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ON LOW RANK PERTURBATIONS OF COMPLEX MATRICES AND SOME DISCRETE METRIC SPACES*

LEV GLEBSKY† AND LUIS MANUEL RIVERA‡

Abstract. In this article, several theorems on perturbations of a complex matrix by a matrix of a given rank are presented. These theorems may be divided into two groups. The first group is about spectral properties of a matrix under such perturbations; the second is about almost-near relations with respect to the rank distance.

Key words. Complex matrices, Rank, Perturbations.

AMS subject classifications. 15A03, 15A18.

1. Introduction.

In this article, we present several theorems on perturbations of a complex matrix by a matrix of a given rank. These theorems may be divided into two groups. The first group is about spectral properties of a matrix under such perturbations; the second is about almost-near relations with respect to the rank distance.

1.1. The first group.

Theorem 2.1 gives necessary and sufficient conditions on the Weyr characteristics, [20], of matrices A and B if $rank(A - B) \leq k$. In one direction the theorem is known; see [12, 13, 14]. For $k = 1$ the theorem is a reformulation of a theorem of Thompson [16] (see Theorem 2.3 of the present article). We prove Theorem 2.1 by induction with respect to k. To this end, we introduce a discrete metric space (the space of Weyr characteristics) and prove that this metric space is geodesic. In fact, the induction (with respect to k) is hidden in this proof (see Proposition 3.6 and Proposition 3.1). Theorem 2.6 with Theorem 2.4 give necessary and sufficient conditions on the spectra of self-adjoint (unitary) matrices A and B if $rank(A - B) \leq k$. The proof of Theorem 2.6 (Theorem 2.7) is similar to the proof of Theorem 2.1: for $k = 1$ it is well known; to proceed further we introduce another discrete metric space and prove that it is geodesic. Theorem 2.6 is proved in [17] for self-adjoint A, B and positive $A - B$; the positivity assumption looks essential in the proof of [17]. It is interesting to compare Theorem 2.6 with results of [17] about singular values under bounded rank perturbations.

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Theorem 2.4 is an analogue of the Cauchy interlacing theorem [2] about spectra of principal submatrices of a self-adjoint matrix. For normal matrices, it seems to be new and related with Theorem 5.2 of [9]. Both results have very similar proofs.

1.2. The second group. A matrix A is almost unitary (self-adjoint, normal) if $rank(A^*A-E)$ ($rank(A-A^*)$, $rank(A^*A-AA^*)$) is small with respect to the size of the matrices. Matrix A is near unitary (self-adjoint, normal) if there exists a unitary (self-adjoint, normal) matrix B with small $rank(A-B)$. Theorem 2.9 says that an almost self-adjoint matrix is a near self-adjoint (this result is trivial). Theorem 2.10 says the same for almost unitary matrices. We don't know if every almost normal matrix is near normal. For a matrix A with a simple spectrum, Theorem 2.11 gives an example of a matrix that almost commutes with A and is far from any matrix that commutes with A.

Some of our motivation is the following. The equation $d(A,B) = rank(A-B)$ defines a metric on the set of $n \times n$-matrices (the *arithmetic distance*, according to L.K. Hua, [19], Definition 3.1). Almost-near questions have been considered for the norm distance $dn(A,B) = \|A-B\|$ with $\|\cdot\|$ being a supremum operator norm. In [11] the following assertion is proved: For any $\delta > 0$ there exists $\epsilon_\delta > 0$ such that if $\|AA^* - A^*A\| \leq \epsilon_\delta$ then there exists a normal B with $\|A-B\| \leq \delta$. The ϵ_δ is independent of the size of the matrices. It is equivalent to the following: close to any pair of almost commuting self-adjoint matrices there exists a pair of commuting self-adjoint matrices (with respect to the norm distance $dn(\cdot,\cdot)$). On the other hand, there are almost commuting (with respect to $dn(\cdot,\cdot)$) unitary matrices, close to which there are no commuting matrices, [3, 5, 18]. Similar questions have been studied for operators in Hilbert spaces (Calkin algebras, [7]). In Hilbert spaces, an operator a is called essentially normal if $aa^* - a^*a$ is a compact operator. In contrast with Theorem 2.10, there exists an essentially unitary operator that is not a compact perturbation of a unitary operator (just infinite 0-Jordan cell; see the example below Theorem 2.10). There is a complete characterization of compact perturbations of normal operators; see [7] and the bibliography therein. So, our Theorems 2.4, 2.10, 2.11 may be considered as the start of an investigation in the direction described with the arithmetic distance instead of the norm distance.

In Section 2, we formulate the main theorems of the article; in Section 3, we define discrete distances needed in the proofs and investigate their properties; in Section 4, we prove the main theorems.

We use the notations $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{Z}^+ = \{1, 2, ...\}$.

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2. Formulation of main results and discussion.

2.1. Spectral properties of general matrices. Let $\eta_m(A,\lambda)$ denote the number of λ-Jordan blocks in A of size greater or equal than m ($m \in \mathbb{Z}^+$):

$$\eta_m(A,\lambda) = \dim\ Ker(\lambda E - A)^m - \dim\ Ker(\lambda E - A)^{m-1}.$$

The function $\eta_{(\cdot)}(A,\cdot) : \mathbb{Z}^+ \times \mathbb{C} \to \mathbb{N}$ is called the *Weyr characteristic* of the matrix A, [15, 20]. It is clear that $\eta_m(A,\lambda)$ is nonzero for finitely many pairs (m,λ) only and $\eta_{m+1}(A,\lambda) \le \eta_m(A,\lambda)$.

THEOREM 2.1. *Let $A \in \mathbb{C}_{n\times n}$ with Weyr characteristic $\eta_m(A,\lambda)$. Then $\nu_m(\lambda)$ is the Weyr characteristic of some $B \in \mathbb{C}_{n\times n}$ with $rank(A-B) \le k$ if and only if the following conditions are satisfied:*

- $\mid \eta_m(A,\lambda) - \nu_m(\lambda) \mid \le k$ *for any* $\lambda \in \mathbb{C}$ *and any* $m \in \mathbb{Z}^+$.
- $\nu_{m+1}(\lambda) \le \nu_m(\lambda)$.
- $\sum_{\lambda\in\mathbb{C}} \sum_{m\in\mathbb{Z}^+} \nu_m(\lambda) = n$.

