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SOME RESULTS ON GROUP INVERSES OF BLOCK MATRICES
OVER SKEW FIELDS

CHANGJIANG BU†, JIEMEI ZHAO†, AND KUIZE ZHANG†

Abstract. In this paper, necessary and sufficient conditions are given for the existence of the group inverse of the block matrix \( \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \) over any skew field, where \( A, B \) are both square and \( rank(B) \geq rank(A) \). The representation of this group inverse and some relative additive results are also given.

Key words. Skew, Block matrix, Group inverse.

AMS subject classifications. 15A09.

1. Introduction. Let \( K \) be a skew field and \( K^{n \times n} \) be the set of all matrices over \( K \). For \( A \in K^{n \times n} \), the matrix \( X \in K^{n \times n} \) is said to be the group inverse of \( A \), if
\[
AXA = A, XAX = X, AX = XA.
\]
We then write \( X = A^\sharp \). It is well known that if \( A^\sharp \) exists, it is unique; see [16].

Research on representations of the group inverse of block matrices is an important effort in generalized inverse theory of matrices; see [14] and [13]. Indeed, generalized inverses are useful tools in areas such as special matrix theory, singular differential and difference equations and graph theory; see [5], [9], [11], [12] and [15]. For example, in [9] it is shown that the adjacency matrix of a bipartite graph can be written in the form of \( \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \), and necessary and sufficient conditions are given for the existence and representation of the group inverse of a block matrix \( \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \).

In 1979, Campbell and Meyer proposed the problem of finding an explicit representation for the Drazin (group) inverse of a \( 2 \times 2 \) block matrix \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) in terms of its sub-blocks, where \( A \) and \( D \) are required to be square matrices; see [5]. In [10] a condition for the existence of the group inverse of \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is given under the as-

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sumption that $A$ and $(I + CA^{-2}B)$ are both invertible over any field; however, the representation of the group inverse is not given. The representation of the group inverse of a block matrix \(egin{pmatrix} A & B \\ 0 & C \end{pmatrix}\) over skew fields has been given in 2001; see [6]. The representation of the Drazin (group) inverse of a block matrix of the form \(egin{pmatrix} A & B \\ C & 0 \end{pmatrix}\) (A is square, 0 is square null matrix) has not been given since it was proposed as a problem by Campbell in 1983; see [4]. However, there are some references in the literature about representations of the Drazin (group) inverse of the block matrices \(egin{pmatrix} A & B \\ C & 0 \end{pmatrix}\) under certain conditions. Some results are on matrices over the field of complex numbers, e.g., in [8]; or when $A = B = I_n$ in [7]; or when $A, B, C \in \{P, P^*, PP^*\}$, $P^2 = P$ and $P^*$ is the conjugate transpose of $P$. Some results are over skew fields, e.g., in [1], when $A = I_n$ and $\text{rank}(CB)^2 = \text{rank}(B) = \text{rank}(C)$; in [3] when $A = B$, $A^2 = A$. In addition, in [2] results are given on the group inverse of the product of two matrices over a skew field, as well as some related properties.

In this paper, we mainly give necessary and sufficient conditions for the existence and the representation of the group inverse of a block matrix \(egin{pmatrix} A & A \\ B & 0 \end{pmatrix}\) or \(egin{pmatrix} A & B \\ A & 0 \end{pmatrix}\), where $A, B \in K^{n \times n}$, $\text{rank}(B) \geq \text{rank}(A)$. We also give a sufficient condition for $AB$ to be similar to $BA$.

Letting $A \in K^{m \times n}$, the order of the maximum invertible sub-block of $A$ is said to be the rank of $A$, denoted by $\text{rank}(A)$; see [17]. Let $A, B \in K^{n \times n}$. If there is an invertible matrix $P \in K^{n \times n}$ such that $B = PAP^{-1}$, then $A$ and $B$ are similar; see [17].

2. Some Lemmas.

**Lemma 2.1.** Let $A, B \in K^{n \times n}$. If $\text{rank}(A) = r$, $\text{rank}(B) = \text{rank}(AB) = \text{rank}(BA)$, then there are invertible matrices $P, Q \in K^{n \times n}$ such that

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q, \quad B = Q^{-1} \begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix} P^{-1},$$

where $B_1 \in K^{r \times r}$, $X \in K^{r \times (n-r)}$, and $Y \in K^{(n-r) \times r}$.

