

2009

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Recommended Citation

Rojo, Oscar and Medina, Luis. (2009), "Spectra of weighted compound graphs of generalized Bethe trees", *Electronic Journal of Linear Algebra*, Volume 18.
DOI: <https://doi.org/10.13001/1081-3810.1292>

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SPECTRA OF WEIGHTED COMPOUND GRAPHS OF GENERALIZED BETHE TREES*

OSCAR ROJO[†] AND LUIS MEDINA[†]

Abstract. A generalized Bethe tree is a rooted tree in which vertices at the same distance from the root have the same degree. Let \mathcal{G}_m be a connected weighted graph on m vertices. Let $\{\mathcal{B}_i : 1 \leq i \leq m\}$ be a set of trees such that, for $i = 1, 2, \dots, m$,

(i) \mathcal{B}_i is a generalized Bethe tree of k_i levels,

(ii) the vertices of \mathcal{B}_i at the level j have degree d_{i,k_i-j+1} for $j = 1, 2, \dots, k_i$, and

(iii) the edges of \mathcal{B}_i joining the vertices at the level j with the vertices at the level $(j + 1)$ have weight w_{i,k_i-j} for $j = 1, 2, \dots, k_i - 1$.

Let $\mathcal{G}_m \{\mathcal{B}_i : 1 \leq i \leq m\}$ be the graph obtained from \mathcal{G}_m and the trees $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_m$ by identifying the root vertex of \mathcal{B}_i with the i th vertex of \mathcal{G}_m . A complete characterization is given of the eigenvalues of the Laplacian and adjacency matrices of $\mathcal{G}_m \{\mathcal{B}_i : 1 \leq i \leq m\}$ together with results about their multiplicities. Finally, these results are applied to the particular case $\mathcal{B}_1 = \mathcal{B}_2 = \dots = \mathcal{B}_m$.

Key words. Weighted graph, Generalized Bethe tree, Laplacian matrix, Adjacency matrix, Spectral radius, Algebraic connectivity.

AMS subject classifications. 5C50, 15A48.

1. Introduction. Let $\mathcal{G} = (V, E)$ be a simple graph with vertex set V and edge set E . A weighted graph \mathcal{G} is a graph in which each edge $e \in E$ has a positive weight $w(e)$. Labelling the vertices of \mathcal{G} by $1, 2, \dots, n$, the Laplacian matrix of \mathcal{G} is the $n \times n$ matrix $L(\mathcal{G}) = (l_{i,j})$ defined by

$$l_{i,j} = \begin{cases} -w(e) & \text{if } i \neq j \text{ and } e \text{ is the edge joining } i \text{ and } j \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j \\ -\sum_{k \neq i} l_{i,k} & \text{if } i = j \end{cases}$$

and the adjacency matrix of \mathcal{G} is the $n \times n$ matrix $A(\mathcal{G}) = (a_{i,j})$ defined by

$$a_{i,j} = \begin{cases} w(e) & \text{if } i \neq j \text{ and } e \text{ is the edge joining } i \text{ and } j \\ 0 & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j \\ 0 & \text{if } i = j \end{cases} .$$

*Received by the editors June 30, 2008. Accepted for publication December 30, 2008. Handling Editor: Michael Neumann.

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$L(\mathcal{G})$ and $A(\mathcal{G})$ are both real symmetric matrices. From this fact and Geršgorin's Theorem, it follows that the eigenvalues of $L(\mathcal{G})$ are nonnegative real numbers. Moreover, since its rows sum to 0, $(0, \mathbf{e})$ is an eigenpair of $L(\mathcal{G})$, where \mathbf{e} is the all ones vector. Fiedler [1] proved that \mathcal{G} is a connected graph if and only if the second smallest eigenvalue of $L(\mathcal{G})$ is positive. This eigenvalue is called the algebraic connectivity of \mathcal{G} .

If $w(e) = 1$ for all $e \in E$, then \mathcal{G} is an unweighted graph. In [4], some of the many results known for the Laplacian matrix of an unweighted graph are given.

A generalized Bethe tree is a rooted tree in which vertices at the same distance from the root have the same degree. In [5], one can find a complete characterization of the spectra of the Laplacian matrix and adjacency matrix of such class of trees.

Let $\{\mathcal{B}_i : 1 \leq i \leq m\}$ be a set of trees such that

(i) \mathcal{B}_i is a generalized Bethe tree of k_i levels,

(ii) the vertices of \mathcal{B}_i at the level j have degree d_{i,k_i-j+1} for $j = 1, 2, \dots, k_i$, and

(iii) the edges of \mathcal{B}_i joining the vertices at the level j with the vertices at the level $(j+1)$ have weight w_{i,k_i-j} for $j = 1, 2, \dots, k_i - 1$.

Let \mathcal{G}_m be a connected weighted graph on m vertices v_1, v_2, \dots, v_m . As usual $v_i \sim v_j$ means that v_i and v_j are adjacent. Let $\varepsilon_{i,j} = \varepsilon_{j,i}$ be the weight of the edge $v_i v_j$ if $v_i \sim v_j$ and let $\varepsilon_{i,j} = \varepsilon_{j,i} = 0$ otherwise.

In this paper, we characterize completely the spectra of the Laplacian and adjacency matrices of the graph $\mathcal{G}_m \{\mathcal{B}_i : 1 \leq i \leq m\}$ obtained from \mathcal{G}_m and the trees $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_m$ by identifying the root vertex of \mathcal{B}_i with v_i . In particular, we apply the results to the case $\mathcal{B}_1 = \mathcal{B}_2 = \dots = \mathcal{B}_m$.

From now on, we write $\mathcal{G}_m \{\mathcal{B}_i\}$ instead of $\mathcal{G}_m \{\mathcal{B}_i : 1 \leq i \leq m\}$.

For $j = 1, 2, 3, \dots, k_i$, let n_{i,k_i-j+1} be the number of vertices at the level j of \mathcal{B}_i . Observe that $n_{i,k_i} = 1$ and $n_{i,k_i-1} = d_{i,k_i}$. We have

$$(1.1) \quad n_{i,k_i-j} = (d_{i,k_i-j+1} - 1) n_{i,k_i-j+1}, \quad 2 \leq j \leq k_i - 1.$$

For $i = 1, 2, \dots, m$, let $d(v_i)$ be the degree of v_i as a vertex of \mathcal{G}_m . The total number of vertices in $\mathcal{G}_m \{\mathcal{B}_i\}$ is $n = \sum_{i=1}^m \sum_{j=1}^{k_i-1} n_{i,j} + m$.

For $i = 1, 2, \dots, m$, let

$$\delta_{i,1} = w_{i,1}, \quad \delta_{i,j} = (d_{i,j} - 1) w_{i,j-1} + w_{i,j} \quad (j = 2, 3, \dots, k_i - 1)$$

$$\delta_{i,k_i} = d_{i,k_i} w_{i,k_i-1} \quad \text{and} \quad \delta_i = \sum_{v_i \sim v_j} \varepsilon_{i,j}.$$

Observe that if \mathcal{B}_i is an unweighted tree, then $\delta_{i,j} = d_{i,j}$, and if \mathcal{G}_m is an unweighted graph, then $\delta_i = d(v_i)$.

We introduce the following additional notation:

$|A|$ is the determinant of A .

0 and I are the all zeros matrix and the identity matrix of appropriate sizes, respectively.

I_r is the identity matrix of size $r \times r$.

\mathbf{e}_r is the all ones column vector of dimension r .

For $1 \leq i \leq m$ and $1 \leq j \leq k_i - 2$, $m_{i,j} = \frac{n_{i,j}}{n_{i,j+1}}$ and $C_{i,j}$ is the block diagonal matrix defined by

$$C_{i,j} = \text{diag} \{ \mathbf{e}_{m_{i,j}}, \mathbf{e}_{m_{i,j}}, \dots, \mathbf{e}_{m_{i,j}} \}$$

with $n_{i,j+1}$ diagonal blocks. The size of $C_{i,j}$ is $n_{i,j} \times n_{i,j+1}$.

For $1 \leq i \leq m$, let $s_i = \sum_{j=1}^{k_i-2} n_{i,j}$ and E_i be the $s_i \times m$ matrix defined by

$$E_i(p, q) = \begin{cases} 1 & \text{if } q = i \text{ and } s_i + 1 \leq p \leq s_i + n_{i,k_i-1} \\ 0 & \text{elsewhere} \end{cases} .$$

We label the vertices of $\mathcal{G}_m \{ \mathcal{B}_i \}$ as follows:

1. Using the labels $1, 2, \dots, \sum_{j=1}^{k_1-1} n_{1,j}$, we label the vertices of \mathcal{B}_1 from the bottom to level 2 and, at each level, in a counterwise sense.
2. Using the labels $\sum_{j=1}^{k_1-1} n_{1,j} + 1, \dots, \sum_{j=1}^{k_1-1} n_{1,j} + \sum_{j=1}^{k_2-1} n_{2,j}$, we label the vertices of \mathcal{B}_2 from the bottom to level 2 and, at each level, in a counterwise sense.
3. We continue labelling the vertices of $\mathcal{B}_3, \mathcal{B}_4, \dots, \mathcal{B}_m$, in this order, as above.
4. Finally, using the labels $n - m + 1, n - m + 2, \dots, n$, we label the vertices of \mathcal{G}_m .