The pole assignment theorem (see, for example, Theorem 6.5.1 of [8]) is an easy corollary of Theorem 2.1.

COROLLARY 2.2 (Pole assignment theorem). *Suppose that $A \in \mathbb{C}_{n\times n}$ has a geometrically simple spectrum (for each eigenvalue there is a unique corresponding Jordan cell in the Jordan normal form of A). Suppose that B runs over all $n\times n$-matrices such that $rank(A-B) = 1$. Then the spectrum of B runs over all multisubsets of $\mathbb{C}$ of size n.*

In one direction, Theorem 2.1 easily follows from the Thompson's theorem formulated below. In [14] a direct proof of Theorem 2.1 (in one direction) is given. For $rank(A-B) = 1$, the above theorem is just a reformulation of Thompson's theorem. We proceed by a "hidden" induction on $rank(A-B)$, using the fact that the space $\mathbb{C}_{n\times n}$ with arithmetic distance and the space of the Weyr characteristics are geodesic metric spaces.

THEOREM 2.3 (Thompson,[16]). *Let $\mathbb{F}$ be a field and let $A \in \mathbb{F}_{n\times n}$ have similarity invariants $h_n(A) \mid h_{n-1}(A) \mid \cdots \mid h_1(A)$. Then, as column n-tuple x and row n-tuple y range over all vectors with entries in $\mathbb{F}$, the similarity invariants assumed by the matrix:*

$$B = A + xy$$

are precisely the monic polynomials $h_n(B) \mid \cdots \mid h_1(B)$ over $\mathbb{F}$ for which

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$degree(h_1(B)\cdots h_n(B)) = n$ and

$$h_n(B) \mid h_{n-1}(A) \mid h_{n-2}(B) \mid h_{n-3}(A) \mid \cdots,$$
$$h_n(A) \mid h_{n-1}(B) \mid h_{n-2}(A) \mid h_{n-3}(B) \mid \cdots.$$

2.2. Spectra of normal matrices. We say that the vector x is an α-eigenvector of A if $Ax = \alpha x$. We denote by $\mathcal{R}(A, \lambda, \epsilon)$ the span of all α-eigenvectors of A with $|\lambda - \alpha| \leq \epsilon$.

THEOREM 2.4. *If A and B are normal matrices, then for any λ, and for any $\epsilon \geq 0$,*

$$\mid dim(\mathcal{R}(A, \lambda, \epsilon)) - dim(\mathcal{R}(B, \lambda, \epsilon)) \mid \leq rank(A - B)$$

Let $O_\epsilon(\lambda) = \{x \in \mathbb{C} \mid |x - \lambda| \leq \epsilon\}$. For finite complex multisets $\mathcal{A}, \mathcal{B}$ (unordered tuples of complex numbers, notation: $\mathcal{A}, \mathcal{B} \subset_M \mathbb{C}$) let

$$dc(\mathcal{A}, \mathcal{B}) = \max_{O \in \mathcal{S}} \{|(|\mathcal{A} \cap O| - |\mathcal{B} \cap O|)|\},$$ [1]

where $\mathcal{S} = \{O_\epsilon(\lambda) \mid \lambda \in \mathbb{C},\ \epsilon > 0\}$. Theorem 2.4 implies that any ball on the complex plain, containing m spectral points of A must contain at least $m - k$ spectral points of B and vice versa. So, we have:

COROLLARY 2.5. *If A and B are normal matrices then $dc(sp(A), sp(B)) \leq rank(A - B)$.*

Does Corollary 2.5 describe all spectra accessible by a rank k perturbation? We show that the answer is "yes" for self-adjoint and unitary matrices.

THEOREM 2.6. *Let A be a self-adjoint (unitary) $n \times n$-matrix. Let $\mathcal{B} \subset_M \mathbb{R}$ ($\mathcal{B} \subset_M S^1$), $|\mathcal{B}| = n$. Then there exists a self-adjoint (unitary) matrix B such that $sp(B) = \mathcal{B}$ and $rank(A - B) = dc(sp(A), \mathcal{B})$.*

By Proposition 3.4 of Section 3, the theorem is equivalent to:

THEOREM 2.7. *Let $l \subset \mathbb{C}$ be a circumference or a straight line. Let A be a normal $n \times n$-matrix, $sp(A) \subset_M l$. Let $\mathcal{B} \subset_M l$, $|\mathcal{B}| = n$. Then there exists a normal matrix B such that $sp(B) = \mathcal{B}$ and $rank(A - B) = dc(sp(A), \mathcal{B})$.*

For self-adjoint matrices, Theorem 2.6 is related to the inverse Cauchy interlacing theorem, see [6]. In the work of Thompson [17], Theorem 2.6 is proved for self-adjoint

[1] If X is a set or multiset then $|X|$ denotes the cardinality of X. If x is a number then $|x|$ denotes the absolute value of x.

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ELA
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A, B and positive $A-B$. In this case we may order the spectrum of A and B, say $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ and $\beta_1 \le \beta_2 \le \cdots \le \beta_b$, respectively. Due to positivity of $A-B$, one has $\alpha_i \ge \beta_i$ and $dc(\alpha, \beta) \le k$ is equivalent to:

$$\beta_i \le \alpha_i \le \beta_{i+k}. \tag{2.1}$$

If $A - B$ is not positive (negative), then the condition $\alpha_i \ge \beta_i$ ($\alpha_i \le \beta_i$) is not valid any more. Now, the condition $dc(\alpha, \beta) \le k$ is not equivalent to inequalities of type (2.1). It is interesting to compare it with results of [17] about singular values under bounded rank perturbations. Small rank perturbations of positive operators are considered in [4].

It is also worth noting the work [10], where the author studies the relationships of the spectra of self-adjoint matrices H_1, H_2, and $H_1 + H_2$.