**Proof.** Since $\text{rank}(A) = r$, there are nonsingular matrices $P, Q \in K^{n \times n}$ such that

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q, \quad B = Q^{-1} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} P^{-1},$$
where \( B_1 \in K^{r \times r} \), \( B_2 \in K^{r \times (n-r)} \), \( B_3 \in K^{(n-r) \times r} \), and \( B_4 \in K^{(n-r) \times (n-r)} \). From \( \text{rank}(B) = \text{rank}(AB) \), we have
\[
B_3 = YB_1, \quad B_4 = YB_2, \quad Y \in K^{(n-r) \times r}.
\]

Since \( \text{rank}(B) = \text{rank}(BA) \), we obtain
\[
B_2 = B_1X, \quad B_4 = B_3X, \quad X \in K^{r \times (n-r)}.
\]

So
\[
B = Q^{-1} \begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix} P^{-1}.
\]

**Lemma 2.2.** \([6]\) Let \( A \in K^{r \times r}, B \in K^{(n-r) \times r} \), \( M = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \in K^{n \times n} \).

Then the group inverse of \( M \) exists if and only if the group inverse of \( A \) exists and \( \text{rank}(A) = \text{rank} \left( \begin{array}{c} A \\ B \end{array} \right) \). If the group inverse of \( M \) exists, then
\[
M^\# = \begin{pmatrix} A^\# & 0 \\ B(A^\#)^2 & 0 \end{pmatrix}.
\]

**Lemma 2.3.** \([6]\) Let \( A \in K^{r \times r}, B \in K^{(n-r) \times r} \), \( M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \in K^{n \times n} \).

Then the group inverse of \( M \) exists if and only if the group inverse of \( A \) exists and \( \text{rank}(A) = \text{rank} \left( \begin{array}{c} A \\ B \end{array} \right) \). If the group inverse of \( M \) exists, then
\[
M^\# = \begin{pmatrix} A^\# & (A^\#)^2B \\ 0 & 0 \end{pmatrix}.
\]

**Lemma 2.4.** \([2]\) Let \( A \in K^{m \times n}, B \in K^{n \times m} \). If \( \text{rank}(A) = \text{rank}(BA) \), \( \text{rank}(B) = \text{rank}(AB) \), then the group inverse of \( AB \) and \( BA \) exist.

**Lemma 2.5.** Let \( A, B \in K^{n \times n} \). If \( \text{rank}(A) = \text{rank}(B) = \text{rank}(AB) = \text{rank}(BA) \), then the following conclusions hold:

(i) \( AB(AB)^\#A = A \),
(ii) \( A(BA)^\#BA = A \),
(iii) \( BA(BA)^\#B = B \),
(iv) \( B(AB)^\#A = BA(BA)^\# \),
(v) \( A(BA)^\# = (BA)^\#A \).
Proof. Suppose \( \text{rank}(A) = r \). By Lemma 2.1, we have
\[
A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q, \quad B = Q^{-1} \begin{pmatrix} B_1 & B_1 X \\ YB_1 & YB_1 X \end{pmatrix} P^{-1},
\]
where \( B_1 \in K^{r \times r} \), \( X \in K^{r \times (n-r)} \), \( Y \in K^{(n-r) \times r} \). Then
\[
AB = P \begin{pmatrix} B_1 & B_1 X \\ 0 & 0 \end{pmatrix} P^{-1}, \quad BA = Q^{-1} \begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix} Q.
\]
Since \( \text{rank}(A) = \text{rank}(B) \), we have that \( B_1 \) is invertible. By using Lemma 2.2 and Lemma 2.3, we get
\[
(AB)^\dagger = P \begin{pmatrix} B_1^{-1} & B_1^{-1} X \\ 0 & 0 \end{pmatrix} P^{-1}, \quad (BA)^\dagger = Q^{-1} \begin{pmatrix} B_1^{-1} & 0 \\ YB_1^{-1} & 0 \end{pmatrix} Q.
\]
Then
\[
\begin{array}{l}
(i) \quad AB(AB)^\dagger A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q = A, \\
(ii) \quad A(BA)^\dagger BA = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q = A, \\
(iii) \quad BA(AB)^\dagger B = Q^{-1} \begin{pmatrix} B_1 & B_1 X \\ YB_1 & YB_1 X \end{pmatrix} P^{-1} = B, \\
(iv) \quad B(AB)^\dagger A = Q^{-1} \begin{pmatrix} I_r & 0 \\ Y & 0 \end{pmatrix} Q = BA(AB)^\dagger, \\
(v) \quad A(BA)^\dagger = P \begin{pmatrix} B_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q = (AB)^\dagger A. \quad \square
\end{array}
\]