Thus, the adjacency matrix $A(\mathcal{G}_m \{ \mathcal{B}_i \})$ and the matrix $L(\mathcal{G}_m \{ \mathcal{B}_i \})$ become

$$(1.2) \quad A(\mathcal{G}_m \{ \mathcal{B}_i \}) = \begin{bmatrix} A_1 & 0 & \cdots & 0 & w_{1,k_1-1} E_1 \\ 0 & A_2 & \cdots & 0 & w_{2,k_2-1} E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m & w_{m,k_m-1} E_m \\ w_{1,k_1-1} E_1^T & w_{2,k_2-1} E_2^T & \cdots & w_{m,k_m-1} E_m^T & A(\mathcal{G}_m) \end{bmatrix}$$

and

$$(1.3) \quad L(\mathcal{G}_m \{\mathcal{B}_i\}) = \begin{bmatrix} L_1 & 0 & \cdots & 0 & -w_{1,k_1-1}E_1 \\ 0 & L_2 & \cdots & 0 & -w_{2,k_2-1}E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & L_m & -w_{m,k_m-1}E_m \\ -w_{1,k_1-1}E_1^T & -w_{2,k_2-1}E_2^T & \cdots & -w_{m,k_m-1}E_m^T & L_{m+1} \end{bmatrix}$$

where, for $i = 1, 2, \dots, m$, A_i and L_i are the following block tridiagonal matrices:

$$(1.4) \quad A_i = \begin{bmatrix} 0 & w_{i,1}C_{i,1} & & & \\ w_{i,1}C_{i,1}^T & 0 & w_{i,2}C_{i,2} & & \\ & w_{i,2}C_{i,2}^T & \ddots & & \\ & & \ddots & 0 & w_{i,k_i-2}C_{i,k_i-2} \\ & & & w_{i,k_i-2}C_{i,k_i-2}^T & 0 \end{bmatrix}$$

and

$$(1.5) \quad L_i = \begin{bmatrix} \delta_{i,1}I_{n_{i,1}} & -w_{i,1}C_{i,1} & & & \\ -w_{i,1}C_{i,1}^T & \delta_{i,2}I_{n_{i,2}} & -w_{i,2}C_{i,2} & & \\ & -w_{i,2}C_{i,2}^T & \ddots & & \\ & & \ddots & \delta_{i,k_i-2}I_{n_{i,k_i-2}} & -w_{i,k_i-2}C_{i,k_i-2} \\ & & & -w_{i,k_i-2}C_{i,k_i-2}^T & \delta_{i,k_i-1}I_{n_{i,k_i-1}} \end{bmatrix}$$

Moreover,

$$(1.6) \quad A(\mathcal{G}_m) = \begin{bmatrix} 0 & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,m} \\ \varepsilon_{1,2} & 0 & \varepsilon_{2,3} & \cdots & \varepsilon_{2,m} \\ \varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & 0 & \varepsilon_{m-1,m} \\ \varepsilon_{1,m} & \varepsilon_{2,m} & \cdots & \varepsilon_{m-1,m} & 0 \end{bmatrix}$$

and

$$(1.7) \quad L_{m+1} = \begin{bmatrix} \delta_{1,k_1} + \delta_1 & -\varepsilon_{1,2} & -\varepsilon_{1,3} & \cdots & -\varepsilon_{1,m} \\ -\varepsilon_{1,2} & \delta_{2,k_2} + \delta_2 & -\varepsilon_{2,3} & \cdots & -\varepsilon_{2,m} \\ -\varepsilon_{1,3} & -\varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \delta_{m-1,k_{m-1}} + \delta_{m-1} & -\varepsilon_{m-1,m} \\ -\varepsilon_{1,m} & -\varepsilon_{2,m} & \cdots & -\varepsilon_{m-1,m} & \delta_{m,k_m} + \delta_m \end{bmatrix}$$

2. Preliminaries.

LEMMA 2.1. *Let*

$$X = \begin{bmatrix} X_1 & 0 & \cdots & 0 & \pm w_{1,k_1-1}E_1 \\ 0 & X_2 & \cdots & 0 & \pm w_{2,k_2-1}E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X_m & \pm w_{m,k_m-1}E_m \\ \pm w_{1,k_1-1}E_1^T & \pm w_{2,k_2-1}E_2^T & \cdots & \pm w_{m,k_m-1}E_m^T & X_{m+1} \end{bmatrix}$$

where, for $i = 1, 2, \dots, m$, X_i is the block tridiagonal matrix

$$X_i = \begin{bmatrix} \alpha_{i,1}I_{n_{i,1}} & \pm w_{i,1}C_{i,1} & & & \\ \pm w_{i,1}C_{i,1}^T & \alpha_{i,2}I_{n_{i,2}} & \pm w_{i,2}C_{i,2} & & \\ & \pm w_{i,2}C_{i,2}^T & \ddots & & \\ & & \ddots & \ddots & \\ & & & \alpha_{i,k_i-2}I_{n_{i,k_i-2}} & \pm w_{i,k_i-2}C_{i,k_i-2} \\ & & & \pm w_{i,k_i-2}C_{i,k_i-2}^T & \alpha_{i,k_i-1}I_{n_{i,k_i-1}} \end{bmatrix},$$

and

$$(2.1) \quad X_{m+1} = \begin{bmatrix} \alpha_1 & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,m} \\ \varepsilon_{1,2} & \alpha_2 & \varepsilon_{2,3} & \cdots & \varepsilon_{2,m} \\ \varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \alpha_{m-1} & \varepsilon_{m-1,m} \\ \varepsilon_{1,m} & \varepsilon_{2,m} & \cdots & \varepsilon_{m-1,m} & \alpha_m \end{bmatrix}$$

or

$$(2.2) \quad X_{m+1} = \begin{bmatrix} \alpha_1 & -\varepsilon_{1,2} & -\varepsilon_{1,3} & \cdots & -\varepsilon_{1,m} \\ -\varepsilon_{1,2} & \alpha_2 & -\varepsilon_{2,3} & \cdots & -\varepsilon_{2,m} \\ -\varepsilon_{1,3} & -\varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \alpha_{m-1} & -\varepsilon_{m-1,m} \\ -\varepsilon_{1,m} & -\varepsilon_{2,m} & \cdots & -\varepsilon_{m-1,m} & \alpha_m \end{bmatrix},$$

respectively. For $i = 1, 2, \dots, m$, let

$$\begin{aligned} \beta_{i,1} &= \alpha_{i,1}, \\ \beta_{i,j} &= \alpha_{i,j} - \frac{n_{i,j-1} w_{i,j-1}^2}{n_{i,j} \beta_{i,j-1}}, \quad j = 2, 3, \dots, k_i - 1, \\ \beta_i &= \alpha_i - n_{i,k_i-1} \frac{w_{i,k_i-1}^2}{\beta_{i,k_i-1}}. \end{aligned}$$

If $\beta_{i,j} \neq 0$ for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k_i - 1$, then

$$(2.3) \quad |X| = \left(\prod_{i=1}^m \prod_{j=1}^{k_i-1} \beta_{i,j}^{n_{i,j}} \right) |Y_{m+1}|,$$

where

$$(2.4) \quad Y_{m+1} = \begin{bmatrix} \beta_1 & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,m} \\ \varepsilon_{1,2} & \beta_2 & \varepsilon_{2,3} & \cdots & \varepsilon_{2,m} \\ \varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \beta_{m-1} & \varepsilon_{m-1,m} \\ \varepsilon_{1,m} & \varepsilon_{2,m} & \cdots & \varepsilon_{m-1,m} & \beta_m \end{bmatrix}$$

or

$$(2.5) \quad Y_{m+1} = \begin{bmatrix} \beta_1 & -\varepsilon_{1,2} & -\varepsilon_{1,3} & \cdots & -\varepsilon_{1,m} \\ -\varepsilon_{1,2} & \beta_2 & -\varepsilon_{2,3} & \cdots & -\varepsilon_{2,m} \\ -\varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \beta_{m-1} & -\varepsilon_{m-1,m} \\ -\varepsilon_{1,m} & -\varepsilon_{2,m} & \cdots & -\varepsilon_{m-1,m} & \beta_m \end{bmatrix},$$

respectively.

Proof. We give a proof for X_{m+1} in (2.1). Suppose $\beta_{1,j} \neq 0$ for all $j = 1, 2, \dots, k_1 - 1$. After some steps of the Gaussian elimination procedure, without row interchanges, we reduce X to the intermediate matrix

$$\begin{bmatrix} R_1 & 0 & \cdots & 0 & \pm w_{1,k_1-1} E_1 \\ 0 & X_2 & \cdots & 0 & \pm w_{2,k_2-1} E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X_m & \pm w_{m,k_m-1} E_m \\ 0 & \pm w_{2,k_2-1} E_2^T & \cdots & \pm w_{m,k_m-1} E_m^T & X_{m+1}^{(1)} \end{bmatrix},$$

where R_1 is the block bidiagonal matrix

$$R_1 = \begin{bmatrix} \beta_{1,1} I_{n_{1,1}} & \pm w_{1,1} C_{1,1} & & & \\ & \beta_{1,2} I_{n_{1,2}} & \ddots & & \\ & & \ddots & \pm w_{1,k_1-2} C_{1,k_1-2} & \\ & & & & \beta_{1,k_1-1} I_{n_{1,k_1-1}} \end{bmatrix}$$

and $X_{m+1}^{(1)}$ is the matrix

$$X_{m+1}^{(1)} = \begin{bmatrix} \beta_1 & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,m} \\ \varepsilon_{1,2} & \alpha_2 & \varepsilon_{2,3} & \cdots & \varepsilon_{2,m} \\ \varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \alpha_{m-1} & \varepsilon_{m-1,m} \\ \varepsilon_{1,m} & \varepsilon_{2,m} & \cdots & \varepsilon_{m-1,m} & \alpha_m \end{bmatrix}.$$

Suppose, in addition, that $\beta_{i,j} \neq 0$ for all $i = 2, 3, \dots, m$ and $j = 1, 2, \dots, k_i - 1$. We continue the Gaussian elimination procedure to finally obtain the upper triangular matrix

$$\begin{bmatrix} R_1 & 0 & \cdots & 0 & \pm w_{1,k_1-1}E_1 \\ 0 & R_2 & \cdots & 0 & \pm w_{2,k_2-1}E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & R_m & \pm w_{m,k_m-1}E_m \\ 0 & 0 & \cdots & 0 & Y_{m+1} \end{bmatrix},$$

where, for $i = 2, 3, \dots, m$,

$$R_i = \begin{bmatrix} \beta_{i,1}I_{n_{i,1}} & \pm w_{i,1}C_{i,1} & & & \\ & \beta_{i,2}I_{n_{i,2}} & \ddots & & \\ & & \ddots & \pm w_{i,k_i-2}C_{i,k_i-2} & \\ & & & \beta_{i,k_i-1}I_{i,n_{k_i-1}} & \end{bmatrix}$$

and Y_{m+1} is as in (2.4). A similar proof yields to Y_{m+1} in (2.5) whenever X_{m+1} is in form (2.2). Thus, (2.3) is proved. \square

From now on, we denote by \tilde{A} the submatrix obtained from A by deleting its last row and its last column. Moreover, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m$, let $F_{i,j}$ be the $k_i \times k_j$ matrix with $F_{i,j}(k_i, k_j) = 1$ and zeroes elsewhere.

