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## THE ANTI-SYMMETRIC ORTHO-SYMMETRIC SOLUTIONS OF THE MATRIX EQUATION $A^T X A = D^*$

QING-FENG XIAO<sup>†</sup>, XI-YAN HU<sup>†</sup>, AND LEI ZHANG<sup>†</sup>

**Abstract.** In this paper, the following problems are discussed.

Problem I. Given matrices  $A \in R^{n \times m}$  and  $D \in R^{m \times m}$ , find  $X \in ASR_P^n$  such that  $A^T X A = D$ , where

$$ASR_P^n = \{X \in ASR^{n \times n} | PX \in SR^{n \times n} \text{ for given } P \in OR^{n \times n} \text{ satisfying } P^T = P\}.$$

Problem II. Given a matrix  $\tilde{X} \in R^{n \times n}$ , find  $\hat{X} \in S_E$  such that

$$\|\tilde{X} - \hat{X}\| = \inf_{X \in S_E} \|\tilde{X} - X\|,$$

where  $\|\cdot\|$  is the Frobenius norm, and  $S_E$  is the solution set of Problem I.

Expressions for the general solution of Problem I are derived. Necessary and sufficient conditions for the solvability of Problem I are provided. For Problem II, an expression for the solution is given as well.

**Key words.** Anti-symmetric ortho-symmetric matrix, Matrix equation, Matrix nearness problem, Optimal approximation, Least-square solutions.

**AMS subject classifications.** 65F15, 65F20.

**1. Introduction.** Let  $R^{n \times m}$  denote the set of all  $n \times m$  real matrices, and let  $OR^{n \times n}$ ,  $SR^{n \times n}$ ,  $ASR^{n \times n}$  denote the set of all  $n \times n$  orthogonal matrices, the set of all  $n \times n$  real symmetric matrices, the set of all  $n \times n$  real skew-symmetric matrices, respectively. The symbol  $I_k$  will stand for the identity matrix of order  $k$ ,  $A^+$  for the Moore-Penrose generalized inverse of a matrix  $A$ , and  $\text{rank}(A)$  for the rank of matrix  $A$ . For matrices  $A, B \in R^{n \times m}$ , the expression  $A * B$  will be the Hadamard product of  $A$  and  $B$ ; also  $\|\cdot\|$  will denote the Frobenius norm. Defining the inner product  $(A, B) = \text{tr}(B^T A)$  for matrices  $A, B \in R^{n \times m}$ ,  $R^{n \times m}$  becomes a Hilbert space. The norm of a matrix generated by this inner product is the Frobenius norm. If  $A = (a_{ij}) \in R^{n \times n}$ , let  $L_A = (l_{ij}) \in R^{n \times n}$  be defined as follows:  $l_{ij} = a_{ij}$  whenever  $i > j$  and  $l_{ij} = 0$  otherwise ( $i, j = 1, 2, \dots, n$ ). Let  $e_i$  be the  $i$ -th column of the identity matrix  $I_n$  ( $i = 1, 2, \dots, n$ ) and set  $S_n = (e_n, e_{n-1}, \dots, e_1)$ . It is easy to see that

$$S_n^T = S_n, \quad S_n^T S_n = I_n.$$

An inverse problem [2-6] arising in the structural modification of the dynamic behaviour of a structure calls for the solution of the matrix equation

$$(1.1) \quad A^T X A = D,$$

where  $A \in R^{n \times m}$ ,  $D \in R^{m \times m}$ , and the unknown  $X$  is required to be real and symmetric, and positive semidefinite or possibly definite. No assumption is made

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about the relative sizes of  $m$  and  $n$ , and it is assumed throughout that  $A \neq 0$  and  $D \neq 0$ .

Equation (1.1) is a special case of the matrix equation

$$(1.2) \quad AXB = C.$$

Consistency conditions for equation (1.2) were given by Penrose [7] (see also [1]). When the equation is consistent, a solution can be obtained using generalized inverses. Khatri and Mitra [8] gave necessary and sufficient conditions for the existence of symmetric and positive semidefinite solutions as well as explicit formulae using generalized inverses. In [9,10] solvability conditions for symmetric and positive definite solutions and general solutions of Equation (1.2) were obtained through the use of the generalized singular value decomposition [11-13].