2.3. Almost-near relations.

Definition 2.8. *A matrix $A \in \mathbb{C}_{n\times n}$ is said to be*

- *k-self-adjoint if $rank(A - A^*) \le k$.*
- *k-unitary if $rank(AA^* - E) \le k$, where E is the unit matrix.*
- *k-normal if $rank(AA^* - A^*A) \le k$.*

Theorem 2.10 (Theorem 2.9) formulated below says that in a k-neighborhood of a k-unitary (k-self-adjoint) matrix there exists a unitary (self-adjoint) matrix. We don't know if a similar result is valid for k-normal matrices. Precisely, we are trying to prove (or disprove) the following:

Conjecture. For any $\delta > 0$, there exists $\epsilon > 0$ such that in a $(\delta \cdot n)$-neighborhood of an $(\epsilon \cdot n)$-normal matrix there exists a normal matrix (δ, ϵ are independent of the size n of the matrices).

Theorem 2.9. *For any $A \in \mathbb{C}_{n\times n}$, there exists a self-adjoint matrix S such that $rank(A - S) = rank(A - A^*)$.*

Proof. Take $S = \frac{1}{2}(A + A^*)$. $\square$

Theorem 2.10. *For any $A \in \mathbb{C}_{n\times n}$, there exists a unitary matrix U such that $rank(A - U) = rank(A^*A - E)$.*

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ELA
http://math.technion.ac.il/iic/ela

A good illustration for this theorem is a 0-Jordan cell:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix},$$

but the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \multicolumn{7}{c}{\dotfill} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is unitary.

Let us return to the almost-commuting matrices. As we mentioned above, there are several results on almost-commuting matrices with respect to norm distance, [3, 5, 11, 18]. For the arithmetic distance, we manage to prove only the following.

THEOREM 2.11. *For every $n \in \mathbb{N}$ such that $n \geq 4$ and every $A \in \mathbb{C}_{n\times n}$ with an algebraically simple spectrum, there exists an $X \in \mathbb{C}_{n\times n}$ such that $rank(AX - XA) = 2$ and $rank(B - X) \geq \frac{n}{2}$ for any matrix B that commutes with A.*

In the above theorem the matrix A is fixed. What happens if one is allowed to change A as well as X?

3. Some discrete geodesic spaces.

A metric space is called *geodesic* if the distance between two points equals the length of a geodesic from one of the points to the other. In the present article, we are interested in integer valued metrics. In this case,

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ELA
http://math.technion.ac.il/iic/ela

if for any x and y, $d(x,y)=k$, there exists a sequence $x=x_0,x_1,x_2,...,x_{k-1},x_k=y$, such that $d(x_i,x_{i+1})=1$, then a metric $d(\cdot,\cdot)$ is geodesic. We will use metric spaces in the proofs of Theorem 2.1 and Theorem 2.7. All these theorems are known to be valid for $rank(A-B)=1$. Then we proceed by induction on $rank(A-B)$ using the fact that $rank(A-B)$ is a geodesic metric.

Proposition 3.1. *Let X and Y be geodesic metric spaces. Let $O_n^X(x)$ denote the closed ball of radius n around x in X. Let $\phi: X \to Y$ be such that $\phi(O_1^X(x)) = O_1^Y(\phi(x))$ for all $x \in X$. Then $\phi(O_n^X(x)) = O_n^Y(\phi(x))$ for any $n \in \mathbb{N}$ and $x \in X$.*

Proof. The proof is by induction. For $n=1$ there is nothing to prove. Step $n \to n+1$: It follows that $O_{n+1}^X(x) = \bigcup_{z\in O_n^X(x)} O_1^X(z)$ (X is geodesic). Now

$$\phi(O_{n+1}^X(x)) = \bigcup_{z\in O_n^X(x)} \phi(O_1^X(z)) = \bigcup_{z\in\phi(O_n^X(x))} O_1^Y(z) =$$

$$\bigcup_{z\in O_n^Y(\phi(x))} O_1^Y(z) = O_{n+1}^Y(\phi(x)). \qquad \square$$

3.1. Arithmetic distance on $\mathbb{C}_{n\times n}$.

Lemma 3.2. *The arithmetic distance, $d(A,B) = rank(A-B)$, is geodesic on the*

- *set of all $n\times n$ matrices;*
- *set of all self-adjoint $n\times n$ matrices;*
- *set of all unitary $n\times n$ matrices.*

Proof. It is clear that a rank k matrix (self-adjoint matrix) may be represented as a sum of k matrices (self-adjoint matrices) of rank 1. The first two items follow from the fact that set of matrices (self-adjoint matrices) is closed with respect to summation. Now consider unitary matrices. Let $rank(U_1-U_2)=k$, that is, $rank(E-U_1^{-1}U_2)=k$. This means that, in a proper basis, $U_1^{-1}U_2 = diag(\lambda_1,\lambda_2,...,\lambda_k,1,1,...,1)$. Now the sequence U_1, $U_1\cdot diag(\lambda_1,1,1,...,1)$, $U_1\cdot diag(\lambda_1,\lambda_2,1,1,...,1)$, ..., $U_1\cdot diag(\lambda_1,\lambda_2,...,\lambda_k,1,1,...,1) = U_2$ gives us the geodesic needed. $\square$

Remark 3.3. The methods used in the above proof are not applicable to normal matrices – the set of normal matrices is not closed under either summation or multiplication. In fact, an example from [6] hints that arithmetic distance might be non geodesic on the set of normal matrices.

