \( \mathcal{P}_0 \equiv \{s \rightarrow 1\}, \mathcal{P}_1^k \equiv \{1 \rightarrow 2 \rightarrow \cdots \rightarrow k\} \) for \( 2 \leq k \leq n_1 - 1 \), \( \mathcal{P}_1 = \mathcal{P}_1^{n_1-1} \), \( \mathcal{P}_2^j \equiv \{1 \rightarrow (n_1+1) \rightarrow \cdots \rightarrow (n_1+j)\} \) for \( n_1+1 \leq j \leq n \), \( \mathcal{P}_2 = \mathcal{P}_2^n \), \( \mathcal{P}_3 \equiv \{n \rightarrow 2 \rightarrow \cdots \rightarrow 1\} \) and \( \mathcal{P}_4 \equiv \{(n_1 - 1) \rightarrow n_1\} \) whose compositions are \((1,0), (k-1,0), (n_1 - 2,0), (0,j-n_1), (0,n-n_1), (1,1)\) and \((1,0)\) respectively. In the same way, we define the cycles \( \mathcal{C}_1 \equiv \{1 \rightarrow 2 \rightarrow \rightarrow 1\}, \mathcal{C}_2 \equiv \{(n_1 - 1) \rightarrow (n_1 - 1)\}, \mathcal{C}_3 \equiv \{n_1 \rightarrow n_1\} \) and \( \mathcal{C}_4 = \mathcal{P}_2 \cup \mathcal{P}_3 \) with respective compositions \((1,1), (1,0), (0,1)\) and \((1,n-n_1+1)\).

**Lemma 3.2.** If there exists an \( s \)-path of composition \((p,q)\) ending in vertex 1,
then there is at least an \( s \)-path of composition \((p + \lambda, q + \delta)\) terminating at a vertex belonging to \( V \setminus \{n_1\} \) for every \( \lambda \) and \( \delta \in \mathbb{Z}_+ \) such that \( 0 \leq \delta \leq n - n_1 + \lambda \).

\textbf{Proof.} Firstly, assuming that we have an \( s \)-path \( P \) of composition \((p, q)\) ending in vertex 1, then the cycle \( C_1 \) allows us to affirm that for all \( \eta \in \mathbb{Z}_+ \) there is also an \( s \)-path of composition \((p + \eta, q + \eta)\) terminating at vertex 1 since this cycle adds \((\eta, \eta)\) to the composition of the initial path after doing \( \eta \) laps.

Combining these \( s \)-paths \( P \cup \eta C_1 \) with both the paths \( P^k \) for \( 2 \leq k \leq n_1 - 1 \) and the path \( P_1 \) and the cycle \( C_2 \), it is immediately seen that for all \( \alpha \in \mathbb{Z}_+ \) we can find \( s \)-paths of composition \((p + \eta + \alpha, q + \eta + \beta)\) ending in a vertex belonging to \( V_1 = \{1, 2, 3, \ldots, n_1 - 1\} \). Thus, taking \( \lambda = \eta + \alpha \) and \( \delta = \eta \), it is inferred that there are \( s \)-paths of composition \((p + \lambda, q + \delta)\) terminating at a vertex belonging to \( V_1 \) for every \( \lambda \in \mathbb{Z}_+ \) and \( 0 \leq \delta \leq \lambda \).

Consequently, if \( n_1 = n \geq 3 \), then the assertion of this theorem is established because, in this case, the inequality \( 0 \leq \delta \leq n - n_1 + \lambda \) is reduced to \( 0 \leq \delta \leq \lambda \) and \( V \setminus \{n_1\} = V_1 \).

Secondly, if \( n > n_1 \geq 3 \), the paths \( P^j \), \( n_1 + 1 \leq j \leq n \), allow us to claim that for all \( \beta \in \mathbb{Z}_+ \) satisfying \( 1 \leq \beta \leq n - n_1 \) there is an \( s \)-path of composition \((p + \eta, q + \eta + \beta)\) terminating at a vertex belonging to \( V_2 = \{n_1 + 1, n_1 + 2, \ldots, n\} \). Choosing \( \lambda = \eta \) and \( \delta = \eta + \beta \), we have an \( s \)-path of composition \((p + \lambda, q + \delta)\) ending in a vertex belonging to \( V_2 \) for all \( \lambda, \delta \in \mathbb{Z}_+ \) with \( \lambda + 1 \leq \delta \leq n - n_1 + \lambda \). Hence, since \( V \setminus \{n_1\} = V_1 \cup V_2 \) the statement of this lemma is easily derived from. \( \square \)