For important results on the inverse problem  $A^T X A = D$  associated with several kinds of different sets  $S$ , for instance, symmetric matrices, symmetric nonnegative definite matrices, bisymmetric (same as persymmetric) matrices, bisymmetric nonnegative definite matrices and so on, we refer the reader to [14-17].

For the case the unknown  $A$  is anti-symmetric ortho-symmetric, [18] has discussed the inverse problem  $AX = B$ . However, for this case, the inverse problem  $A^T X A = D$  has not been dealt with yet. This problem will be considered here.

DEFINITION 1.1. A matrix  $P \in R^{n \times n}$  is said to be a *symmetric orthogonal matrix* if  $P^T = P$ ,  $P^T P = I_n$ .

In this paper, without special statement, we assume that  $P$  is a given symmetric orthogonal matrix.

DEFINITION 1.2. A matrix  $X \in R^{n \times n}$  is said to be a *anti-symmetric ortho-symmetric matrix* if  $X^T = -X$ ,  $(PX)^T = PX$ . We denote the set of all  $n \times n$  anti-symmetric ortho-symmetric matrices by  $ASR_p^n$ .

The problem studied in this paper can now be described as follows.

Problem I. Given matrices  $A \in R^{n \times m}$  and  $D \in R^{m \times m}$ , find an anti-symmetric ortho-symmetric matrix  $X$  such that

$$A^T X A = D.$$

In this paper, we discuss the solvability of this problem and an expression for its solution is presented.

The optimal approximation problem of a matrix with the above-given matrix restriction comes up in the processes of test or recovery of a linear system due to incomplete data or revising given data. A preliminary estimate  $\tilde{X}$  of the unknown matrix  $X$  can be obtained by the experimental observation values and the information of statistical distribution. The optimal estimate of  $X$  is a matrix  $\hat{X}$  that satisfies the given matrix restriction for  $X$  and is the best approximation of  $\tilde{X}$ , see [19-21].

In this paper, we will also consider the so-called optimal approximation problem associated with  $A^T X A = D$ . It reads as follows.

Problem II. Given matrix  $\tilde{X} \in R^{n \times n}$ , find  $\hat{X} \in S_E$  such that

$$\|\tilde{X} - \hat{X}\| = \inf_{X \in S_E} \|\tilde{X} - X\|,$$

where  $S_E$  is the solution set of Problem I.

We point out that if Problem I is solvable, then Problem II has a unique solution, and in this case an expression for the solution can be derived.

The paper is organized as follows. In Section 2, we obtain the general form of  $S_E$  and the sufficient and necessary conditions under which Problem I is solvable mainly by using the structure of  $ASR_P^n$  and orthogonal projection matrices. In Section 3, the expression for the solution of the matrix nearness problem II will be provided.

**2. The expression of the general solution of problem I.** In this section we first discuss some structure properties of symmetric orthogonal matrices. Then, given such a matrix  $P$ , we consider structural properties of the subset  $ASR_P^n$  of  $R^{n \times n}$ . Finally, we present necessary and sufficient conditions for the existence of and the expressions for the anti-symmetric ortho-symmetric (with respect to the given  $P$ ) solutions of problem I.

LEMMA 2.1. Assume  $P$  is a symmetric orthogonal matrix of size  $n$ , and let

$$(2.1) \quad P_1 = \frac{1}{2}(I_n + P), \quad P_2 = \frac{1}{2}(I_n - P).$$

Then  $P_1$  and  $P_2$  are orthogonal projection matrices satisfying  $P_1 + P_2 = I_n$ ,  $P_1 P_2 = 0$ .