Proposition 3.4. *Let $\phi(x) = (ax+b)(cx+d)^{-1}$ be a Möbius transformation of $\mathbb{C}_{n\times n}$ ($a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$). Suppose that ϕ is defined on A, B. Then $rank(A-B) = rank(\phi(A)-\phi(B))$.*

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ELA
http://math.technion.ac.il/iic/ela

Proof. A Möbius transformation is a composition of linear transformations $A \to aA + b$ $(a, b \in \mathbb{C})$ and taking inverse $A \to A^{-1}$. Those transformations (if defined) clearly conserve arithmetic distance, for example, $rank(A^{-1} - B^{-1}) = rank(A^{-1}(B - A)B^{-1}) = rank(A - B)$. ($A^{-1}$ and B^{-1} are of full rank.) □

3.2. Distance on the spaces of the Weyr characteristics. Having in mind the Weyr characteristics of complex matrices, we introduce the spaces $\Im_n$ of the Weyr characteristics: $\Im_n$ is the space of functions $\mathbb{Z}^+ \times \mathbb{C} \to \mathbb{N}$, $(i, \lambda) \to \eta_i(\lambda)$ such that

- $\eta_i(\lambda) \neq 0$ for finitely many (i, λ) only, and $\sum\limits_{\lambda \in \mathbb{C}} \sum\limits_{i \in \mathbb{Z}^+} \eta_i(\lambda) = n$.
- $\eta_i(\lambda) \geq \eta_{i+1}(\lambda)$.

On $\Im_n$ we define a metric $d(\eta, \mu) = \max\limits_{(i,\lambda)}\{|\eta_i(\lambda) - \mu_i(\lambda)|\}$. First of all let us note that $d(\cdot, \cdot)$ is indeed a metric. Trivially, $d(\eta, \mu) = 0$ implies $\eta = \mu$ and $d(\cdot, \cdot)$ satisfies the triangle inequality since it is supremum (maximum) of semimetrics. It is clear that $d(\mu, \nu)$ is also well defined for μ and ν in different spaces of Weyr characteristics (for different n). We will need the following

Proposition 3.5. *Let $\nu \in \Im_m$ and $n > m$. Then there exists $\mu \in \Im_n$ such that $d(\nu, \eta) \geq d(\mu, \eta)$ for any $\eta \in \Im_n$.*

Proof. Let $\nu_i(\lambda_0) \neq 0$ and $\nu_{i+1}(\lambda_0) = 0$. We can take $\mu_{i+1}(\lambda_0) = \mu_{i+2}(\lambda_0) = \cdots = \mu_{i+n-m}(\lambda_0) = 1$ and $\mu_j(\lambda) = \nu_j(\lambda)$ for all other pairs (j, λ). □

Proposition 3.6. *$\Im_n$ are geodesic metric spaces.*

Proof. Let $\eta, \mu \in \Im_n$ and $d(\eta, \mu) = k > 1$. It suffices to find $\nu \in \Im_n$ such that either $d(\eta, \nu) = 1$ and $d(\nu, \mu) = k - 1$, or $d(\eta, \nu) = k - 1$ and $d(\nu, \mu) = 1$. Moreover, by Proposition 3.5 it suffices to find $\nu \in \Im_m$ for $m \leq n$. Let $S_+ = \{(j, \lambda) \in \mathbb{Z}^+ \times \mathbb{C} \mid \eta_j(\lambda) - \mu_j(\lambda) = k\}$ and $S_- = \{(j, \lambda) \in \mathbb{Z}^+ \times \mathbb{C} \mid \eta_j(\lambda) - \mu_j(\lambda) = -k\}$. Suppose that $|S_+| \geq |S_-|$ (if not, we can interchange $\eta \leftrightarrow \mu$). Now let:

$$\nu_i(\lambda) = \begin{cases} \eta_i(\lambda) - 1 & \text{if} \quad (i, \lambda) \in S_+ \\ \eta_i(\lambda) + 1 & \text{if} \quad (i, \lambda) \in S_- \\ \eta_i(\lambda) & \text{if} \quad (i, \lambda) \notin S_+ \cup S_- \end{cases}.$$

We have to show that $\nu \in \Im_m$ for $m = n - |S_+| + |S_-|$. It remains to show that $\nu_{j+1}(\lambda) \leq \nu_j(\lambda)$. Suppose that $\nu_{j+1}(\lambda) > \nu_j(\lambda)$. There are three possibilities:

a) $\eta_{j+1}(\lambda) = \eta_j(\lambda)$, $(j, \lambda) \in S_+$ and $(j+1, \lambda) \notin S_+$, but then $k > \eta_{j+1}(\lambda) - \mu_{j+1}(\lambda) \geq \eta_{j+1}(\lambda) - \mu_j(\lambda) = \eta_j(\lambda) - \mu_j(\lambda) = k$, a contradiction.

b) $\eta_{j+1}(\lambda) = \eta_j(\lambda)$, $(j + 1, \lambda) \in S_-$ and $(j, \lambda) \notin S_-$, but then $-k < \eta_j(\lambda) - \mu_j(\lambda) \leq \eta_j(\lambda) - \mu_{j+1}(\lambda) = \eta_{j+1}(\lambda) - \mu_{j+1}(\lambda) = -k$, a contradiction.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 302-316, June 2009

c) $\eta_{j+1}(\lambda) = \eta_j(\lambda) - 1$, $(j, \lambda) \in S_+$ and $(j+1, \lambda) \in S_-$, but then $-k = \eta_{j+1}(\lambda) - \mu_{j+1}(\lambda) \geq \eta_{j+1}(\lambda) - \mu_j(\lambda) = \eta_j(\lambda) - \mu_j(\lambda) - 1 = k - 1$, so $-k \geq k - 1$ and $1/2 \geq k$, a contradiction with $k > 1$.

Now, by construction, $d(\nu, \eta) = 1$ and $d(\nu, \mu) = k - 1$. ∎

3.3. Distances dc and $\tilde{dc}$ on finite multisets of complex numbers. The language of multisets is very convenient to deal with spectra. We need only finite multisets. For a multiset $\mathcal{A}$ let $set(\mathcal{A})$ denote the set of elements of $\mathcal{A}$ (forgetting multiplicity). It is clear that a multiset can also be considered as the multiplicity function $\chi_{\mathcal{A}} : set(A) \to \mathbb{N}$, for any $x \notin \mathcal{A}$ we will suppose $\chi_{\mathcal{A}}(x) = 0$. (For all cases, considered here, $set(\mathcal{A}) \subset \mathbb{C}$, so we can consider $\chi_{\mathcal{A}} : \mathbb{C} \to \mathbb{N} = \{0, 1, 2, ...\}$.) We need the following generalizations of set-theoretical operations to multisets:

- Difference of two multisets $\mathcal{A} \setminus \mathcal{B}$, $\chi_{\mathcal{A}\setminus\mathcal{B}}(x) = \max\{0, \chi_{\mathcal{A}}(x) - \chi_{\mathcal{B}}(x)\}$.
- Intersection $\mathcal{A} \cap X$ of a set X and a multiset $\mathcal{A}$,

$$\chi_{\mathcal{A}\cap X}(x) = \begin{cases} \chi_{\mathcal{A}}(x), & \text{if} \quad x \in X \\ 0, & \text{if} \quad x \notin X \end{cases} .$$

- Union $\mathcal{A} \uplus \mathcal{B}$, $\chi_{\mathcal{A}\uplus\mathcal{B}}(x) = \chi_{\mathcal{A}}(x) + \chi_{\mathcal{B}}(x)$.