\textbf{Remark 3.3.} In the event that \( n > n_1 \), we stress that for a fixed value \( p \in \mathbb{N} \) the \( s \)-paths consisting exactly of \( p \)-\( 1 \)-arcs which terminate at vertex 1 are only those equal to \( P_0 \cup \alpha C_1 \cup \beta C_4 \) with \( \alpha + \beta + 1 = p \) (except for some other rearrangements of the same cycles, for instance, \( P_0 \cup C_4 \cup C_1 \cup (\beta - 1)C_4 \cup (\alpha - 1)C_1 \)). With respect to their lengths, the longest path is clearly \( P_0 \cup (p - 1)C_4 \). Then, \( P_0 \cup (p - 1)C_4 \cup P_2 \) is the greatest in length among those \( s \)-paths consisting precisely of \( p \)-\( 1 \)-arcs which end in a vertex belonging to \( V \setminus \{n_1\} \) and this vertex is \( n \).

Analogously, \( P_0 \cup (p - 1)C_1 \) is the longest \( s \)-path among those of composition consisting precisely of \( p \)-\( 1 \)-arcs which terminate at \( V \setminus \{n_1\} \), namely at vertex 1, when \( n = n_1 \).

\textbf{Lemma 3.4.} \textit{Let \( p \) be any natural number, then there is an \( s \)-path of composition \((p, \beta), \beta \in \mathbb{Z}_+ \), ending in a vertex belonging to \( V \setminus \{n_1\} \) if and only if \( 0 \leq \beta < (n - n_1 + 1)p \). Besides that, there is a unique \( s \)-path of composition \((p, (n - n_1 + 1)p - 1) \) terminating at a vertex belonging to \( V \setminus \{n_1\} \).

\textbf{Proof.} Case \( n > n_1 \geq 3 \): The proof is by induction on \( p \). Initially, Lemma 3.2
enables us to assert that we can find $s$-paths of composition $(1, \beta)$ with $0 \leq \beta < n - n_1 - 1$ ending in a vertex of $V \setminus \{n_1\}$ because the path $P_0$ terminates at vertex 1 and has composition $(1, 0)$. Conversely, according to the 2-D influence digraph given in Fig. 3.2, the $s$-paths of composition $(1, \beta)$ are only $P_0$ (with $\beta = 0$) and $P_0 \cup P_2^{n_1+\beta}$ where $1 \leq \beta \leq n - n_1$. We observe that each of them ends in a vertex belonging to $V \setminus \{n_1\}$ and in addition, for $\beta = n - n_1$, the corresponding $s$-path is exactly $P_0 \cup P_2$ whose final vertex is $n$. Hence, the assertion of this lemma for $p = 1$ is proven.

Let us assume that the same statement holds for $p = \alpha$, that is,

- There exist $s$-paths of composition $(\alpha, \beta)$ ending in a vertex of $V \setminus \{n_1\}$ if and only if

\begin{equation}
0 \leq \beta < (n - n_1 + 1)\alpha.
\end{equation}

- There is a unique $s$-path of composition $(\alpha, (n - n_1 + 1)\alpha - 1)$ ending in a vertex belonging to $V \setminus \{n_1\}$. In fact, this $s$-path is precisely $P_0 \cup (\alpha - 1)C_4 \cup P_2$ which terminates at vertex $n$.

Let us take now the case in which $p = \alpha + 1$. We know that for every $t \in \mathbb{N}$ there exist $s$-paths $P_0 \cup (t - 1)C_4$ of composition $(t, (t - 1)(n - n_1 + 1))$ terminating at vertex 1. In particular for each $t \in \mathbb{N}$ satisfying $1 \leq t \leq \alpha + 1$, we can apply successively Lemma 3.2 to each one of these paths, taking $p = t$, $q = (t - 1)(n - n_1 + 1)$ and setting $\lambda = \alpha + 1 - t$, that is, such that $p + \lambda = \alpha + 1$. Hence, we can find at least an $s$-path of composition $(\alpha + 1, \beta)$ ending in a vertex of $V \setminus \{n_1\}$ for each $\beta \in \mathbb{Z}_+$ such that $q \leq \beta \leq q + n - n_1 + \lambda$, so:

\begin{equation}
(t - 1)(n - n_1 + 1) \leq \beta \leq t(n - n_1 + 1) + \alpha - t.
\end{equation}

Plugging the different values of $t$ into equation (3.3), we can assure that there is at least an $s$-path of composition $(\alpha + 1, \beta)$ for each $0 \leq \beta < (n - n_1 + 1)(\alpha + 1)$ terminating at a vertex belonging to $V \setminus \{n_1\}$.