3. Conclusions.

**Theorem 3.1.** Let \( M = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix} \), where \( A, B \in K^{n \times n} \), \( \text{rank}(B) \geq \text{rank}(A) = r \). Then

\( (i) \) The group inverse of \( M \) exists if and only if \( \text{rank}(A) = \text{rank}(B) = \text{rank}(AB) = \text{rank}(BA) \).

\( (ii) \) If the group inverse of \( M \) exists, then \( M^\dagger = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \), where
\[
\begin{align*}
M_{11} &= (AB)^\dagger A - (AB)^\dagger A^2(BA)^\dagger B, \\
M_{12} &= (AB)^\dagger A, \\
M_{21} &= (BA)^\dagger B - B(AB)^\dagger A^2(BA)^\dagger + B(AB)^\dagger A(AB)^\dagger A^2(BA)^\dagger B, \\
M_{22} &= -B(AB)^\dagger A^2(BA)^\dagger.
\end{align*}
\]
Proof. (i) It is obvious that the condition is sufficient. Now we show that the condition is necessary.

\[ \text{rank}(M) = \text{rank} \begin{pmatrix} A & A \\ B & 0 \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \text{rank}(A) + \text{rank}(B), \]

\[ \text{rank}(M^2) = \text{rank} \begin{pmatrix} A^2 + AB & A^2 \\ BA & BA \end{pmatrix} = \text{rank} \begin{pmatrix} AB & A^2 \\ 0 & BA \end{pmatrix}. \]

Since the group inverse of \( M \) exists if and only if \( \text{rank}(M) = \text{rank}(M^2) \), we have

\[ \text{rank}(A) + \text{rank}(B) = \text{rank}(M^2) \]

\[ \leq \text{rank}(AB) + \text{rank} \begin{pmatrix} A^2 \\ BA \end{pmatrix} \]

\[ \leq \text{rank}(AB) + \text{rank}(A), \]

\[ \text{rank}(A) + \text{rank}(B) = \text{rank}(M^2) \]

\[ \leq \text{rank} \begin{pmatrix} AB & A^2 \\ BA & BA \end{pmatrix} + \text{rank}(BA) \]

\[ \leq \text{rank}(BA) + \text{rank}(A). \]

Then \( \text{rank}(B) \leq \text{rank}(AB) \), and \( \text{rank}(B) \leq \text{rank}(BA) \). Therefore

\[ \text{rank}(B) = \text{rank}(AB) = \text{rank}(BA). \]

From \( \text{rank}(B) = \text{rank}(AB) \leq \text{rank}(A) \), and \( \text{rank}(A) \leq \text{rank}(B) \), we have

\[ \text{rank}(A) = \text{rank}(B). \]

Since \( \text{rank}(A) + \text{rank}(B) \leq \text{rank} \begin{pmatrix} AB & A^2 \\ BA & BA \end{pmatrix} + \text{rank}(BA) \), and \( \text{rank} \begin{pmatrix} AB & A^2 \\ BA & BA \end{pmatrix} \leq \text{rank}(A) \), we get \( \text{rank} \begin{pmatrix} AB & A^2 \\ BA & BA \end{pmatrix} = \text{rank}(A) \). Thus

\[ \text{rank} \begin{pmatrix} AB & A^2 \\ BA & BA \end{pmatrix} = \text{rank}(AB). \]

Then there exists a matrix \( U \in K^{n \times n} \) such that \( ABU = A^2 \). Then

\[ \text{rank}(M^2) = \text{rank} \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} = \text{rank}(AB) + \text{rank}(BA). \]