Let $O_r(a) = \{x \in \mathbb{C} : |x - a| \leq r\}$ and let $\mathcal{S} = \{O_r(a) \mid a \in \mathbb{C},\ r \in \mathbb{R}^+\}$ be the set of all balls. For $\mathcal{A}, \mathcal{B} \subset_M \mathbb{C}$, let $dc(\mathcal{A}, \mathcal{B}) = \max\limits_{O\in\mathcal{S}}\{|(|\mathcal{A} \cap O| - |\mathcal{B} \cap O|)|\}$. Let us extend $\mathcal{S}$ to $\tilde{\mathcal{S}}$, which includes the complements of open balls and semiplanes: $\tilde{\mathcal{S}} = \mathcal{S} \cup \{\{x \in \mathbb{C} : |x - a| \geq r\} \mid a \in \mathbb{C},\ r \in \mathbb{R}^+\} \cup \{\{x \in \mathbb{C} : Im(\frac{x-b}{a}) \geq 0\} \mid a, b \in \mathbb{C}\}$. Introduce the new metric $\tilde{dc}(\mathcal{A}, \mathcal{B}) = \max\limits_{O\in\tilde{\mathcal{S}}}\{|(|\mathcal{A} \cap O| - |\mathcal{B} \cap O|)|\}$.

Proposition 3.7.

- *dc and $\tilde{dc}$ are metrics on the set of finite multisubsets of $\mathbb{C}$.*
- *$dc(\mathcal{A}, \mathcal{B}) = dc(\mathcal{A} \setminus \mathcal{B}, \mathcal{B} \setminus \mathcal{A})$, $\tilde{dc}(\mathcal{A}, \mathcal{B}) = \tilde{dc}(\mathcal{A} \setminus \mathcal{B}, \mathcal{B} \setminus \mathcal{A})$.*
- *If $|\mathcal{A}| = |\mathcal{B}|$, then $\tilde{dc}(\mathcal{A}, \mathcal{B}) = dc(\mathcal{A}, \mathcal{B})$.*

Proof

- The same as for spaces of Weyr characteristics.
- Let $\sum_O(\mathcal{A}, \mathcal{B}) = |\mathcal{A}\cap O| - |\mathcal{B}\cap O| = \sum\limits_{x\in O}(\chi_{\mathcal{A}}(x) - \chi_{\mathcal{B}}(x))$. Then $\sum_O(\mathcal{A}\setminus\mathcal{B}, \mathcal{B}\setminus\mathcal{A}) = \sum\limits_{x\in O}(\max\{0, \chi_{\mathcal{A}}(x) - \chi_{\mathcal{B}}(x)\} - \max\{0, \chi_{\mathcal{B}}(x) - \chi_{\mathcal{A}}(x)\}) = \sum\limits_{x\in O}(\chi_{\mathcal{A}}(x) - \chi_{\mathcal{B}}(x))$. Now the item follows by definition of dc ($\tilde{dc}$).
- Since $\mathcal{A}$ and $\mathcal{B}$ are finite multisets, for any semiplane p we can find a ball c such that $\sum_p(\mathcal{A}, \mathcal{B}) = \sum_c(\mathcal{A}, \mathcal{B})$. Also for any closed ball c_c, there exists an open ball c_o such that $\sum_{c_c}(\mathcal{A}, \mathcal{B}) = \sum_{c_o}(\mathcal{A}, \mathcal{B})$. Now, under the stated assumption, $\sum_O(\mathcal{A}, \mathcal{B}) = -\sum_{\mathbb{C}\setminus O}(\mathcal{A}, \mathcal{B})$ and the result follows. ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 302-316, June 2009

ELA
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Proposition 3.8. *Let $\phi(x) = \frac{ax+b}{cx+d}$ be a Möbius transformation of $\mathbb{C}$ ($a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$). Suppose, that ϕ is defined on $set(\mathcal{A}) \cup set(\mathcal{B})$. Then:*

$$\tilde{dc}(\mathcal{A}, \mathcal{B}) = \tilde{dc}(\phi(\mathcal{A}), \phi(\mathcal{B})).$$

Proof. A Möbius transformation defines a bijection on $\tilde{\mathcal{S}}$. □

We don't know if the metric dc is geodesic on the multisets with fixed cardinality, but its restriction to any circle or line is.

Proposition 3.9. *Let $l \subset \mathbb{C}$ be a circumference or a straight line. Let $\mathcal{A}, \mathcal{B} \subset_M l$, $|\mathcal{A}| = |\mathcal{B}| = n$ and $\tilde{dc}(\mathcal{A}, \mathcal{B}) = k \geq 2$. Then there exists $\mathcal{C} \subset_M l$, $|\mathcal{C}| = n$ such that $\tilde{dc}(\mathcal{A}, \mathcal{C}) = 1$ and $\tilde{dc}(\mathcal{C}, \mathcal{B}) = k - 1$.*