*Proof.* By direct computation.  $\square$

LEMMA 2.2. Assume  $P_1$  and  $P_2$  are defined as (2.1) and  $\text{rank}(P_1) = r$ . Then  $\text{rank}(P_2) = n - r$ , and there exist unit column orthogonal matrices  $U_1 \in R^{n \times r}$  and  $U_2 \in R^{n \times (n-r)}$  such that

$$P_1 = U_1 U_1^T, \quad P_2 = U_2 U_2^T, \quad P = U_1 U_1^T - U_2 U_2^T, \quad U_1^T U_2 = 0.$$

*Proof.* Since  $P_1$  and  $P_2$  are orthogonal projection matrices satisfying  $P_1 + P_2 = I_n$ ,  $P_1 P_2 = 0$ , the column space  $R(P_2)$  of the matrix  $P_2$  is the orthogonal complement of the column space  $R(P_1)$  of the matrix  $P_1$ , in other words,  $R^n = R(P_1) \oplus R(P_2)$ . Hence, if  $\text{rank}(P_1) = r$ , then  $\text{rank}(P_2) = n - r$ . On the other hand,  $\text{rank}(P_1) = r$ ,  $\text{rank}(P_2) = n - r$ , and  $P_1, P_2$  are orthogonal projection matrices. Thus there exist unit column orthogonal matrices  $U_1 \in R^{n \times r}$  and  $U_2 \in R^{n \times (n-r)}$  such that  $P_1 = U_1 U_1^T$ ,  $P_2 = U_2 U_2^T$ . Using  $R^n = R(P_1) \oplus R(P_2)$ , we have  $U_1^T U_2 = 0$ . Substituting  $P_1 = U_1 U_1^T$ ,  $P_2 = U_2 U_2^T$  into (2.1), we have  $P = U_1 U_1^T - U_2 U_2^T$ .  $\square$

Elaborating on Lemma 2.2 and its proof, we note that  $U = (U_1, U_2)$  is an orthogonal matrix and that the symmetric orthogonal matrix  $P$  can be expressed as

$$(2.2) \quad P = U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^T.$$

LEMMA 2.3. The matrix  $X \in ASR_P^n$  if and only if  $X$  can be expressed as

$$(2.3) \quad X = U \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T,$$

where  $F \in R^{r \times (n-r)}$  and  $U$  is the same as (2.2).

*Proof.* Assume  $X \in ASR_P^n$ . By Lemma 2.2 and the definition of  $ASR_P^n$ , we have

$$P_1 X P_1 = \frac{I+P}{2} X \frac{I+P}{2} = \frac{1}{4}(X + PX + XP + PXP) = \frac{1}{4}(PX + XP),$$

$$P_2 X P_2 = \frac{I-P}{2} X \frac{I-P}{2} = \frac{1}{4}(X - PX - XP + PXP) = -\frac{1}{4}(PX + XP).$$

Hence,

$$\begin{aligned} X &= (P_1 + P_2)X(P_1 + P_2) = P_1XP_1 + P_1XP_2 + P_2XP_1 + P_2XP_2 \\ &= P_1XP_2 + P_2XP_1 = U_1U_1^T XU_2U_2^T + U_2U_2^T XU_1U_1^T. \end{aligned}$$

Let  $F = U_1^T XU_2$ ,  $G = U_2^T XU_1$ , it is easy to verify that  $F^T = -G$ . Then we have

$$X = U_1FU_2^T + U_2GU_1^T = U \begin{pmatrix} 0 & F \\ G & 0 \end{pmatrix} U^T = U \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T$$

Conversely, for any  $F \in R^{r \times (n-r)}$ , let

$$X = U \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T.$$

It is easy to verify that  $X^T = -X$ . Using (2.2), we have

$$\begin{aligned} PXP &= PU \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T P \\ &= U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^T U \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T U \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} U^T \\ &= U \begin{pmatrix} 0 & -F \\ F^T & 0 \end{pmatrix} U^T = -X. \end{aligned}$$

This implies that  $X = U \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T \in ASR_P^n$ .  $\square$

LEMMA 2.4. Let  $A \in R^{n \times n}$ ,  $D \in ASR^{n \times n}$  and assume  $A - A^T = D$ . Then there is precisely one  $G \in SR^{n \times n}$  such that

$$A = L_D + G,$$

and  $G = \frac{1}{2}(A + A^T) - \frac{1}{2}(L_D + L_D^T)$ .