So we get

\[ \text{rank}(A) = \text{rank}(B) = \text{rank}(AB) = \text{rank}(BA). \]
(ii) Let \( X = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \). We will prove that the matrix \( X \) satisfies the conditions of the group inverse. Firstly, we compute

\[
MX = \begin{pmatrix} AM_{11} + AM_{21} & AM_{12} + AM_{22} \\ BM_{11} & BM_{12} \end{pmatrix},
\]

\[
XM = \begin{pmatrix} M_{11}A + M_{12}B & M_{11}A \\ M_{21}A + M_{22}B & M_{21}A \end{pmatrix}.
\]

Applying Lemma 2.5 (i), (ii), and (v), we have

\[
AM_{11} + AM_{21} = A(AB)^\dagger A - A(AB)^\dagger A^2(BA)^\dagger B + A(BA)^\dagger B - AB(AB)^\dagger A^2(BA)^\dagger
\]
\[
+ AB(AB)^\dagger A(AB)^\dagger A^2(BA)^\dagger B = A(AB)^\dagger B,
\]

\[
M_{11}A + M_{12}B = (AB)^\dagger A^2 - (AB)^\dagger A^2(BA)^\dagger BA + (AB)^\dagger AB
\]
\[
= (AB)^\dagger A^2 - (AB)^\dagger A^2 + (AB)^\dagger AB
\]
\[
= A(AB)^\dagger B.
\]

Using Lemma 2.5 (i), (ii), and (v), we get

\[
AM_{12} + AM_{22} = A(AB)^\dagger A - AB(AB)^\dagger A^2(BA)^\dagger
\]
\[
= A(AB)^\dagger A - A^2(BA)^\dagger
\]
\[
= 0,
\]

\[
M_{11}A = (AB)^\dagger A^2 - (AB)^\dagger A^2(BA)^\dagger BA
\]
\[
= (AB)^\dagger A^2 - (AB)^\dagger A^2
\]
\[
= 0.
\]

From Lemma 2.5 (ii), we obtain

\[
BM_{11} = B(AB)^\dagger A - B(AB)^\dagger A^2(BA)^\dagger B,
\]

\[
M_{21}A + M_{22}B = (BA)^\dagger BA - B(AB)^\dagger A^2(BA)^\dagger A + B(AB)^\dagger A^2(BA)^\dagger A(BA)^\dagger BA
\]
\[
- B(AB)^\dagger A^2(BA)^\dagger B
\]
\[
= (BA)^\dagger BA - [B(AB)^\dagger A^2(BA)^\dagger A - B(AB)^\dagger A^2(BA)^\dagger A]
\]
\[
- B(AB)^\dagger A^2(BA)^\dagger B
\]
\[
= B(AB)^\dagger A - B(AB)^\dagger A^2(BA)^\dagger B.
\]
Using Lemma 2.5 (ii), we have

\[ BM_{12} = B(AB)^\sharp A, \]
\[ M_{21}A = (BA)^\sharp BA - B(AB)^\sharp A^2(BA)^\sharp A + B(AB)^\sharp A^2(BA)^\sharp A(AB)^\sharp BA \]
\[ = B(AB)^\sharp A. \]

So

\[ MX = XM = \begin{pmatrix} A(AB)^\sharp B & 0 \\ B(AB)^\sharp A - B(AB)^\sharp A^2(BA)^\sharp B & B(AB)^\sharp A \end{pmatrix}. \]

Secondly,

\[
\begin{align*}
MXM &= \begin{pmatrix} A & A \\ B & 0 \end{pmatrix} \begin{pmatrix} A(AB)^\sharp B & 0 \\ B(AB)^\sharp A - B(AB)^\sharp A^2(BA)^\sharp B & B(AB)^\sharp A \end{pmatrix} \\
&= \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & 0 \end{pmatrix}.
\end{align*}
\]

Applying Lemma 2.5 (i) and (iii), we compute

\[
\begin{align*}
X_{11} &= A^2(BA)^\sharp B + AB(AB)^\sharp A - AB(AB)^\sharp A^2(BA)^\sharp B \\
&= AB(AB)^\sharp A \\
&= A, \\
X_{12} &= AB(AB)^\sharp A = A, \\
X_{21} &= BA(AB)^\sharp B = B.
\end{align*}
\]

Hence

\[ MXM = \begin{pmatrix} A & A \\ B & 0 \end{pmatrix}. \]