Proof. By Proposition 3.8, it suffices to prove the assertion for the unit circle. Let us start with the case in which $set(\mathcal{A}) \cap set(\mathcal{B}) = \emptyset$. Let $\Gamma = set(\mathcal{A}) \cup set(\mathcal{B}) \subset S^1$. Let $|\Gamma| = r$. We cyclically (anticlockwise) order $\Gamma = \{\gamma_0, \gamma_1, ..., \gamma_{r-1}\}$ by elements of $\mathbb{Z}_r$. To construct $\mathcal{C}$ we move each element of $\mathcal{A}$ to the next element in Γ, precisely, $set(\mathcal{C}) \subseteq \Gamma$ and

$$\chi_{\mathcal{C}}(\gamma_i) = \max\{0, \chi_{\mathcal{A}}(\gamma_i) - 1\} + \chi_{set(\mathcal{A})}(\gamma_{i-1}),$$

in other words

$$\chi_{\mathcal{C}}(\gamma_i) = \begin{cases} \chi_{\mathcal{A}}(\gamma_i) - 1 & \text{if} \quad \gamma_i \in set(\mathcal{A}) \text{ and } \gamma_{i-1} \notin set(\mathcal{A}) \\ 1 & \text{if} \quad \gamma_i \notin set(\mathcal{A}) \text{ and } \gamma_{i-1} \in set(\mathcal{A}) \\ \chi_{\mathcal{A}}(\gamma_i) & \text{for} \quad \text{the other cases} \end{cases}$$

Check that $\mathcal{C}$ satisfies our needs: For $x, y \in \Gamma$ let $[x, y]$ denote the closed segment of S^1, starting from x and going anticlockwise to y (so $[x, y] \cup [y, x] = S^1$). For $X, Y \subset_M \Gamma$ one has $\tilde{dc}(X, Y) = \max\{|(|X \cap [\alpha, \beta]| - |Y \cap [\alpha, \beta]|)| \ : \ \alpha, \beta \in set(X) \cup set(Y)\}$. Let $\sum_{[\alpha,\beta]}(X, Y) = |X \cap [\alpha, \beta]| - |Y \cap [\alpha, \beta]|$. Note that if $X \cap Y = \emptyset$ and $\sum_{[\gamma_i,\gamma_j]}(X, Y) = \tilde{dc}(X, Y)$, then without loss of generality we may suppose that $\gamma_i, \gamma_j \in X$ and $\gamma_{i-1}, \gamma_{j+1} \notin X$.

Now, $\tilde{dc}(\mathcal{A}, \mathcal{C}) = \tilde{dc}(\mathcal{A} \setminus \mathcal{C}, \mathcal{C} \setminus \mathcal{A}) = 1$, since $\mathcal{A} \setminus \mathcal{C} = F_1 = \{\gamma_i \mid \gamma_i \in set(\mathcal{A}) \wedge \gamma_{i-1} \notin set(\mathcal{A})\}$ and $\mathcal{C} \setminus \mathcal{A} = F_2 = \{\gamma_i \mid \gamma_i \notin set(\mathcal{A}) \wedge \gamma_{i-1} \in set(\mathcal{A})\}$ are interlacing sets on S^1. (In defining F_1 and F_2, we use the fact that $set(\mathcal{A}) \cap set(\mathcal{B}) = \emptyset$. The following consideration is useful. An interval $[\gamma_i, \gamma_j] \cap \Gamma \subset set(\mathcal{A})$ is said to be $\mathcal{A}$-maximal if $\gamma_{i-1}, \gamma_{j+1} \in \mathcal{B}$. In the same way we define $\mathcal{B}$-maximal intervals. Then Γ is the union of interlacing $\mathcal{A}$-maximal and $\mathcal{B}$-maximal intervals. Any $\mathcal{A}$-maximal interval contains exactly one point of F_1; any $\mathcal{B}$-maximal interval contains exactly one point of F_2.) If we suppose further that $\tilde{dc}(\mathcal{C}, \mathcal{B}) = \tilde{dc}(\mathcal{C} \setminus \mathcal{B}, \mathcal{B} \setminus \mathcal{C}) \geq k$, then there exists $[\gamma_i, \gamma_j]$ such that either:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 18, pp. 302-316, June 2009

1. $\sum_{[\gamma_i,\gamma_j]}(\mathcal{C}\setminus\mathcal{B},\mathcal{B}\setminus\mathcal{C}) = \tilde{d}c(\mathcal{C}\setminus\mathcal{B},\mathcal{B}\setminus\mathcal{C}) \geq k$,
   or
2. $\sum_{[\gamma_i,\gamma_j]}(\mathcal{C}\setminus\mathcal{B},\mathcal{B}\setminus\mathcal{C}) \leq -k$.

It suffices to consider the first case only. One checks that $\mathcal{C}\setminus\mathcal{B} = \mathcal{A}\setminus F_1$ and $\mathcal{B}\setminus\mathcal{C} = \mathcal{B}\setminus F_2$, so $set(\mathcal{C}\setminus\mathcal{B}) \subseteq set(\mathcal{A})$ and $set(\mathcal{B}\setminus\mathcal{C}) \subseteq set(\mathcal{B})$. It follows that there exist i, j, satisfying the inequality of item 1, with $\gamma_i, \gamma_j \in \mathcal{A}$ and $\gamma_{i-1}, \gamma_{j+1} \notin \mathcal{A}$. This means that $|[\gamma_i,\gamma_j]\cap F_1| = |[\gamma_i,\gamma_j]\cap F_2|+1$. So, $\sum_{[\gamma_i,\gamma_j]}(\mathcal{A},\mathcal{B}) = \sum_{[\gamma_i,\gamma_j]}(\mathcal{C}\setminus\mathcal{B},\mathcal{B}\setminus\mathcal{C})+1 \geq k+1$, a contradiction. The second case may be reduced to the first ($\gamma_i \leftrightarrow \gamma_j$).