In the proof of the following lemma, we will use the facts $\begin{vmatrix} A & 0 \\ C & B \end{vmatrix} = \begin{vmatrix} A & C \\ 0 & B \end{vmatrix} = |A| |B|$ and $\begin{vmatrix} A & 0 & B \\ C & D & E \\ F & 0 & G \end{vmatrix} = \begin{vmatrix} A & B \\ F & G \end{vmatrix} |D| = \begin{vmatrix} A & C & B \\ 0 & D & 0 \\ F & E & G \end{vmatrix}$ for matrices of appropriate sizes.

LEMMA 2.2. For $i = 1, 2, \dots, m$, let B_i be a matrix of size $k_i \times k_i$ and $\mu_{i,j}$ be

arbitrary scalars. Then it holds that

$$(2.6) \quad \begin{vmatrix} B_1 & \mu_{1,2}F_{1,2} & \cdots & \mu_{1,m-1}F_{1,m-1} & \mu_{1,m}F_{1,m} \\ \mu_{2,1}F_{1,2}^T & B_2 & \cdots & \cdots & \mu_{2,m}F_{2,m} \\ \mu_{3,1}F_{1,3}^T & \mu_{3,2}F_{2,3}^T & \ddots & & \vdots \\ \vdots & \vdots & & B_{m-1} & \mu_{m-1,m}F_{m-1,m} \\ \mu_{m,1}F_{1,m}^T & \mu_{m,2}F_{2,m}^T & \cdots & \mu_{m,m-1}F_{m-1,m}^T & B_m \end{vmatrix} \\ = \begin{vmatrix} |B_1| & \mu_{1,2}|\widetilde{B}_2| & \cdots & \mu_{1,m-1}|\widetilde{B}_{m-1}| & \mu_{1,m}|\widetilde{B}_m| \\ \mu_{2,1}|\widetilde{B}_1| & |B_2| & \cdots & \cdots & \mu_{2,m}|\widetilde{B}_m| \\ \mu_{3,1}|\widetilde{B}_1| & \mu_{3,2}|\widetilde{B}_2| & \ddots & & \vdots \\ \vdots & \vdots & & |B_{m-1}| & \mu_{m-1,m}|\widetilde{B}_m| \\ \mu_{m,1}|\widetilde{B}_1| & \mu_{m,2}|\widetilde{B}_2| & \cdots & \mu_{m,m-1}|\widetilde{B}_{m-1}| & |B_m| \end{vmatrix}.$$

Proof. Let us write $B_i = \begin{bmatrix} \widetilde{B}_i & \mathbf{b}_{i,1} \\ \mathbf{b}_{i,2}^T & b_i \end{bmatrix}$. We use induction on m . For $m = 2$, we have

$$\begin{vmatrix} B_1 & \mu_{1,2}F_{1,2} \\ \mu_{2,1}F_{1,2}^T & B_2 \end{vmatrix} = \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} \\ 0 & 0 & \widetilde{B}_2 & \mathbf{b}_{2,1} \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 \end{vmatrix} \\ = \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & 0 \\ 0 & 0 & \widetilde{B}_2 & \mathbf{b}_{2,1} \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 \end{vmatrix} + \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} \\ 0 & 0 & \widetilde{B}_2 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & 0 \end{vmatrix} \\ = \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} \\ \mathbf{b}_{1,2}^T & b_1 \end{vmatrix} \begin{vmatrix} \widetilde{B}_2 & \mathbf{b}_{2,1} \\ \mathbf{b}_{2,2}^T & b_2 \end{vmatrix} + \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} \\ 0 & 0 & \widetilde{B}_2 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & 0 \end{vmatrix} \\ = \begin{vmatrix} |B_1| & 0 \\ \mu_{2,1}|\widetilde{B}_1| & |B_2| \end{vmatrix} + \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 \\ \mathbf{b}_{1,2}^T & b_1 & \mu_{1,2} \\ 0 & \mu_{2,1} & 0 \end{vmatrix} \\ = \begin{vmatrix} |B_1| & 0 \\ \mu_{2,1}|\widetilde{B}_1| & |B_2| \end{vmatrix} - \begin{vmatrix} \widetilde{B}_2 & \mu_{1,2} \\ 0 & \mu_{2,1} \end{vmatrix} \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} \\ 0 & \mu_{2,1} \end{vmatrix}.$$

Clearly,

$$\begin{vmatrix} 0 & \mu_{1,2}|\widetilde{B}_2| \\ \mu_{2,1}|\widetilde{B}_1| & |B_2| \end{vmatrix} = -|\widetilde{B}_2|\mu_{1,2} \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} \\ 0 & \mu_{2,1} \end{vmatrix}.$$

Thus,

$$\begin{aligned} \begin{vmatrix} B_1 & \mu_{1,2}F_{1,2} \\ \mu_{2,1}F_{1,2}^T & B_2 \end{vmatrix} &= \begin{vmatrix} |B_1| & 0 \\ \mu_{2,1}|\widetilde{B}_1| & |B_2| \end{vmatrix} + \begin{vmatrix} 0 & \mu_{1,2}|\widetilde{B}_2| \\ \mu_{2,1}|\widetilde{B}_1| & |B_2| \end{vmatrix} \\ &= \begin{vmatrix} |B_1| & \mu_{1,2}|\widetilde{B}_2| \\ \mu_{2,1}|\widetilde{B}_1| & |B_2| \end{vmatrix}. \end{aligned}$$

We have proved (2.6) for $m = 2$. Let $m \geq 3$, and assume that (2.6) is true for $m - 1$. The meaning of the scalars t, t_1 and t_2 below will be clear from the context. By linearity on the last column,

$$\begin{aligned} t := & \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} & \cdots & 0 & \mu_{1,m-1} & 0 & \mu_{1,m} \\ 0 & 0 & \widetilde{B}_2 & \mathbf{b}_{2,1} & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & \cdots & \cdots & 0 & \mu_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & & \widetilde{B}_{m-1} & \mathbf{b}_{m-1,1} & 0 & 0 \\ 0 & \mu_{m-1,1} & \vdots & \vdots & & \mathbf{b}_{m-1,2}^T & b_{m-1} & 0 & \mu_{m-1,m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \widetilde{B}_m & \mathbf{b}_{m,1} \\ 0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} & \mathbf{b}_{m,2}^T & b_m \end{vmatrix} \\ = & \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} & \cdots & 0 & \mu_{1,m-1} & 0 & 0 \\ 0 & 0 & \widetilde{B}_2 & \mathbf{b}_{2,1} & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & & \widetilde{B}_{m-1} & \mathbf{b}_{m-1,1} & 0 & 0 \\ 0 & \mu_{m-1,1} & \vdots & \vdots & & \mathbf{b}_{m-1,2}^T & b_{m-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \widetilde{B}_m & \mathbf{b}_{m,1} \\ 0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} & \mathbf{b}_{m,2}^T & b_m \end{vmatrix} \end{aligned}$$

$$+ \begin{pmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} & \cdots & 0 & \mu_{1,m-1} & 0 & \mu_{1,m} \\ 0 & 0 & \widetilde{B}_2 & \mathbf{b}_{2,1} & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & \cdots & \cdots & 0 & \mu_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & & \widetilde{B}_{m-1} & \mathbf{b}_{m-1,1} & 0 & 0 \\ 0 & \mu_{m-1,1} & \vdots & \vdots & & \mathbf{b}_{m-1,2}^T & b_{m-1} & 0 & \mu_{m-1,m} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \widetilde{B}_m & 0 \\ 0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} & \mathbf{b}_{m,2}^T & 0 \end{pmatrix} := t_1 + t_2.$$

By definition,

$$t_1 = \begin{pmatrix} B_1 & \mu_{1,2}F_{1,2} & \cdots & \cdots & \mu_{1,m-1}F_{1,m-1} & 0 \\ \mu_{2,1}F_{1,2}^T & B_2 & \mu_{2,3}F_{2,3} & \cdots & \mu_{2,m-1}F_{2,m-1} & 0 \\ \vdots & \mu_{3,2}F_{2,3}^T & \ddots & & \vdots & \vdots \\ \mu_{m-2,1}F_{1,m-2}^T & \vdots & & \ddots & \mu_{m-2,m-1}F_{m-2,m-1} & \vdots \\ \mu_{m-1,1}F_{1,m-1}^T & \mu_{m-1,2}F_{2,m-1}^T & \cdots & \cdots & B_{m-1} & 0 \\ \mu_{m,1}F_{1,m}^T & \mu_{m,2}F_{2,m}^T & \cdots & \cdots & \mu_{m,m-1}F_{m-1,m}^T & B_m \end{pmatrix},$$

and hence,

$$t_1 = \begin{pmatrix} B_1 & \mu_{1,2}F_{1,2} & \cdots & \cdots & \mu_{1,m-1}F_{1,m-1} \\ \mu_{2,1}F_{1,2}^T & B_2 & \mu_{2,3}F_{2,3} & \cdots & \mu_{2,m-1}F_{2,m-1} \\ \vdots & \mu_{3,2}F_{2,3}^T & \ddots & & \vdots \\ \mu_{m-2,1}F_{1,m-2}^T & \vdots & & \ddots & \mu_{m-2,m-1}F_{m-2,m-1} \\ \mu_{m-1,1}F_{1,m-1}^T & \mu_{m-1,2}F_{2,m-1}^T & \cdots & \cdots & B_{m-1} \end{pmatrix} |B_m|.$$

We apply the induction hypothesis on the first factor obtaining that

$$t_1 = \begin{pmatrix} |B_1| & \mu_{1,2}|\widetilde{B}_2| & \cdots & \cdots & \mu_{1,m-1}|\widetilde{B}_{m-1}| \\ \mu_{2,1}|\widetilde{B}_1| & |B_2| & \mu_{2,3}|\widetilde{B}_3| & \cdots & \mu_{2,m-1}|\widetilde{B}_{m-1}| \\ \vdots & \mu_{3,2}|\widetilde{B}_2| & \ddots & & \vdots \\ \mu_{m-2,1}|\widetilde{B}_1| & \vdots & & \ddots & \mu_{m-2,m-1}|\widetilde{B}_{m-1}| \\ \mu_{m-1,1}|\widetilde{B}_1| & \mu_{m-1,2}|\widetilde{B}_2| & \cdots & \cdots & |B_{m-1}| \end{pmatrix} |B_m|.$$