*Proof.* For given  $A \in R^{n \times n}$ ,  $D \in ASR^{n \times n}$  and  $A - A^T = D$ , it is easy to verify that there exist unique  $G = \frac{1}{2}(A + A^T) - \frac{1}{2}(L_D + L_D^T) \in SR^{n \times n}$ , and we have  $A = \frac{1}{2}(A - A^T) + \frac{1}{2}(A + A^T) = \frac{1}{2}(L_D - L_D^T) + \frac{1}{2}(A + A^T) = L_D + \frac{1}{2}(A + A^T) - \frac{1}{2}(L_D + L_D^T) = L_D + G$ .  $\square$

Let  $A \in R^{n \times m}$  and  $D \in R^{m \times m}$ ,  $U$  defined in (2.2). Set

$$(2.4) \quad U^T A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \quad A_1 \in R^{r \times m}, \quad A_2 \in R^{(n-r) \times m}.$$

The generalized singular value decomposition (see [11,12,13]) of the matrix pair  $[A_1^T, A_2^T]$  is

$$(2.5) \quad A_1^T = M \Sigma_{A_1} W^T, \quad A_2^T = M \Sigma_{A_2} V^T,$$

where  $W \in C^{m \times m}$  is a nonsingular matrix,  $W \in OR^{r \times r}$ ,  $V \in OR^{(n-r) \times (n-r)}$  and

$$(2.6) \quad \Sigma_{A_1} = \begin{pmatrix} I_k & & & \\ & S_1 & & \\ & & O_1 & \\ & \dots & & \\ & & O & \end{pmatrix} \begin{matrix} k \\ s \\ t-k-s \\ m-t \end{matrix},$$

$$(2.7) \quad \Sigma_{A_2} = \begin{pmatrix} O_2 & & & \\ & S_2 & & \\ & & I_{t-k-s} & \\ & \dots & & \\ & & O & \end{pmatrix} \begin{matrix} k \\ s \\ t-k-s \\ m-t \end{matrix},$$

where

$$t = \text{rank}(A_1^T, A_2^T), \quad k = t - \text{rank}(A_2^T),$$

$$s = \text{rank}(A_1^T) + \text{rank}(A_2^T) - t$$

$$S_1 = \text{diag}(\alpha_1, \dots, \alpha_s), \quad S_2 = \text{diag}(\beta_1, \dots, \beta_s),$$

with  $1 > \alpha_1 \geq \dots \geq \alpha_s > 0$ ,  $0 < \beta_1 \leq \dots \leq \beta_s < 1$ , and  $\alpha_i^2 + \beta_i^2 = 1$ ,  $i = 1, \dots, s$ .  
 $O$ ,  $O_1$  and  $O_2$  are corresponding zero submatrices.

Then we can immediately obtain the following theorem about the general solution to Problem I.

**THEOREM 2.5.** *Given  $A \in R^{n \times m}$  and  $D \in R^{m \times m}$ ,  $U$  defined in (2.2), and  $U^T A$  has the partition form of (2.4), the generalized singular value decomposition of the matrix pair  $[A_1^T, A_2^T]$  as (2.5). Partition the matrix  $M^{-1} D M^{-T}$  as*

$$M^{-1} D M^{-T} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix} \begin{matrix} k \\ s \\ t-k-s \\ m-t \end{matrix},$$

$$\begin{matrix} k & s & t-k-s & m-t \end{matrix},$$

then the problem I has a solution  $X \in ASR_P^n$  if and only if

$$(2.8) \quad D^T = -D, D_{11} = O, D_{33} = O, D_{41} = O, D_{42} = O, D_{43} = O, D_{44} = O.$$

In that case it has the general solution

$$(2.9) \quad X = U \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T,$$

where

$$(2.10) \quad F = W \begin{pmatrix} X_{11} & D_{12} S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1} (L_{D_{22}} + G) S_2^{-1} & S_1^{-1} D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} V^T,$$

with  $X_{11} \in R^{r \times (n-r+k-t)}$ ,  $X_{21} \in R^{s \times (n-r+k-t)}$ ,  $X_{31} \in R^{(r-k-s) \times (n-r+k-t)}$ ,  $X_{32} \in R^{(r-k-s) \times s}$ ,  $X_{33} \in R^{(r-k-s) \times (t-k-s)}$  and  $G \in SR^{s \times s}$  are arbitrary matrices.