If $set(\mathcal{A})\cap set(\mathcal{B}) \neq \emptyset$ then we can find $\mathcal{C}'$ for $\mathcal{A}\setminus\mathcal{B}$ and $\mathcal{B}\setminus\mathcal{A}$ and then take $\mathcal{C} = \mathcal{C}' \uplus X$, where $X = \mathcal{A}\setminus(\mathcal{A}\setminus\mathcal{B}) = \mathcal{B}\setminus(\mathcal{B}\setminus\mathcal{A})$[2]. $\square$

Here are some other simple properties of dc. Let $X, Y \subset_M \mathbb{C}$ and $dc(X,Y) = 1$. Then:

- if $|X| = |Y| = 2$ then $X\cup Y$ lies on a circle (a line);
- if $X\cap Y = \emptyset$ and $p_0, p_1, ..., p_{n-1}$ are the vertexes of the convex hull of $X\cup Y$ then $p_0, p_1, ..., p_{n-1}$ is an interlacing sequence, that is, $p_i \in X$ if and only if $p_{i\pm 1} \in Y$;
- if $|X| = |Y| \geq 3$ and X lies on a circle (a line) l, then Y lies on the same line l.

We also can show that $\{X \subset_M \mathbb{C} \mid |X| = 2\}$ is geodesic (with respect to $dc = \tilde{d}c$). The *line capacity* of $X \subset \mathbb{C}$ (notation: $c(X)$) is the minimal k such that there exist circles (lines) $l_1, l_2, ..., l_k$, containing X ($X \subset l_1\cup l_2\cup ...\cup l_k$). It is easy to construct nonintersecting multisets M_1, M_2 with $dc(M_1,M_2) = c(M_1\cup M_2)-1$. But we have not been able to find nonintersecting M_1, M_2 with $dc(M_1,M_2) < c(M_1\cup M_2)-1$.

Question. Does $dc(M_1,M_2) \geq c(M_1\cup M_2)-k$ for any nonintersecting finite multisets M_1, M_2 and some k independent of M_i?

4. Proofs of Theorems.

4.1. Proof of Theorem 2.1. It suffices to prove the theorem for $k = 1$. Indeed, assume that the theorem is valid for $k = 1$. Then we may consider the Weyr characteristics as a map $\eta : \mathbb{C}_{n\times n} \to \Im_n$ that satisfies Proposition 3.1. So, Theorem 2.1 follows, because it states that $\eta(O_k(A)) = O_k(\eta(A))$.

Let us prove Theorem 2.1 for $k = 1$. For a given eigenvalue $\lambda \in sp(A)$ the sequence $q_1(A,\lambda) \geq q_2(A,\lambda) \geq \cdots$ of sizes of the λ-Jordan blocks in the Jordan normal form of A is known as the *Segre characteristic* of A relative to λ [15, 20].

[2]We use this strange expression for X just because we have not defined an intersection of multisets.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 302-316, June 2009

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$$\begin{array}{ccccccccc}
 & b_1 & b_2 & b_3 & b_4 & \cdots & b_k & & \\
a_1 & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet & & \\
a_2 & \bullet & \bullet & \bullet & \cdots & \bullet & & & \\
a_3 & \bullet & \bullet & \cdots & \bullet & & & & \\
\vdots & & & & & & & & \vdots \\
a_{n-1} & \bullet & \cdots & \bullet & & & & & \\
a_n & \bullet & & & & & & &
\end{array}$$

Fig. 4.1. *Ferrers diagram.*

The similarity invariant factors of $A \in \mathbb{C}_{n\times n}$ are sequence of monic polynomials in x, $h_n(A) \mid h_{n-1}(A) \mid h_{n-2}(A) \mid \cdots \mid h_1(A)$. It is known that $h_i(A) = \prod_\lambda (\lambda - x)^{q_i(A,\lambda)}$. So, by Thompson's theorem 2.3, $rank(A-B) = 1$ if and only if

$$\begin{aligned} &q_1(B,\lambda) \geq q_2(A,\lambda) \geq q_3(B,\lambda) \geq \cdots, \\ &q_1(A,\lambda) \geq q_2(B,\lambda) \geq q_3(A,\lambda) \geq \cdots. \end{aligned} \tag{4.1}$$

For fixed B and λ, the Weyr characteristic $\eta_i(B,\lambda)$ is the conjugate partition of the Segre characteristic $q_i(B,\lambda)$ [15]. So, the theorem follows from

Proposition 4.1. *Let $a_1 \geq a_2 \geq \cdots$ be the conjugate partition for $b_1 \geq b_2 \geq \cdots$ and let $a'_1 \geq a'_2 \geq \cdots$ be the conjugate partition for $b'_1 \geq b'_2 \geq \cdots$. Then $|b_i - b'_i| \leq 1$ for all i if and only if $a_{i+1} \leq a'_i \leq b_{i-1}$.*

Proof. The Ferrers diagram for a is the set $F_a = \{(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid j \leq a_i\}$; see Figure 4.1. The conjugate partition b is defined by the formula $b_j = |\{(x,y) \in F_B^\lambda \mid x = j\}|$. The inequality $a'_i \geq a_{i+1}$ is equivalent to the statement $\forall i$ $(i+1,j) \in F_a \to (i,j) \in F_{a'}$ (Figure 4.1). The statement is equivalent to the inequality $b'_j \geq b_j - 1$. Similarly, $a_i \geq a'_{i+1}$ with $1 \leq i \leq n-1$ is equivalent to $b_j \geq b'_j - 1$. ☐

The referee pointed out that this proposition was proved by Ross A. Lippert, 2005, using the inequalities $b_{a_i} \geq i$ and $i \geq b_{a_i+1}$.

4.2. Proof of Theorem 2.4. Let $X^\perp$ be the orthogonal complement of a subspace X and let P_X be the orthogonal projection on X.

Lemma 4.2. *Let $N : L \to L$ be a normal operator and let X be a subspace of L such that $\|(N-\lambda)x\| \leq \epsilon\|x\|$ for any $x \in X$. Then (we write $\mathcal{R}(\lambda,\epsilon)$ for $\mathcal{R}(N,\lambda,\epsilon)$)*

1. *$P_{\mathcal{R}(\lambda,\epsilon)}x \neq 0$ for any $x \in X$, $x \neq 0$.*
2. *$\|P_{\mathcal{R}(\lambda,a\epsilon)}x\| \geq \sqrt{1 - \frac{1}{a^2}}\|x\|$ for any $x \in X$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 302-316, June 2009

3. $dim(R(\lambda,\epsilon)) \geq dim(X)$

Proof. It is clear that (1) implies (3).