Finally,

\[
\begin{align*}
XMX &= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A(AB)^\sharp B & 0 \\ B(AB)^\sharp A - B(AB)^\sharp A^2(BA)^\sharp B & B(AB)^\sharp A \end{pmatrix} \\
&= \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}.
\end{align*}
\]

Then

\[
\begin{align*}
Y_{11} &= (AB)^\sharp A^2(BA)^\sharp B - (AB)^\sharp A^2(BA)^\sharp BA(AB)^\sharp B + (AB)^\sharp AB(AB)^\sharp A \\
&- (AB)^\sharp AB(AB)^\sharp A^2(BA)^\sharp B \\
&= (AB)^\sharp A - (AB)^\sharp A^2(BA)^\sharp B \\
&= M_{11},
\end{align*}
\]
and

\[ Y_{12} = M_{12}B(AB)^{\dagger}A \]
\[ = (AB)^{\dagger}AB(AB)^{\dagger}A \]
\[ = (AB)^{\dagger}A = M_{12}. \]

We can easily get

\[ Y_{21} = M_{21}A(AB)^{\dagger}B + M_{22}B(AB)^{\dagger}A - M_{22}B(AB)^{\dagger}A^2(AB)^{\dagger}B \]
\[ = M_{21}; \]
\[ Y_{22} = M_{22}B(AB)^{\dagger}A = M_{22}. \]

So we have \( X = M^{\dagger}. \] 

**Theorem 3.2.** Let \( M = \begin{pmatrix} A & B \\ A & 0 \end{pmatrix} \), where \( A, B \in K^{n \times n} \), \( \text{rank}(B) \geq \text{rank}(A) = r \). Then

(i) The group inverse of \( M \) exists if and only if \( \text{rank}(A) = \text{rank}(B) = \text{rank}(AB) = \text{rank}(BA) \).

(ii) If the group inverse of \( M \) exists, then \( M^{\dagger} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \), where

\[ Z_{11} = (AB)^{\dagger}A - B(AB)^{\dagger}A^2(AB)^{\dagger}, \]
\[ Z_{12} = B(AB)^{\dagger} - (AB)^{\dagger}A^2(AB)^{\dagger}B + B(AB)^{\dagger}A^2(AB)^{\dagger}A(AB)^{\dagger}B, \]
\[ Z_{21} = (AB)^{\dagger}A, \]
\[ Z_{22} = - (AB)^{\dagger}A^2(AB)^{\dagger}B. \]

**Proof.** Let \( X = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \). By Lemma 2.5, we have

\[ MX = XM = \begin{pmatrix} B(AB)^{\dagger}A & A(AB)^{\dagger}B - B(AB)^{\dagger}A^2(AB)^{\dagger}B \\ 0 & A(AB)^{\dagger}B \end{pmatrix}. \]

Furthermore, we can prove \( MXM = M, XMX = X \) easily. Thus, \( X = M^{\dagger}. \]

**Theorem 3.3.** Let \( A, B \in K^{n \times n} \), if \( \text{rank}(B) = \text{rank}(AB) = \text{rank}(BA) \). Then \( AB \) and \( BA \) are similar.

**Proof.** Suppose \( \text{rank}(A) = r \), using Lemma 2.1, there are invertible matrices \( P, Q \in K^{n \times n} \) such that

\[ A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q, \quad B = Q^{-1} \begin{pmatrix} B_1 & B_1X \\ YB_1 & YB_1X \end{pmatrix} P^{-1}. \]
where $B_1 \in K^{r \times r}, X \in K^{r \times (n-r)}, Y \in K^{(n-r) \times r}$. Hence

$$AB = P \begin{pmatrix} B_1 & B_1X \\ 0 & 0 \end{pmatrix} P^{-1} = P \begin{pmatrix} I_r & -X \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_r & X \\ 0 & I_{n-r} \end{pmatrix} P^{-1},$$

$$BA = Q^{-1} \begin{pmatrix} B_1 & 0 \\ YB_1 & 0 \end{pmatrix} Q = Q^{-1} \begin{pmatrix} I_r & 0 \\ Y & I_{n-r} \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_r & 0 \\ -Y & I_{n-r} \end{pmatrix} Q.$$

So $AB$ and $BA$ are similar. □

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