*Proof.* The necessity. Assume Eq.(1.1) has a solution  $X \in ASR_P^n$ . By the definition of  $ASR_P^n$ , it is easy to verify that  $D^T = -D$ , and we have from Lemma 2.3 that  $X$  can be expressed as

$$(2.11) \quad X = U \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} U^T.$$

where  $F \in R^{r \times (n-r)}$ .

Note that  $U$  is an orthogonal matrix, and the definition of  $A_i (i = 1, 2)$ , Eq.(1.1) is equivalent to

$$(2.12) \quad A_1^T F A_2 - A_2^T F A_1 = D.$$

Substituting (2.5) into (2.12), then we have

$$(2.13) \quad \Sigma_{A_1} (W^T F V) \Sigma_{A_2}^T - \Sigma_{A_2} (W^T F V)^T \Sigma_{A_1}^T = M^{-1} D M^{-T},$$

Partition the matrix  $W^T F V$  as

$$(2.14) \quad W^T F V = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix},$$

where  $X_{11} \in R^{r \times (n-r+k-t)}$ ,  $X_{22} \in R^{s \times s}$ ,  $X_{33} \in R^{(r-k-s) \times (t-k-s)}$ .

Taking  $W^T F V$  and  $M^{-1} D M^{-T}$  into (2.13), we have

$$(2.15) \quad \begin{pmatrix} 0 & X_{12} S_2 & X_{13} & 0 \\ -S_2 X_{21}^T & S_1 X_{22} S_2 - (S_1 X_{22} S_2)^T & S_1 X_{23} & 0 \\ -X_{13}^T & -X_{23}^T S_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{pmatrix}.$$

Therefore (2.15) holds if and only if (2.8) holds and

$$X_{12} = D_{12} S_2^{-1}, \quad X_{13} = D_{13}, \quad X_{23} = S_1^{-1} D_{23}$$

and

$$S_1 X_{22} S_2 - (S_1 X_{22} S_2)^T = D_{22}.$$

It follows from Lemma 2.4 that  $X_{22} = S_1^{-1} (L_{D_{22}} + G) S_2^{-1}$ , where  $G \in SR^{s \times s}$  is arbitrary matrix. Substituting the above into (2.14), (2.11), thus we have formulation (2.9) and (2.10).

The sufficiency. Let

$$F_G = W \begin{pmatrix} X_{11} & D_{12} S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1} (L_{D_{22}} + G) S_2^{-1} & S_1^{-1} D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} V^T.$$

Obviously,  $F_G \in R^{r \times (n-r)}$ . By Lemma 2.3 and

$$X_0 = U \begin{pmatrix} 0 & F_G \\ -F_G^T & 0 \end{pmatrix} U^T,$$

we have  $X_0 \in ASR_P^n$ . Hence

$$\begin{aligned} A^T X_0 A &= A^T U U^T X_0 U U^T A = \begin{pmatrix} A_1^T & A_2^T \end{pmatrix} \begin{pmatrix} 0 & F_G \\ -F_G & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \\ &= \begin{pmatrix} A_2^T (-F_G) & A_1^T F_G \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = A_2^T (-F_G) A_1 + A_1^T F_G A_2 \\ &= M \begin{pmatrix} 0 & D_{12} S_2^T & D_{13} & 0 \\ -S_2 D_{12}^T & L_{D_{22}} - \tilde{L}_{D_{22}}^T & D_{23} & 0 \\ -D_{13}^T & -D_{23}^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} M^T = D. \end{aligned}$$

This implies that  $X_0 = U \begin{pmatrix} 0 & F_G \\ -F_G^T & 0 \end{pmatrix} U^T \in ASR_P^n$  is the anti-symmetric ortho-symmetric solution of Eq. (1.1). The proof is completed.  $\square$

**3. The expression of the solution of Problem II.** To prepare for an explicit expression for the solution of the matrix nearness problem II, we first verify the following lemma.