Now it is easy to see that

$$(2.7) \quad t_1 = \begin{vmatrix} |B_1| & \mu_{1,2} |\widetilde{B}_2| & \cdots & \cdots & \mu_{1,m-1} |\widetilde{B}_{m-1}| & 0 \\ \mu_{2,1} |\widetilde{B}_1| & |B_2| & \mu_{2,3} |\widetilde{B}_3| & \cdots & \mu_{2,m-1} |\widetilde{B}_{m-1}| & 0 \\ \vdots & \mu_{3,2} |\widetilde{B}_2| & \ddots & & \vdots & \vdots \\ \mu_{m-2,1} |\widetilde{B}_1| & \vdots & & \ddots & \mu_{m-2,m-1} |\widetilde{B}_{m-1}| & \vdots \\ \mu_{m-1,1} |\widetilde{B}_1| & \mu_{m-1,2} |\widetilde{B}_2| & \cdots & \cdots & |B_{m-1}| & 0 \\ \mu_{m,1} |\widetilde{B}_1| & \mu_{m,2} |\widetilde{B}_2| & \cdots & \cdots & \mu_{m,m-1} |\widetilde{B}_{m-1}| & |B_m| \end{vmatrix}.$$

We have

$$t_2 = \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mathbf{b}_{1,2}^T & b_1 & 0 & \mu_{1,2} & \cdots & 0 & \mu_{1,m-1} & \mu_{1,m} \\ 0 & 0 & \widetilde{B}_2 & \mathbf{b}_{2,1} & \cdots & \cdots & \cdots & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & \cdots & \cdots & \mu_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & & & \vdots \\ 0 & 0 & \vdots & \vdots & \widetilde{B}_{m-1} & \mathbf{b}_{m-1,1} & 0 & \\ 0 & \mu_{m-1,1} & \vdots & \vdots & \mathbf{b}_{m-1,2}^T & b_{m-1} & \mu_{m-1,m} & \\ 0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} & 0 \end{vmatrix} |\widetilde{B}_m|.$$

Expanding along the last column, we get

$$t_2 = (s_1 + s_2 + \cdots + s_{m-1}) |\widetilde{B}_m|$$

where, for $i = 1, 2, \dots, m-1$, the summand s_i is the cofactor of the entry $\mu_{i,m}$. In particular,

$$s_1 = (-1)^{k_2 + \cdots + k_{m-1} + 1} \mu_{1,m} \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \widetilde{B}_2 & \mathbf{b}_{2,1} & \cdots & 0 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & 0 & \mu_{2,m-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots & \widetilde{B}_{m-1} & \mathbf{b}_{m-1,1} & \\ 0 & \mu_{m-1,1} & \vdots & \vdots & \mathbf{b}_{m-1,2}^T & b_{m-1} & \\ 0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} \end{vmatrix}.$$

After $k_{m-1} + k_{m-2} + \dots + k_2$ row interchanges, we obtain

$$s_1 = -\mu_{1,m} \begin{vmatrix} \widetilde{B}_1 & \mathbf{b}_{1,1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mu_{m,1} & 0 & \mu_{m,2} & \cdots & 0 & \mu_{m,m-1} \\ 0 & 0 & \widetilde{B}_2 & \mathbf{b}_{2,1} & \cdots & 0 & 0 \\ 0 & \mu_{2,1} & \mathbf{b}_{2,2}^T & b_2 & \cdots & 0 & \mu_{2,m-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \widetilde{B}_{m-1} & \mathbf{b}_{m-1,1} \\ 0 & \mu_{m-1,1} & 0 & \mu_{m-1,2} & \cdots & \mathbf{b}_{m-1,2}^T & b_{m-1} \end{vmatrix}.$$

We apply the induction hypothesis to get that

$$s_1 = -\mu_{1,m} \begin{vmatrix} \mu_{m,1} |\widetilde{B}_1| & \mu_{m,2} |\widetilde{B}_2| & \cdots & \mu_{m,m-1} |\widetilde{B}_{m-1}| \\ \mu_{2,1} |\widetilde{B}_1| & |B_2| & \cdots & \mu_{2,m-1} |\widetilde{B}_{m-1}| \\ \vdots & \mu_{3,2} |\widetilde{B}_2| & \ddots & \vdots \\ \vdots & \vdots & & \mu_{m-1,m-1} |\widetilde{B}_{m-1}| \\ \mu_{m-1,1} |\widetilde{B}_1| & \mu_{m-1,2} |\widetilde{B}_2| & \cdots & |B_{m-1}| \end{vmatrix}.$$

After $m - 2$ row interchanges, we have

$$(2.8) \quad s_1 = (-1)^{m-1} \mu_{1,m} \begin{vmatrix} \mu_{2,1} |\widetilde{B}_1| & |B_2| & \cdots & \mu_{2,m-1} |\widetilde{B}_{m-1}| \\ \mu_{3,1} |\widetilde{B}_1| & \mu_{3,2} |\widetilde{B}_2| & \cdots & \vdots \\ \vdots & \vdots & \ddots & \mu_{m-1,m-1} |\widetilde{B}_{m-1}| \\ \mu_{m-1,1} |\widetilde{B}_1| & \mu_{m-1,2} |\widetilde{B}_2| & \cdots & |B_{m-1}| \\ \mu_{m,1} |\widetilde{B}_1| & \mu_{m,2} |\widetilde{B}_2| & \cdots & \mu_{m,m-1} |\widetilde{B}_{m-1}| \end{vmatrix}.$$

We may write

$$(2.9) \quad s_1 = \begin{vmatrix} |B_1| & \mu_{1,2} |\widetilde{B}_2| & \cdots & \mu_{1,m-1} |\widetilde{B}_{m-1}| & \mu_{1,m} \\ \mu_{2,1} |\widetilde{B}_1| & |B_2| & \cdots & \mu_{2,m-1} |\widetilde{B}_{m-1}| & 0 \\ \mu_{3,1} |\widetilde{B}_1| & \mu_{3,2} |\widetilde{B}_2| & \ddots & \vdots & \vdots \\ \vdots & \vdots & & |B_{m-1}| & 0 \\ \mu_{m,1} |\widetilde{B}_1| & \mu_{m,2} |\widetilde{B}_2| & \cdots & \mu_{m,m-1} |\widetilde{B}_{m-1}| & 0 \end{vmatrix}.$$

In fact, expanding (2.9) along the last row we obtain (2.8). Similarly,

$$s_2 = \begin{vmatrix} |B_1| & \mu_{1,2} |\widetilde{B}_2| & \cdots & \mu_{1,m-1} |\widetilde{B}_{m-1}| & 0 \\ \mu_{2,1} |\widetilde{B}_1| & |B_2| & \cdots & \mu_{2,m-1} |\widetilde{B}_{m-1}| & \mu_{2,m} \\ \mu_{3,1} |\widetilde{B}_1| & \mu_{3,2} |\widetilde{B}_2| & \ddots & \vdots & 0 \\ \vdots & \vdots & & |B_{m-1}| & \vdots \\ \mu_{m,1} |\widetilde{B}_1| & \mu_{m,2} |\widetilde{B}_2| & \cdots & \mu_{m,m-1} |\widetilde{B}_{m-1}| & 0 \end{vmatrix}$$

$$\vdots$$

$$s_{m-1} = \begin{vmatrix} |B_1| & \mu_{1,2} |\widetilde{B}_2| & \cdots & \mu_{1,m-1} |\widetilde{B}_{m-1}| & 0 \\ \mu_{2,1} |\widetilde{B}_1| & |B_2| & \cdots & \mu_{2,m-1} |\widetilde{B}_{m-1}| & \vdots \\ \mu_{3,1} |\widetilde{B}_1| & \mu_{3,2} |\widetilde{B}_2| & \ddots & \vdots & 0 \\ \vdots & \vdots & & |B_{m-1}| & \mu_{m-1,m} \\ \mu_{m,1} |\widetilde{B}_1| & \mu_{m,2} |\widetilde{B}_2| & \cdots & \mu_{m,m-1} |\widetilde{B}_{m-1}| & 0 \end{vmatrix}.$$

Therefore,

$$(2.10) \quad t_2 = \begin{vmatrix} |B_1| & \mu_{1,2} |\widetilde{B}_2| & \cdots & \mu_{1,m-1} |\widetilde{B}_{m-1}| & \mu_{1,m} |\widetilde{B}_m| \\ \mu_{2,1} |\widetilde{B}_1| & |B_2| & \cdots & \mu_{2,m-1} |\widetilde{B}_{m-1}| & \mu_{2,m} |\widetilde{B}_m| \\ \mu_{3,1} |\widetilde{B}_1| & \mu_{3,2} |\widetilde{B}_2| & \ddots & \vdots & \vdots \\ \vdots & \vdots & & |B_{m-1}| & \mu_{m-1,m} |\widetilde{B}_m| \\ \mu_{m,1} |\widetilde{B}_1| & \mu_{m,2} |\widetilde{B}_2| & \cdots & \mu_{m,m-1} |\widetilde{B}_{m-1}| & 0 \end{vmatrix}.$$

By (2.7) and (2.10), we have

$$t = t_1 + t_2 = \begin{vmatrix} |B_1| & \mu_{1,2} |\widetilde{B}_2| & \cdots & \mu_{1,m-1} |\widetilde{B}_{m-1}| & \mu_{1,m} |\widetilde{B}_m| \\ \mu_{2,1} |\widetilde{B}_1| & |B_2| & \cdots & \mu_{2,m-1} |\widetilde{B}_{m-1}| & \mu_{2,m} |\widetilde{B}_m| \\ \mu_{3,1} |\widetilde{B}_1| & \mu_{3,2} |\widetilde{B}_2| & \ddots & \vdots & \vdots \\ \vdots & \vdots & & |B_{m-1}| & \mu_{m-1,m} |\widetilde{B}_m| \\ \mu_{m,1} |\widetilde{B}_1| & \mu_{m,2} |\widetilde{B}_2| & \cdots & \mu_{m,m-1} |\widetilde{B}_{m-1}| & |B_m| \end{vmatrix}.$$

This completes the proof. \square

3. The spectrum of the Laplacian matrix.

DEFINITION 3.1. For $i = 1, 2, \dots, m$, let

$$P_{i,0}(\lambda) = 1 \quad \text{and} \quad P_{i,1}(\lambda) = \lambda - \delta_{i,1}$$

and, for $j = 2, 3, \dots, k_i - 1$,

$$(3.1) \quad P_{i,j}(\lambda) = (\lambda - \delta_{i,j}) P_{i,j-1}(\lambda) - \frac{n_{i,j-1}}{n_{i,j}} w_{i,j-1}^2 P_{i,j-2}(\lambda).$$