Let us suppose that there exists an $s$-path of composition $(\alpha + 1, \beta)$ ending in a vertex of $V \setminus \{n_1\}$ with $\beta \geq (n - n_1 + 1)(\alpha + 1)$ and that such a vertex is $v$. If $v \in V_1$ then we could construct an $s$-path of composition $(\alpha, \beta)$ for this same $\beta$ simply removing the last 1-arc of the initial path. This contradicts our inductive assumption. If $v \in V_2$ then $v = n_1 + k$ where $k \in \{1, \ldots, n - n_1\}$. Hence, the path constructed by eliminating the last $k$ 2-arcs of the starting path terminates at vertex 1. Moreover, it has composition $(\alpha + 1, \beta - k)$ with $\beta - k \geq (n - n_1 + 1)(\alpha + 1) - k > \alpha(n - n_1 + 1)$ but this is impossible since $P' := P_0 \cup \alpha C_4$ of composition $(\alpha + 1, \alpha(n - n_1 + 1))$ is the longest among those $s$-paths consisting of $\alpha + 1$ 1-arcs and ending in vertex 1 (see remark 3.3).
We also conclude from remark 3.3 that \( P' \sqcup P_2 = P_0 \sqcup \alpha C_4 \sqcup P_2 \) is the greatest in length among those \( s \)-paths consisting exactly of \( \alpha + 1 \) 1-arcs which terminate at vertex \( n \). Thus, the proof is complete.

**Case \( n = n_1 \geq 3 \):** The same conclusion can be drawn for this situation, with the \( s \)-paths \( P_0 \sqcup (p - 1)C_4 \) replaced by the \( s \)-paths \( P_0 \sqcup (p - 1)C_1 \). It deserves to be mentioned that the unique \( s \)-path of composition \((\alpha, (n - n_1)\alpha - 1) = (\alpha, \alpha - 1)\) ends in vertex 1 instead of vertex \( n \).

The following theorem provides us an expression of the reachability index for the specific class of systems we are considering in this paper.

**Theorem 3.5.** Let \((A_1, A_2, B_1, B_2)\) be a positive 2-D system given as in (3.1). Then, it is (positively) locally reachable and its local reachability index \( I_{LR} \) is:

\[
I_{LR} = n_1(n - n_1 + 2).
\]

**Proof.** Case \( n > n_1 \geq 3 \): Let us examine the deterministically reachable vertices \( v \) and their corresponding determination indices \( I_D(v) \). In particular, let us check that every vertex is deterministically reachable and that \( I_D(n_1) \) provides the maximum determination index, that is, \( I_D(n_1) = I_{LR} \).

We start by noting that we can find a unique path of composition \((\ell, 0)\) ending in \( \ell \), for all \( \ell \in V_1 = \{1, \ldots, n_1 - 1\} \). Then each \( \ell, \ell \in V_1 \), is deterministically reachable with composition \((\ell, 0)\). In fact,

\[
(A_1^{\ell - 1} \sqcup 0 A_2)B_1 = (A_1^{\ell - 1} \sqcup 0 A_2)\alpha e_1 = \beta e_\ell.
\]

where \( \beta > 0 \).

Likewise, every vertex \( \ell \) with \( \ell \in V_2 = \{n_1 + 1, \ldots, n\} \) is deterministically reachable with composition \((1, \ell - n_1)\) since

\[
(A_1^0 \sqcup \ell - n_1 A_2)B_1 = (A_1^0 \sqcup \ell - n_1 A_2)\alpha e_1 = \gamma e_\ell.
\]

where \( \gamma > 0 \). That is, there exists a unique path of composition \((1, \ell - n_1)\) terminating at \( \ell \), for all \( \ell \in V_2 \). Hence, obviously

\[
I_D(1) = 1, I_D(2) = 2, \ldots, I_D(n_1 - 1) = n_1 - 1, \quad I_D(n_1 + 1) = 2, I_D(n_1 + 2) = 3, \ldots, I_D(n) = n - n_1 + 1.
\]

Therefore, to complete the local reachability analysis of this system, it is necessary to analyze whether the vertex \( n_1 \) is deterministically reachable or not as well as to evaluate its determination index \( I_D(n_1) \) if possible. For this immediate purpose
we must calculate the shortest length among those s-paths of 2-D influence digraph deterministically reaching \( n_1 \).

In order to obtain an s-path ending in vertex \( n_1 \), we emphasize that an s-path may end in vertex \( n_1 \) only if it contains at least \( n_1 \) 1-arcs. Notice that the s-path \( P_0 \sqcup P_1 \sqcup P_4 \) of composition \((n_1,0)\) terminates at vertex \( n_1 \). Furthermore, in accordance with Lemma 3.4, there is an s-path of composition \((n_1 + \lambda, \beta)\) ending in \( V \setminus \{n_1\} \) if and only if \( 0 \leq \beta < (n_1 + \lambda)(n - n_1 + 1) \). Hence, if there exists an s-path of composition \((n_1 + \lambda, \beta)\) with \( \beta \geq (n_1 + \lambda)(n - n_1 + 1) \), then this one can terminate uniquely at vertex \( n_1 \). Note that it is straightforward that if there is an s-path of composition \((n_1, n_1(n - n_1 + 1))\) then it is the shortest among those s-paths of the 2-D influence digraph deterministically reaching \( n_1 \). Such an s-path of composition \((n_1, n_1(n - n_1 + 1))\) deterministically reaching \( n_1 \) is \( P_0 \sqcup P_1 \sqcup P_4 \sqcup (n_1(n - n_1 + 1))C_3 \). Thus, \( I_D(n_1) = n_1 + (n - n_1 + 1)n_1 = n_1(n - n_1 + 2) \). Consequently, all vertices are deterministically reachable and the system is locally reachable. It is obvious that the determination index \( I_D(n_1) \) is greater than the determination indices found in (3.3). Hence,