LEMMA 3.1. *Suppose that  $E, F \in R^{s \times s}$ , and let  $S_a = \text{diag}(a_1, \dots, a_s) > 0$ ,  $S_b = \text{diag}(b_1, \dots, b_s) > 0$ . Then there exist a unique  $S_s \in SR^{s \times s}$  and a unique  $S_r \in ASR^{s \times s}$  such that*

$$(3.1) \quad \|S_a S S_b - E\|^2 + \|S_a S S_b - F\|^2 = \min$$

and

$$(3.2) \quad S_s = \Phi * [S_a(E + F)S_b + S_b(E + F)^T S_a],$$

$$(3.3) \quad S_r = \Phi * [S_a(E + F)S_b - S_b(E + F)^T S_a],$$

where

$$(3.4) \quad \Phi = (\psi_{ij}) \in SR^{s \times s}, \psi_{ij} = \frac{1}{2(a_i^2 b_j^2 + a_j^2 b_i^2)}, 1 \leq i, j \leq s.$$

*Proof.* We prove only the existence of  $S_r$  and (3.3). For any  $S = (s_{ij}) \in ASR^{s \times s}$ ,  $E = (e_{ij}), F = (f_{ij}) \in R^{s \times s}$ , since  $s_{ii} = 0, s_{ij} = -s_{ji}$ ,

$$\begin{aligned} \|S_a S S_b - E\|^2 + \|S_a S S_b - F\|^2 &= \sum_{1 \leq i, j \leq s} [(a_i b_j s_{ij} - e_{ij})^2 + (a_i b_j s_{ij} - f_{ij})^2] \\ &= \sum_{1 \leq i < j \leq s} [(a_i b_j s_{ij} - e_{ij})^2 + (-a_j b_i s_{ij} - e_{ji})^2 + (a_i b_j s_{ij} - f_{ij})^2 + (-a_j b_i s_{ij} - f_{ji})^2] + \\ &\quad + \sum_{1 \leq i \leq s} (e_{ij}^2 + e_{ij}^2). \end{aligned}$$

Hence, there exists a unique solution  $S_r = (s_{ij}) \in ASR^{s \times s}$  for (3.1) such that

$$s_{ij} = \frac{a_i b_j (e_{ij} + f_{ij}) - a_j b_i (e_{ji} + f_{ji})}{2(a_i^2 b_j^2 + a_j^2 b_i^2)}, \quad 1 \leq i, j \leq s.$$



This amounts to the same as (3.3).  $\square$

**THEOREM 3.2.** *Let  $\tilde{X} \in R^{n \times n}$ , the generalized singular value decomposition of the matrix pair  $[A_1^T, A_2^T]$  as (2.5), let*

$$(3.5) \quad U^T \tilde{X} U = \begin{pmatrix} Z_{11}^* & Z_{12}^* \\ Z_{21}^* & Z_{22}^* \end{pmatrix},$$

$$(3.6) \quad W^T Z_{12}^* V = \begin{pmatrix} X_{11}^* & X_{12}^* & X_{13}^* \\ X_{21}^* & X_{22}^* & X_{23}^* \\ X_{31}^* & X_{32}^* & X_{33}^* \end{pmatrix}, \quad W^T Z_{21}^* V = \begin{pmatrix} Y_{11}^* & Y_{12}^* & Y_{13}^* \\ Y_{21}^* & Y_{22}^* & Y_{23}^* \\ Y_{31}^* & Y_{32}^* & Y_{33}^* \end{pmatrix},$$

if Problem I is solvable, then Problem II has a unique solution  $\hat{X}$ , which can be expressed as

$$(3.7) \quad \hat{X} = U \begin{pmatrix} 0 & \tilde{F} \\ -\tilde{F}^T & 0 \end{pmatrix} U^T,$$