Moreover, for $i = 1, 2, \dots, m$, let

$$P_i(\lambda) = (\lambda - \delta_{i,k_i} - \delta_i) P_{i,k_i-1}(\lambda) - n_{i,k_i-1} w_{i,k_i-1}^2 P_{i,k_i-2}(\lambda)$$

and

$$\Omega_i = \{j : 1 \leq j \leq k_i - 1 : n_{i,j} > n_{i,j+1}\}.$$

THEOREM 3.2. *The following hold:*

(a)

$$(3.2) \quad |\lambda I - L(\mathcal{G}_m \{\mathcal{B}_i\})| = P(\lambda) \prod_{i=1}^m \prod_{j \in \Omega_i} P_{i,j}^{n_{i,j} - n_{i,j+1}}(\lambda),$$

where

$$P(\lambda) = \begin{vmatrix} P_1(\lambda) & \varepsilon_{1,2} P_{2,k_2-1}(\lambda) & \cdots & \varepsilon_{1,m} P_{m,k_m-1}(\lambda) \\ \varepsilon_{1,2} P_{1,k_1-1}(\lambda) & P_2(\lambda) & \cdots & \varepsilon_{2,m} P_{m,k_m-1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \varepsilon_{m-1,m} P_{m,k_m-1}(\lambda) \\ \varepsilon_{1,m} P_{1,k_1-1}(\lambda) & \varepsilon_{2,m} P_{2,k_2-1}(\lambda) & \cdots & P_m(\lambda) \end{vmatrix}.$$

(b) *The set of eigenvalues of $L(\mathcal{G}_m \{\mathcal{B}_i\})$ is*

$$\sigma(L(\mathcal{G}_m \{\mathcal{B}_i\})) = (\cup_{i=1}^m \cup_{j \in \Omega_i} \{\lambda : P_{i,j}(\lambda) = 0\}) \cup \{\lambda : P(\lambda) = 0\}.$$

Proof. (a) $L(\mathcal{G}_m \{\mathcal{B}_i\})$ is given by (1.3), (1.5) and (1.7). We apply Lemma 2.1 to the matrix $X = \lambda I - L(\mathcal{G}_m \{\mathcal{B}_i\})$. For this matrix,

$$\alpha_{i,j} = \lambda - \delta_{i,j} \quad (1 \leq i \leq m, 1 \leq j \leq k_i - 1) \quad \text{and} \quad \alpha_i = \lambda - \delta_{i,k_i} - \delta_i \quad (1 \leq i \leq m).$$

Let $\beta_{i,j}, \beta_i$ be as in Lemma 2.1. We first suppose that $\lambda \in \mathbb{R}$ is such that $P_{i,j}(\lambda) \neq 0$ for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k_i - 1$. For brevity, we write $P_{i,j}(\lambda) = P_{i,j}$ and $P(\lambda) = P$. Then we have

$$\begin{aligned} \beta_{i,1} &= \lambda - \delta_{i,1} = \frac{P_{i,1}}{P_{i,0}} \neq 0, \\ \beta_{i,2} &= (\lambda - \delta_{i,2}) - \frac{n_{i,1}}{n_{i,2}} \frac{w_{i,1}^2}{\beta_{i,1}} = (\lambda - \delta_{i,2}) - \frac{n_{i,1}}{n_{i,2}} w_{i,1}^2 \frac{P_{i,0}}{P_{i,1}} \\ &= \frac{(\lambda - \delta_{i,2}) P_{i,1} - \frac{n_{i,1}}{n_{i,2}} w_{i,1}^2 P_{i,0}}{P_{i,1}} = \frac{P_{i,2}}{P_{i,1}} \neq 0, \\ &\vdots \\ \beta_{i,k_i-1} &= (\lambda - \delta_{i,k_i-1}) - \frac{n_{k_i-2}}{n_{k_i-1}} \frac{w_{i,k_i-2}^2}{\beta_{i,k_i-2}} = (\lambda - \delta_{i,k_i-1}) - \frac{n_{i,k_i-2}}{n_{i,k_i-1}} w_{i,k_i-2}^2 \frac{P_{i,k_i-3}}{P_{i,k_i-2}} \\ &= \frac{(\lambda - \delta_{i,k_i-1}) P_{i,k_i-2} - \frac{n_{i,k_i-2}}{n_{i,k_i-1}} w_{i,k_i-2}^2 P_{i,k_i-3}}{P_{i,k_i-2}} = \frac{P_{i,k_i-1}}{P_{i,k_i-2}} \neq 0. \end{aligned}$$

Moreover, for $i = 1, \dots, m$,

$$\begin{aligned} \beta_i &= \lambda - \delta_{i,k_i} - \delta_i - n_{i,k_i-1} \frac{w_{i,k_i-1}^2}{\beta_{i,k_i-1}} = \lambda - \delta_{i,k_i} - \delta_i - n_{i,k_i-1} w_{i,k_i-1}^2 \frac{P_{i,k_i-2}}{P_{i,k_i-1}} \\ &= \frac{(\lambda - \delta_{i,k_i} - \delta_i) P_{i,k_i-1} - n_{i,k_i-1} w_{i,k_i-1}^2 P_{i,k_i-2}}{P_{i,k_i-1}} = \frac{P_i}{P_{i,k_i-1}}. \end{aligned}$$

By (2.3), it follows that

$$\begin{aligned} &\det(\lambda I - L(\mathcal{G}_m \{B_i\})) \\ &= \left(\prod_{i=1}^m \prod_{j=1}^{k_i-1} \beta_{i,j}^{n_{i,j}} \right) |C_{m+1}| \\ &= \left(\prod_{i=1}^m \frac{P_{i,1}^{n_{i,1}} P_{i,2}^{n_{i,2}} P_{i,3}^{n_{i,3}} \dots P_{i,k_i-2}^{n_{i,k_i-2}} P_{i,k_i-1}^{n_{i,k_i-1}}}{P_{i,0}^{n_{i,1}} P_{i,1}^{n_{i,2}} P_{i,2}^{n_{i,3}} \dots P_{i,k_i-3}^{n_{i,k_i-2}} P_{i,k_i-2}^{n_{i,k_i-1}}} \right) |C_{m+1}| \\ &= \left(\prod_{i=1}^m P_{i,1}^{n_{i,1}-n_{i,2}} P_{i,2}^{n_{i,2}-n_{i,3}} \dots P_{i,k_i-2}^{n_{i,k_i-2}-n_{i,k_i-1}} P_{i,k_i-1}^{n_{i,k_i-1}} \right) |C_{m+1}|, \end{aligned}$$

where

$$|C_{m+1}| = \begin{vmatrix} \frac{P_1}{P_{1,k_1-1}} & \varepsilon_{1,2} & \varepsilon_{1,3} & \cdots & \varepsilon_{1,m} \\ \varepsilon_{1,2} & \frac{P_2}{P_{2,k_2-1}} & \varepsilon_{2,3} & \cdots & \varepsilon_{2,m} \\ \varepsilon_{1,3} & \varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \frac{P_{m-1}}{P_{m-1,k_{m-1}-1}} & \varepsilon_{m-1,m} \\ \varepsilon_{1,m} & \varepsilon_{2,m} & \cdots & \varepsilon_{m-1,m} & \frac{P_m}{P_{m,k_m-1}} \end{vmatrix}$$

$$\begin{aligned}
 &= \frac{1}{\prod_{i=1}^m P_{i,k_i-1}} \begin{vmatrix} P_1 & \varepsilon_{1,2}P_{2,k_2-1} & \cdots & \cdots & \varepsilon_{1,m}P_{m,k_m-1} \\ \varepsilon_{1,2}P_{1,k_1-1} & P_2 & \cdots & \cdots & \varepsilon_{2,m}P_{m,k_m-1} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \varepsilon_{m-1,m}P_{m,k_m-1} \\ \varepsilon_{1,m}P_{1,k_1-1} & \varepsilon_{2,m}P_{2,k_2-1} & \cdots & \cdots & P_m \end{vmatrix} \\
 &= \frac{1}{\prod_{i=1}^m P_{i,k_i-1}} P(\lambda).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 &|\lambda I - L(\mathcal{G}_m \{\mathcal{B}_i\})| \\
 &= \left(\prod_{i=1}^m P_{i,1}^{n_{i,1}-n_{i,2}} P_{i,2}^{n_{i,2}-n_{i,3}} \cdots P_{i,k_i-2}^{n_{i,k_i-2}-n_{i,k_i-1}} P_{i,k_i-1}^{n_{i,k_i-1}} \right) \frac{1}{\prod_{i=1}^m P_{i,k_i-1}} P(\lambda) \\
 &= \left(\prod_{i=1}^m P_{i,1}^{n_{i,1}-n_{i,2}} P_{i,2}^{n_{i,2}-n_{i,3}} \cdots P_{i,k_i-2}^{n_{i,k_i-2}-n_{i,k_i-1}} P_{i,k_i-1}^{n_{i,k_i-1}-n_{i,k_i}} \right) P(\lambda) \\
 &= P(\lambda) \prod_{i=1}^m \prod_{j=1}^{k_i-1} P_{i,j}^{n_{i,j}-n_{i,j+1}} \\
 &= P(\lambda) \prod_{i=1}^m \prod_{j \in \Omega_i} P_{i,j}^{n_{i,j}-n_{i,j+1}}(\lambda).
 \end{aligned}$$

We have used the fact that $n_{i,k_i} = 1$. Thus, (3.2) is proved for all $\lambda \in \mathbb{R}$ such that $P_{i,j}(\lambda) \neq 0$, for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k_i - 1$. Now, we consider $\lambda_0 \in \mathbb{R}$ such that $P_{l,s}(\lambda_0) = 0$ for some $1 \leq l \leq m$ and $1 \leq s \leq k_l - 1$. Since the zeros of any nonzero polynomial are isolated, there exists a neighborhood $N(\lambda_0)$ of λ_0 such that $P_{i,j}(\lambda) \neq 0$ for all $\lambda \in N(\lambda_0) - \{\lambda_0\}$, and for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k_i - 1$. Hence,

$$|\lambda I - L(\mathcal{G}_m \{\mathcal{B}_i\})| = P(\lambda) \prod_{i=1}^m \prod_{j \in \Omega_i} P_{i,j}^{n_{i,j}-n_{i,j+1}}(\lambda)$$

for all $\lambda \in N(\lambda_0) - \{\lambda_0\}$. By continuity, taking the limit as λ tends to λ_0 , we obtain

$$|\lambda_0 I - L(\mathcal{G}_m \{\mathcal{B}_i\})| = P(\lambda_0) \prod_{i=1}^m \prod_{j \in \Omega_i} P_{i,j}^{n_{i,j}-n_{i,j+1}}(\lambda_0).$$

Therefore, (3.2) holds for all $\lambda \in \mathbb{R}$.