\[
I_{LR} = I_D(n_1) = n_1 + n_1(n - n_1 + 1) = n_1(n - n_1 + 2),
\]

which completes the proof.

Case \( n = n_1 \geq 3 \): The details are left to the reader since similar considerations as formerly indicated show that

\[
I_D(1) = 1, I_D(2) = 2, \ldots, I_D(n - 1) = n - 1, I_D(n) = I_D(n_1) = 2n.
\]

and hence the theorem follows.

From this result we conclude:

**Corollary 3.6.** The maximum local reachability index \( I_{LR} \) for a positive 2-D system given as in (3.1) for a natural number \( n_1 \) is attained at

\[
\begin{align*}
n_1 &= \frac{n}{2} + 1 & \text{if } n \text{ is an even natural number and} \\
n_1 &= \frac{n + 1}{2} \quad \text{or} \quad \frac{n + 3}{2} & \text{if } n \text{ is an odd natural number.}
\end{align*}
\]

Hence, the maximum \( I_{LR} \) is achieved for those systems satisfying \( n_1 = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and can be written uniformly like:

\[
I_{LR} = \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \left( n + 1 - \left\lfloor \frac{n}{2} \right\rfloor \right).
\]

Moreover, \( I_{LR} \) is upper bounded by \( \frac{(n+2)^2}{4} \).
An Upper Bound on the Reachability Index of a Class of Positive 2-D Systems

Proof. For a fixed natural number \( n \), we obtain that \( n_1 = \frac{(n+2)}{2} \) is the point of maximum of the function \( I_{LR} = I_D(n_1) = n_1(n - n_1 + 2) \), since its derivative with respect to \( n_1 \), \( \frac{dI_D(n_1)}{dn_1} \), is positive for \( n_1 < (n + 2)/2 \) and negative for \( n_1 > (n + 2)/2 \). Thus, if \( n \) is an even natural number then, \( n_1 = \frac{(n+2)}{2} = \frac{n}{2} + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1 \), but if \( n \) is an odd natural number then \( n_1 \) cannot take the value \( \frac{(n+2)}{2} \), since it is not a natural number and in this case, we deduce that the vertex \( n_1 \) has to be \( \frac{n}{2} + 1 \) or \( \frac{n+3}{2} = \left\lfloor \frac{n}{2} \right\rfloor + 2 \), studying the closest natural values in this parabola to the aforementioned one corresponding to \( n_1 \) which is the desired initial assertion. In fact,
\[
I_D\left(\frac{n+1}{2}\right) = \frac{n+1}{2}(n - n + 1 \cdot 2) = \frac{(n+1)(n+3)}{2} = \frac{n+3}{2}(n - n + 3 \cdot 2 + 2) = I_D\left(\frac{n+3}{2}\right).
\]

In addition, it is clear that if \( n \) is an even natural number, an upper bound on \( I_{LR} \) given in expression (3.7) is \( \frac{(n+2)^2}{4} \). In the same way, if \( n \) is an odd natural number, an upper bound on \( I_{LR} \) given in expression (3.7) is \( \frac{(n+1)(n+3)}{4} \). Hence \( I_{LR} \leq \frac{(n+2)^2}{4} \) since \( (n+2)^2 = (n+1)(n+3) + 1 \) which proves this corollary. 

<table>
<thead>
<tr>
<th>Dimension</th>
<th>( I_{LR} )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>10</th>
<th>11</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \left(\frac{n}{2} + 1\right)(n + 1 - \left\lfloor \frac{n}{2} \right\rfloor) )</td>
<td>9</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>36</td>
<td>42</td>
<td>72</td>
<td>121</td>
</tr>
</tbody>
</table>

Minimum values corresponding to the upper bounds on \( I_{LR} \) for any positive 2-D system.

Remark 3.7. As a consequence of the preceding corollary, we can state that an upper bound on \( I_{LR} \) for any (positively) locally reachable 2-D system must be greater than or equal to \( \frac{(n+2)^2}{4} \).

The study of an upper bound on the whole is still an open problem.

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REFERENCES