where

$$\tilde{F} = W \begin{pmatrix} \frac{1}{2}(X_{11}^* - Y_{11}^*) & D_{12} S_2^{-1} & D_{13} \\ \frac{1}{2}(X_{21}^* - Y_{21}^*) & S_1^{-1}(L_{D_{22}} + \tilde{G}) S_2^{-1} & S_1^{-1} D_{23} \\ \frac{1}{2}(X_{31}^* - Y_{31}^*) & \frac{1}{2}(X_{32}^* - Y_{32}^*) & \frac{1}{2}(X_{33}^* - Y_{33}^*) \end{pmatrix} V^T,$$

$$\tilde{G} = \Phi * [S_1^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1} L_{D_{22}} S_2^{-1}) S_2^{-1} + S_2^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1} L_{D_{22}} S_2^{-1})^T S_1^{-1}],$$

with

$$\Phi = (\psi_{ij}) \in SR^{s \times s}, \quad \psi_{ij} = \frac{a_i^2 a_j^2 b_i^2 b_j^2}{2(a_i^2 b_j^2 + a_j^2 b_i^2)}, \quad 1 \leq i, j \leq s.$$

*Proof.* Using the invariance of the Frobenius norm under unitary transformations, from (2.9), (3.5) and (3.6) we have

$$\begin{aligned} \|X - \tilde{X}\|^2 &= \|Z_{11}^*\|^2 + \|F - Z_{12}^*\|^2 + \|-F^T - Z_{21}^*\|^2 + \|Z_{22}^*\|^2 \\ &= \left\| \begin{pmatrix} X_{11} & D_{12} S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1}(L_{D_{22}} + G) S_2^{-1} & S_1^{-1} D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} - W^T Z_{12}^* V \right\|^2 + \|Z_{11}^*\|^2 \\ &+ \left\| \begin{pmatrix} X_{11} & D_{12} S_2^{-1} & D_{13} \\ X_{21} & S_1^{-1}(L_{D_{22}} + G) S_2^{-1} & S_1^{-1} D_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} + W^T Z_{21}^* V \right\|^2 + \|Z_{22}^*\|^2. \end{aligned}$$

Thus

$$\|\tilde{X} - \hat{X}\| = \inf_{X \in S_E} \|\tilde{X} - X\|$$

is equivalent to

$$\|X_{11} - X_{11}^*\|^2 + \|X_{11} + Y_{11}^*\|^2 = \min, \quad \|X_{21} - X_{21}^*\|^2 + \|X_{21} + Y_{21}^*\|^2 = \min,$$

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$$\| X_{31} - X_{31}^* \|^2 + \| X_{31} + Y_{31}^* \|^2 = \min, \quad \| X_{32} - X_{32}^* \|^2 + \| X_{32} + Y_{32}^* \|^2 = \min,$$

$$\| X_{33} - X_{33}^* \|^2 + \| X_{33} + Y_{33}^* \|^2 = \min,$$

$$\| S_1^{-1} G S_2^{-1} - (X_{22}^* - S_1^{-1} L_{D_{22}} S_2^{-1}) \|^2 + \| S_1^{-1} G S_2^{-1} + (Y_{22}^* + S_1^{-1} L_{D_{22}} S_2^{-1}) \|^2 = \min.$$

From Lemma 3.1 we have  $X_{11} = \frac{1}{2}(X_{11}^* - Y_{11}^*)$ ,  $X_{21} = \frac{1}{2}(X_{21}^* - Y_{21}^*)$ ,  $X_{31} = \frac{1}{2}(X_{31}^* - Y_{31}^*)$ ,  $X_{32} = \frac{1}{2}(X_{32}^* - Y_{32}^*)$ ,  $X_{33} = \frac{1}{2}(X_{33}^* - Y_{33}^*)$  and

$$G = \Phi * [S_1^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1}L_{D_{22}}S_2^{-1})S_2^{-1} + S_2^{-1}(X_{22}^* - Y_{22}^* - 2S_1^{-1}L_{D_{22}}S_2^{-1})^T S_1^{-1}].$$

Taking  $X_{11}, X_{21}, X_{31}, X_{32}, X_{33}$  and  $G$  into (2.9), (2.10), we obtain that the solution of (the matrix nearness) Problem II can be expressed as (3.7).  $\square$

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