(b) It is an immediate consequence of part (a). \square

LEMMA 3.5. Let $r = \sum_{i=1}^m k_i$, and let G be the $r \times r$ symmetric matrix defined by

$$G = \begin{bmatrix} T_1 & -\varepsilon_{1,2}F_{1,2} & -\varepsilon_{1,3}F_{1,3} & \cdots & -\varepsilon_{1,m}F_{1,m} \\ -\varepsilon_{1,2}F_{1,2}^T & T_2 & -\varepsilon_{2,3}F_{2,3} & \cdots & -\varepsilon_{2,m}F_{2,m} \\ -\varepsilon_{1,3}F_{1,3}^T & -\varepsilon_{2,3}F_{2,3}^T & \ddots & & \vdots \\ \vdots & \vdots & & T_{m-1} & -\varepsilon_{m-1,m}F_{m-1,m} \\ -\varepsilon_{1,m}F_{1,m}^T & -\varepsilon_{2,m}F_{2,m}^T & \cdots & \cdots & T_m \end{bmatrix}.$$

Then

$$|\lambda I - G| = P(\lambda).$$

Proof. We have

$$\lambda I - G = \begin{bmatrix} \lambda I - T_1 & \varepsilon_{1,2}F_{1,2} & \varepsilon_{1,3}F_{1,3} & \cdots & \varepsilon_{1,m}F_{1,m} \\ \varepsilon_{1,2}F_{1,2}^T & \lambda I - T_2 & \varepsilon_{2,3}F_{2,3} & \cdots & \varepsilon_{2,m}F_{2,m} \\ \varepsilon_{1,3}F_{1,3}^T & \varepsilon_{2,3}F_{2,3}^T & \ddots & & \vdots \\ \vdots & \vdots & & \lambda I - T_{m-1} & \varepsilon_{m-1,m}F_{m-1,m} \\ \varepsilon_{1,m}F_{1,m}^T & \varepsilon_{2,m}F_{2,m}^T & \cdots & \varepsilon_{m-1,m}F_{m-1,m}^T & \lambda I - T_m \end{bmatrix}$$

We apply Lemma 2.2 to $\lambda I - G$ to get

$$|\lambda I - G| = \begin{vmatrix} |\lambda I - T_1| & \varepsilon_{1,2} \widetilde{|\lambda I - T_2|} & \cdots & \cdots & \varepsilon_{1,m} \widetilde{|\lambda I - T_m|} \\ \varepsilon_{1,2} \widetilde{|\lambda I - T_1|} & |\lambda I - T_2| & \cdots & \cdots & \varepsilon_{2,m} \widetilde{|\lambda I - T_m|} \\ \vdots & \vdots & \ddots & & \vdots \\ \varepsilon_{1,m-1} \widetilde{|\lambda I - T_1|} & \vdots & & |\lambda I - T_{m-1}| & \varepsilon_{m-1,m} \widetilde{|\lambda I - T_m|} \\ \varepsilon_{1,m} \widetilde{|\lambda I - T_1|} & \varepsilon_{2,m} \widetilde{|\lambda I - T_2|} & \cdots & \varepsilon_{m-1,m} \widetilde{|\lambda I - T_{m-1}|} & |\lambda I - T_m| \end{vmatrix}.$$

We observe that $\widetilde{|\lambda I - T_i|} = |\lambda I - T_{i,k_i-1}|$. We now use Lemma 3.4 to obtain

$$\begin{aligned} |\lambda I - G| &= \begin{vmatrix} P_1(\lambda) & \varepsilon_{1,2}P_{2,k_2-1}(\lambda) & \cdots & \cdots & \varepsilon_{1,m}P_{m,k_m-1}(\lambda) \\ \varepsilon_{1,2}P_{1,k_1-1}(\lambda) & P_2(\lambda) & \cdots & \cdots & \varepsilon_{2,m}P_{m,k_m-1}(\lambda) \\ \varepsilon_{1,3}P_{1,k_1-1}(\lambda) & \varepsilon_{2,3}P_{2,k_2-1}(\lambda) & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \varepsilon_{m-1,m}P_{m,k_m-1}(\lambda) \\ \varepsilon_{1,m}P_{1,k_1-1}(\lambda) & \varepsilon_{2,m}P_{2,k_2-1}(\lambda) & \cdots & \cdots & P_m(\lambda) \end{vmatrix} \\ &= P(\lambda). \end{aligned}$$

The proof is complete. \square

THEOREM 3.6.

(a) $\sigma(L(\mathcal{G}_m \{\mathcal{B}_i\})) = (\cup_{i=1}^m \cup_{j \in \Omega_i} \sigma(T_{i,j})) \cup \sigma(G).$

(b) *The multiplicity of each eigenvalue of $T_{i,j}$, as an eigenvalue of $L(\mathcal{G}_m \{\mathcal{B}_i\})$, is $n_{i,j} - n_{i,j+1}$ for $j \in \Omega_i$.*

(c) *The matrix G is singular.*

Proof. It is known that the eigenvalues of a symmetric tridiagonal matrix with nonzero codiagonal entries are simple [2]. This fact, Theorem 3.2, Lemma 3.4 and Lemma 3.5 yield (a) and (b). One can easily check that $|T_{i,j}| = \omega_{i,1}\omega_{i,2} \cdots \omega_{i,j} > 0$ for $1 \leq i \leq m$ and $1 \leq j \leq k_i - 1$. This fact and part (a) imply that 0 is an eigenvalue of G . Hence, G is a singular matrix. \square

THEOREM 3.7. *The spectral radius of $L(\mathcal{G}_m \{\mathcal{B}_i\})$ is the largest eigenvalue of the matrix G .*

Proof. Since $L(\mathcal{G}_m \{\mathcal{B}_i\})$ is a positive semidefinite matrix, its spectral radius is its largest eigenvalue. By Theorem 3.6, the eigenvalues of $L(\mathcal{G}_m \{\mathcal{B}_i\})$ are the eigenvalues of the matrices $T_{i,j}$ for $i \in \Omega_i$ and $1 \leq j \leq k_i - 1$ together with the eigenvalues of G . Since the eigenvalues of the matrices $T_{i,j}$ interlace the eigenvalues of G , we conclude that the spectral radius of $L(\mathcal{G}_m \{\mathcal{B}_i\})$ is the largest eigenvalue of G . \square

Next, we summarize the above results for the particular case of unweighted trees \mathcal{B}_i and unweighted graph \mathcal{G}_m .

THEOREM 3.8. *If each \mathcal{B}_i is an unweighted generalized Bethe tree and \mathcal{G}_m is an unweighted graph, then the following hold:*

(a)

$$\sigma(L(\mathcal{G}_m \{\mathcal{B}_i\})) = (\cup_{i=1}^m \cup_{j \in \Omega_i} \sigma(T_{i,j})) \cup \sigma(G),$$

where G is the matrix defined in Lemma 3.5 with

$$T_i = \begin{bmatrix} 1 & \sqrt{d_{i,2} - 1} & & & & \\ \sqrt{d_{i,2} - 1} & d_{i,2} & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \sqrt{d_{i,k_i-1} - 1} & \\ & & & & \sqrt{d_{i,k_i-1} - 1} & d_{i,k_i} \\ & & & & & \sqrt{d_{i,k_i}} \\ & & & & & & d_{i,k_i} + d(v_i) \end{bmatrix}$$

of size $k_i \times k_i$ and $T_{i,j}$ is the $j \times j$ leading principal submatrix of T_i .

(b) The multiplicity of each eigenvalue of $T_{i,j}$, as an eigenvalue of $L(\mathcal{G}_m \{\mathcal{B}_i\})$, is $n_{i,j} - n_{i,j+1}$ for $j \in \Omega_i$.

(c) The matrix G is singular.

(d) The spectral radius of $L(\mathcal{G}_m \{\mathcal{B}_i\})$ is the largest eigenvalue of the matrix G .

4. The spectrum of the adjacency matrix. By (1.2), we have

$$A(\mathcal{G}_m \{\mathcal{B}_i\}) = \begin{bmatrix} A_1 & 0 & \cdots & 0 & w_{1,k_1-1}E_1 \\ 0 & A_2 & \cdots & 0 & w_{2,k_2-1}E_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m & w_{m,k_m-1}E_m \\ w_{1,k_1-1}E_1^T & w_{2,k_2-1}E_2^T & \cdots & w_{m,k_m-1}E_m^T & A(\mathcal{G}_m) \end{bmatrix},$$

where the diagonal blocks A_i ($1 \leq i \leq m$) are given by (1.4) and $A(\mathcal{G}_m)$ is given by (1.6).

We may apply Lemma 2.1 to $X = \lambda I - A(\mathcal{G}_m \{\mathcal{B}_i\})$. For this matrix, $\alpha_{i,j} = \lambda$ for $1 \leq i \leq m$ and $1 \leq j \leq k_i - 1$, and $\alpha_i = \lambda$ for $1 \leq i \leq m$.

DEFINITION 4.1. For $i = 1, 2, \dots, m$, let

$$Q_{i,0}(\lambda) = 1, \quad Q_{i,1}(\lambda) = \lambda$$

and, for $j = 2, 3, \dots, k_i - 1$, let

$$Q_{i,j}(\lambda) = \lambda Q_{i,j-1}(\lambda) - \frac{n_{i,j-1}}{n_{i,j}} w_{i,j-1}^2 Q_{i,j-2}(\lambda).$$

Moreover, for $i = 1, 2, \dots, m$, let

$$Q_i(\lambda) = \lambda Q_{i,k_i-1}(\lambda) - n_{i,k_i-1} w_{i,j-1}^2 Q_{i,k_i-2}(\lambda).$$

THEOREM 4.2. The following hold:

(a)

$$|\lambda I - A(\mathcal{G}_m \{\mathcal{B}_i\})| = Q(\lambda) \prod_{i=1}^m \prod_{j \in \Omega_i} Q_{i,j}^{n_{i,j} - n_{i,j+1}}(\lambda).$$

(b)

$$\sigma(A(\mathcal{G}_m \{\mathcal{B}_i\})) = (\cup_{i=1}^m \cup_{j \in \Omega_i} \{\lambda : Q_{i,j}(\lambda) = 0\}) \cup \{\lambda : Q(\lambda) = 0\},$$

where

$$Q(\lambda) = \begin{vmatrix} Q_1(\lambda) & -\varepsilon_{1,2}Q_{2,k_2-1}(\lambda) & \cdots & -\varepsilon_{1,m}Q_{m,k_m-1}(\lambda) \\ -\varepsilon_{1,2}Q_{1,k_1-1}(\lambda) & Q_2(\lambda) & \cdots & -\varepsilon_{2,m}Q_{m,k_m-1}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ -\varepsilon_{1,m}Q_{1,k_1-1}(\lambda) & -\varepsilon_{2,m}Q_{2,k_2-1}(\lambda) & \cdots & Q_m(\lambda) \end{vmatrix}.$$

Proof. Similar to the proof of Theorem 3.2. \square

DEFINITION 4.3. For $i = 1, 2, \dots, m$, let S_i be the $k_i \times k_i$ symmetric matrix defined by

$$S_i = \begin{bmatrix} 0 & w_{i,1}\sqrt{d_{i,2}-1} & & & \\ w_{i,1}\sqrt{d_{i,2}-1} & 0 & \ddots & & \\ & \ddots & \ddots & w_{i,k_i-2}\sqrt{d_{i,k_i-1}-1} & \\ & & \ddots & 0 & w_{i,k_i-1}\sqrt{d_{i,k_i}} \\ & & & w_{i,k_i-1}\sqrt{d_{i,k_i}} & 0 \end{bmatrix}.$$

Moreover, for $i = 1, 2, \dots, m$ and for $j = 1, 2, \dots, k_i - 1$, let $S_{i,j}$ be the $j \times j$ leading principal submatrix of S_i .

LEMMA 4.4. For $i = 1, 2, \dots, m$ and for $j = 1, 2, \dots, k_i - 1$, we have

$$|\lambda I - S_{i,j}| = Q_{i,j}(\lambda).$$

Moreover, for $i = 1, 2, \dots, m$,

$$|\lambda I - S_i| = Q_i(\lambda).$$

Proof. Similar to the proof of Lemma 3.4. \square

LEMMA 4.5. Let $r = \sum_{i=1}^m k_i$ and H be the $r \times r$ symmetric matrix defined by

$$H = \begin{bmatrix} S_1 & \varepsilon_{1,2}F_{1,2} & \varepsilon_{1,3}F_{1,3} & \cdots & \varepsilon_{1,m}F_{1,m} \\ \varepsilon_{1,2}F_{1,2}^T & S_2 & \varepsilon_{2,3}F_{2,3} & \cdots & \varepsilon_{2,m}F_{2,m} \\ \varepsilon_{1,3}F_{1,3}^T & \varepsilon_{2,3}F_{2,3}^T & \ddots & & \vdots \\ \vdots & \vdots & & S_{m-1} & \varepsilon_{m-1,m}F_{m-1,m} \\ \varepsilon_{1,m}F_{1,m}^T & \varepsilon_{2,m}F_{2,m}^T & \cdots & \varepsilon_{m-1,m}F_{m-1,m}^T & S_m \end{bmatrix}.$$

Then

$$|\lambda I - H| = Q(\lambda).$$

Proof. We have

$$\lambda I - H = \begin{bmatrix} \lambda I - S_1 & -\varepsilon_{1,2}F_{1,2} & \cdots & \cdots & & -\varepsilon_{1,m}F_{1,m} \\ -\varepsilon_{1,2}F_{1,2}^T & \lambda I - S_2 & \cdots & \cdots & & -\varepsilon_{2,m}F_{2,m} \\ -\varepsilon_{1,3}F_{1,3}^T & -\varepsilon_{2,3}F_{2,3}^T & \ddots & & & \vdots \\ \vdots & \vdots & & \lambda I - S_{m-1} & & -\varepsilon_{m-1,m}F_{m-1,m} \\ -\varepsilon_{1,m}F_{1,m}^T & -\varepsilon_{2,m}F_{2,m}^T & \cdots & -\varepsilon_{m-1,m}F_{m-1,m}^T & & \lambda I - S_m \end{bmatrix}.$$

We apply Lemma 2.2 to $\lambda I - H$ to obtain that

$$|\lambda I - H| = \begin{vmatrix} |\lambda I - S_1| & -\varepsilon_{1,2}|\widetilde{\lambda I - S_2}| & \cdots & \cdots & -\varepsilon_{1,m}|\widetilde{\lambda I - S_m}| \\ -\varepsilon_{1,2}|\widetilde{\lambda I - S_1}| & |\lambda I - S_2| & \cdots & \cdots & -\varepsilon_{2,m}|\widetilde{\lambda I - S_m}| \\ \vdots & \vdots & \ddots & & \vdots \\ -\varepsilon_{1,m-1}|\widetilde{\lambda I - S_1}| & \vdots & & |\lambda I - S_{m-1}| & -\varepsilon_{m-1,m}|\widetilde{\lambda I - S_m}| \\ -\varepsilon_{1,m}|\widetilde{\lambda I - S_1}| & -\varepsilon_{2,m}|\widetilde{\lambda I - S_2}| & \cdots & -\varepsilon_{m-1,m}|\widetilde{\lambda I - S_{m-1}}| & |\lambda I - S_m| \end{vmatrix}.$$

We observe that $\widetilde{\lambda I - S_i} = \lambda I - S_{i,k_i-1}$. We now use Lemma 4.4 to obtain $|\lambda I - H| = Q(\lambda)$. \square

THEOREM 4.6.

(a) $\sigma(A(\mathcal{G}_m\{\mathcal{B}_i\})) = (\cup_{i=1}^m \cup_{j \in \Omega_i} \sigma(S_{i,j})) \cup \sigma(H)$.

(b) *The multiplicity of each eigenvalue of the matrix $S_{i,j}$, as an eigenvalue of $A(\mathcal{G}_m\{\mathcal{B}_i\})$, is $n_{i,j} - n_{i,j+1}$ for $j \in \Omega_i$.*

(c) *The largest eigenvalue of H is the spectral radius of $A(\mathcal{G}_m\{\mathcal{B}_i\})$.*

Proof. The proofs of (a) and (b) are similar to the proof of Theorem 3.6. Finally, (c) follows from part (a) and the interlacing property of the eigenvalues. \square

The following theorem summarizes the above results for the case of unweighted trees \mathcal{B}_i and unweighted graph \mathcal{G}_m .

THEOREM 4.7. *If each \mathcal{B}_i is an unweighted generalized Bethe tree and \mathcal{G}_m is an unweighted graph, then we have:*

(a) $\sigma(A(\mathcal{G}_m\{\mathcal{B}_i\})) = (\cup_{i=1}^m \cup_{j \in \Omega_i} \sigma(S_{i,j})) \cup \sigma(H)$, where H is the matrix defined

in Lemma 4.5 with

$$S_i = \begin{bmatrix} 0 & \sqrt{d_{i,2}-1} & & & & \\ \sqrt{d_{i,2}-1} & 0 & & \ddots & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & \sqrt{d_{i,k_i-1}-1} & & \\ & & & & 0 & \sqrt{d_{i,k_i}} \\ & & & & \sqrt{d_{i,k_i}} & 0 \end{bmatrix}$$

of size $k_i \times k_i$ and $S_{i,j}$ is the $j \times j$ leading principal submatrix of S_i .

(b) The multiplicity of each eigenvalue of the matrix $S_{i,j}$, as an eigenvalue of $A(\mathcal{G}_m \{ \mathcal{B}_i \})$, is $n_{i,j} - n_{i,j+1}$ for $j \in \Omega_i$.

(c) The spectral radius of $A(\mathcal{G}_m \{ \mathcal{B}_i \})$ is the largest eigenvalue of the matrix H .

5. Unweighted compound graphs of copies of a generalized Bethe tree.

In this section, we assume that \mathcal{G}_m is any connected unweighted graph and that $\mathcal{B}_1 = \mathcal{B}_2 = \dots = \mathcal{B}_m = \mathcal{B}$, where \mathcal{B} is an unweighted generalized Bethe of k levels in which d_{k-j+1} and n_{k-j+1} are the degree of the vertices and the number of them at the level j . Then

$$\begin{aligned} k_1 &= k_2 = \dots = k_m = k, \\ \Omega_1 &= \dots = \Omega_m = \Omega = \{ j : 1 \leq j \leq k-1, n_{j+1} > n_j \}, \\ F_{i,j} &= F, \end{aligned}$$

where F is a $k \times k$ matrix whose entries are 0 except $F(k, k) = 1$. Moreover, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k-1$, the matrix $T_{i,j}$ is the $j \times j$ leading principal submatrix of

$$T_i = \begin{bmatrix} 1 & \sqrt{d_2-1} & & & & \\ \sqrt{d_2-1} & d_2 & & \ddots & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & \sqrt{d_{k-1}-1} & & \\ & & & & \sqrt{d_{k-1}-1} & \sqrt{d_k} \\ & & & & \sqrt{d_k} & d_k + d(v_i) \end{bmatrix},$$

and

$$G = \begin{bmatrix} T_1 & -\varepsilon_{1,2}F & -\varepsilon_{1,3}F & \dots & -\varepsilon_{1,m}F \\ -\varepsilon_{1,2}F & T_2 & -\varepsilon_{2,3}F & \dots & -\varepsilon_{2,m}F \\ -\varepsilon_{1,3}F & -F & \ddots & & \vdots \\ \vdots & \vdots & & T_{m-1} & -\varepsilon_{m-1,m}F \\ -\varepsilon_{1,m}F & -\varepsilon_{2,m}F & \dots & -\varepsilon_{m-1,m}F & T_m \end{bmatrix}.$$

We recall that the Kronecker product [8] of two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ of sizes $m \times m$ and $n \times n$, respectively, is defined to be the $(mn) \times (mn)$ matrix $A \otimes B = (a_{i,j}B)$. For matrices A, B, C and D of appropriate sizes,

$$(A \otimes B)(C \otimes D) = (AC \otimes BD).$$

We write $\mathcal{G}_m \{\mathcal{B}\}$ instead of $\mathcal{G}_m \{\mathcal{B}_i\}$.

From now on

$$B = \begin{bmatrix} 1 & \sqrt{d_2 - 1} & & & & \\ \sqrt{d_2 - 1} & d_2 & & \ddots & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & \sqrt{d_{k-1} - 1} & & \\ & & & & d_{k-1} & \sqrt{d_k} \\ & & & & \sqrt{d_k} & d_k \end{bmatrix}.$$

THEOREM 5.1. *If \mathcal{B} is an unweighted generalized Bethe tree and \mathcal{G}_m is an unweighted graph, then*

$$\sigma(L(\mathcal{G}_m \{\mathcal{B}\})) = (\cup_{j \in \Omega} \sigma(B_j)) \cup (\cup_{s=1}^m \sigma(B + \gamma_s F))$$

where, for $j = 1, 2, \dots, k-1$, B_j is the $j \times j$ leading principal submatrix of

$$B + \gamma_s F = \begin{bmatrix} 1 & \sqrt{d_2 - 1} & & & & \\ \sqrt{d_2 - 1} & d_2 & & \ddots & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & \sqrt{d_{k-1} - 1} & & \\ & & & & d_{k-1} & \sqrt{d_k} \\ & & & & \sqrt{d_k} & d_k + \gamma_s \end{bmatrix},$$

and $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{m-1} > \gamma_m = 0$ are the Laplacian eigenvalues of \mathcal{G}_m .

Proof. By Theorem 3.8,

$$\sigma(L(\mathcal{G}_m \{\mathcal{B}_i\})) = (\cup_{i=1}^m \cup_{j \in \Omega_i} \sigma(T_{i,j})) \cup \sigma(G).$$

For $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k-1$, $T_{i,j} = B_j$. As a consequence,

$$\sigma(L(\mathcal{G}_m \{\mathcal{B}\})) = (\cup_{j \in \Omega} \sigma(B_j)) \cup \sigma(G).$$

It remains to prove that $\sigma(G) = \cup_{s=1}^m \sigma(L(B + \gamma_s F))$. We may write

$$G = I_m \otimes B + L(\mathcal{G}_m) \otimes F,$$

where

$$L(\mathcal{G}_m) = \begin{bmatrix} d(v_1) & -\varepsilon_{1,2} & -\varepsilon_{1,3} & \cdots & -\varepsilon_{1,m} \\ -\varepsilon_{1,2} & d(v_2) & -\varepsilon_{2,3} & \cdots & -\varepsilon_{2,m} \\ -\varepsilon_{1,3} & -\varepsilon_{2,3} & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & -\varepsilon_{m-1,m} \\ -\varepsilon_{1,m} & -\varepsilon_{2,m} & \cdots & -\varepsilon_{m-1,m} & d(v_m) \end{bmatrix}$$

is the Laplacian matrix of \mathcal{G}_m . Let $\gamma_1, \gamma_2, \dots, \gamma_m$ be the eigenvalues $L(\mathcal{G}_m)$, and let

$$V = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_{m-1} \quad \mathbf{v}_m]$$

be an orthogonal matrix whose columns $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are eigenvectors corresponding to the eigenvalues $\gamma_1, \gamma_2, \dots, \gamma_m$. Therefore,

$$\begin{aligned} (V \otimes I_k) G (V^T \otimes I_k) &= (V \otimes I_k) (I_m \otimes B + L(\mathcal{G}_m) \otimes F) (V^T \otimes I_k) \\ &= I_m \otimes B + (VL(\mathcal{G}_m)V^T) \otimes F. \end{aligned}$$

We have

$$\begin{aligned} (VL(\mathcal{G}_m)V^T) \otimes F &= \begin{bmatrix} \gamma_1 & & & & \\ & \gamma_2 & & & \\ & & & & \\ & & & \gamma_{m-1} & \\ & & & & \gamma_m \end{bmatrix} \otimes F \\ &= \begin{bmatrix} \gamma_1 F & & & & \\ & \gamma_2 F & & & \\ & & & & \\ & & & \gamma_{m-1} F & \\ & & & & \gamma_m F \end{bmatrix}, \end{aligned}$$

and hence,

$$(V \otimes I_k) G (V^T \otimes I_k) = \begin{bmatrix} B + \gamma_1 F & & & & \\ & B + \gamma_2 F & & & \\ & & & & \\ & & & B + \gamma_{m-1} F & \\ & & & & B + \gamma_m F \end{bmatrix}.$$

Since G and $(V \otimes I_k) G (V^T \otimes I_k)$ are similar matrices, we conclude that

$$\sigma(G) = \cup_{s=1}^m \sigma(B + \gamma_s F).$$

The proof is complete. \square

COROLLARY 5.2.

(a) For $j \in \Omega$, the multiplicity of each eigenvalue of B_j , as an eigenvalue of $L(\mathcal{G}_m \{\mathcal{B}\})$, is $m(n_j - n_{j+1})$.

(b) $\det(B + \gamma_s F) = \gamma_s$. In particular, B is a singular matrix.

(c) The spectral radius of $L(\mathcal{G}_m \{\mathcal{B}\})$ is the largest eigenvalue of $B + \gamma_1 F$.

(d) The algebraic connectivity of $L(\mathcal{G}_m \{\mathcal{B}\})$ is the smallest eigenvalue of $B + \gamma_{m-1} F$.

Proof. (a) We have $T_{i,j} = B_j$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k-1$. Then the result is an immediate consequence of part (b) of Theorem 3.8.

(b) It follows easily applying the Gaussian elimination procedure.

(c), (d) The eigenvalues of $L(\mathcal{G}_m \{\mathcal{B}\})$ are the eigenvalues of the matrices B_j for $j \in \Omega$ together with the eigenvalues of the matrices $B + \gamma_s F$. The eigenvalues of each B_j interlace the eigenvalues of any $B + \gamma_s F$. Then the spectral radius of $L(\mathcal{G}_m \{\mathcal{B}\})$ is the maximum of the spectral radii of the matrices $B + \gamma_s F$ and the algebraic connectivity of $L(\mathcal{G}_m \{\mathcal{B}\})$ is the minimum eigenvalue in $\cup_{s=1}^{m-1} \sigma(B + \gamma_s F)$. For $s = 1, 2, \dots, m$,

$$B + \gamma_1 F = (B + \gamma_s F) + (\gamma_1 - \gamma_s) F.$$

We use the fact that the spectral radius of an irreducible nonnegative matrix increases when any of its entries increases [7], to conclude part (c). For $s = 1, 2, \dots, m-1$,

$$B + \gamma_s F = (B + \gamma_{m-1} F) + (\gamma_s - \gamma_{m-1}) F.$$

We use now the fact that the eigenvalues of a Hermitian matrix do not decrease if a positive semidefinite matrix is added to it [3], to conclude part (d). \square

We consider now the adjacency matrix of $\mathcal{G}_m \{\mathcal{B}\}$. For $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k-1$, the matrices $S_{i,j}$, S_i and H in Theorem 4.7 become

$$S_{i,j} = U_j = \begin{bmatrix} 0 & \sqrt{d_2 - 1} & & & & & \\ \sqrt{d_2 - 1} & 0 & & \ddots & & & \\ & & \ddots & & & & \\ & & & \ddots & & & \\ & & & \sqrt{d_{j-1} - 1} & & & \\ & & & & 0 & & \\ & & & & \sqrt{d_j - 1} & & \\ & & & & & \sqrt{d_j - 1} & \\ & & & & & & 0 \end{bmatrix},$$

$$S_1 = S_2 = \dots = S_m = S = \begin{bmatrix} 0 & \sqrt{d_2 - 1} & & & & \\ \sqrt{d_2 - 1} & 0 & \ddots & & & \\ & \ddots & \ddots & & & \\ & & \sqrt{d_{k-1} - 1} & & & \\ & & & \sqrt{d_{k-1} - 1} & 0 & \sqrt{d_k} \\ & & & & \sqrt{d_k} & 0 \end{bmatrix}$$

and

$$H = \begin{bmatrix} S & \varepsilon_{1,2}F & \varepsilon_{1,3}F & \dots & \varepsilon_{1,m}F \\ \varepsilon_{1,2}F & S & \varepsilon_{2,3}F & \dots & \varepsilon_{2,m}F \\ \varepsilon_{1,3}F & F & \ddots & & \vdots \\ \vdots & \vdots & & S & \varepsilon_{m-1,m}F \\ \varepsilon_{1,m}F & \varepsilon_{2,m}F & \dots & \varepsilon_{m-1,m}F & S \end{bmatrix}.$$

THEOREM 5.3. *If \mathcal{B} is an unweighted generalized Bethe tree and \mathcal{G}_m is an unweighted graph, then*

$$\sigma(A(\mathcal{G}_m\{\mathcal{B}\})) = (\cup_{j \in \Omega} \sigma(U_j)) \cup (\cup_{s=1}^m \sigma(S + \gamma_s F)),$$

where $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_{m-1} \geq \gamma_m$ are the eigenvalues of the adjacency matrix of \mathcal{G}_m .

Proof. The proof uses Theorem 4.7 and is similar to the proof of Theorem 5.1. \square

COROLLARY 5.4.

(a) *For $j \in \Omega$, the multiplicity of each eigenvalue of U_j , as an eigenvalue of $A(\mathcal{G}_m\{\mathcal{B}\})$, is $m(n_j - n_{j+1})$.*

(b) *The spectral radius of $A(\mathcal{G}_m\{\mathcal{B}\})$ is the largest eigenvalue of $S + \gamma_1 F$.*

Proof. Similar to the proof of Corollary 5.2. \square

Acknowledgment. The authors wish to thank the referee for the comments which led to an improved version of the paper.

REFERENCES

- [1] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Math. J.*, 23:298–305, 1973.
- [2] G.H. Golub and C.F. Van Loan. *Matrix Computations*, 2nd edition. Johns Hopkins University Press, Baltimore, 1989.

- [3] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.
- [4] R. Merris. Laplacian Matrices of Graphs: A Survey. *Linear Algebra Appl.*, 197/198:143–176, 1994.
- [5] O. Rojo and R. Soto. The spectra of the adjacency matrix and Laplacian matrix for some balanced trees. *Linear Algebra Appl.*, 403:97–117, 2005.
- [6] L.N. Trefethen and D. Bau, III. *Numerical Linear Algebra*. SIAM, Philadelphia, 1997.
- [7] R.S. Varga. *Matrix Iterative Analysis*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.
- [8] F. Zhang. *Matrix Theory*. Springer-Verlag, New York, 1999.