1. Let $e_1, e_2, ..., e_n$ be a diagonal orthonormal basis for N and let $\lambda_1, \lambda_2, ...\lambda_n$ be corresponding eigenvalues ($Ne_i = \lambda_i e_i$). Let $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n \in X$ with $\|x\| = \sum_{i=1}^{n} |\alpha_i|^2 = 1$. Now, $\|(N-\lambda)x\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 |\lambda_i - \lambda|^2 \leq \epsilon^2$ implies that $\sum_{i \mid |\lambda_i - \lambda| > \epsilon} |\alpha_i|^2 < 1$. So, $P_{\mathcal{R}(\lambda,\epsilon)} x = \sum_{i \mid |\lambda_i - \lambda| \leq \epsilon} \alpha_i e_i \neq 0$.

2. Similarly,

$$\sum_{i \mid |\lambda_i - \lambda| > a\epsilon} |\alpha_i|^2 < \frac{1}{a^2} \text{ and } \sum_{i \mid |\lambda_i - \lambda| \leq a\epsilon} |\alpha_i|^2 \geq (1 - \frac{1}{a^2}),$$

so, $\|P_{\mathcal{R}(\lambda,a\epsilon)} x\| \geq \sqrt{1 - \frac{1}{a^2}} \|x\|$. □

Now we are ready to prove Theorem 2.4. Let $rank(A - B) = r$ and let $X = \mathcal{R}(A, \lambda, \epsilon) \cap \ker(A - B)$. Then $dim(X) \geq dim(\mathcal{R}(A, \lambda, \epsilon)) - r$ and $A|_X = B|_X$. So, $\|(B - \lambda)|_X\| \leq \epsilon$ and, by Lemma 4.2, $dim(\mathcal{R}(B, \lambda, \epsilon)) \geq dim(\mathcal{R}(A, \lambda, \epsilon)) - r$. Theorem 2.4 follows by symmetry.

4.3. Proofs of Theorem 2.6 and Theorem 2.7.

- It suffices to prove Theorem 2.7 for self-adjoint matrices. Indeed, let $sp(A), \mathcal{B} \subset_M l$, $|\mathcal{B}| = n$ for a circle (line) $l \subset \mathbb{C}$. Then there exists a Möbius transformation ϕ, defined on $sp(A) \cup \mathcal{B}$, that maps l to the real line. Then $\phi(A)$ is self-adjoint and we can apply Theorem 2.6 to $\phi(A)$ and $\phi(\mathcal{B})$ to find $\tilde{B}$ with $sp(\tilde{B}) = \phi(\mathcal{B})$ and $rank(\phi(A) - \tilde{B}) = \tilde{dc}(\phi(sp(A)), \phi(\mathcal{B}))$. Now take $B = \phi^{-1}(\tilde{B})$ and the result follows, since Möbius transformations preserve arithmetic distance on $C_{n\times n}$ as well as the distance $\tilde{dc}$ on multisets (Proposition 3.4 and Proposition 3.8).
- It suffices to prove Theorem 2.7 for $dc(sp(A), \mathcal{B}) = 1$ and the rest follows from Proposition 3.1, Proposition 3.9, and Lemma 3.2.
- Also without loss of generality we may assume that $set(sp(A)) \cap set(\mathcal{B}) = \emptyset$. If $X = sp(A) \setminus (sp(A) \setminus \mathcal{B})$ we can write $A = A_1 \oplus A_2$ with $sp(A_1) = X$ and $sp(A_2) = sp(A) \setminus X$. We can find B_2 with $sp(B_2) = \mathcal{B} \setminus X$ and $rank(A_2 - B_2) = 1$. Now, take $B = A_1 \oplus B_2$.
- Let $\mathcal{A}, \mathcal{B} \subset_M \mathbb{R}$, $set(\mathcal{A}) \cap set(\mathcal{B}) = \emptyset$, $|\mathcal{A}| = |\mathcal{B}|$, and $dc(\mathcal{A}, \mathcal{B}) = 1$. Then $\mathcal{A}$ and $\mathcal{B}$ are interlacing sets. This means that for $\mathcal{A} = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and $\mathcal{B} = \{\beta_1, \beta_2, ..., \beta_n\}$ we have $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots$ or $\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \cdots$.

Theorem 2.7 follows from:

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 18, pp. 302-316, June 2009

Lemma 4.3. *Let $A \in \mathbb{C}_{n\times n}$ be self-adjoint and have a simple spectrum. Let $\mathcal{B} \subset \mathbb{R}$ with $|\mathcal{B}| = n$. If $sp(A)$ and $\mathcal{B}$ are interlacing then there exists a self-adjoint B with $sp(B) = \mathcal{B}$ and $rank(A - B) = 1$.*

Proof. This is a special case of Theorem 2 in [17]. □

4.4. Proof of Theorem 2.10. Let $rank(A^*A - E) = r$, so there exists a subspace $X \subset \mathbb{C}^n$ with $dim(X) = n - r$ and such that $A^*A|_X = E|_X$. Consider $A|_X : X \to Y = A(X)$, so that $A^*(Y) = X$. It follows that $(A|_X)^* = A^*|_Y : Y \to X$, so $A|_X : X \to Y$ is a unitary operator. Choose any unitary operator $B : X^\perp \to Y^\perp$ ($B^*B = E_{X^\perp}$). Then $U = A|_X \oplus B$ meets the requirements.

4.5. Proof of Theorem 2.11. We use the fact that the matrix $M = [x_{ij}]$ with $x_{ij} = \frac{1}{\alpha_i - \lambda_j}$ is nonsingular if all $\lambda_1, ..., \lambda_k, \alpha_1, ..., \alpha_k$ are different [1], p.119.

Proof. Consider the matrix equation

$$AX - XA = [c_{ij}], \tag{4.2}$$

with $c_{ij} = i + j\mathrm{mod}2$. Let $A = diag(\lambda_1, \lambda_2, ..., \lambda_n)$ be a diagonal matrix with a simple spectrum. The solution $X = [x_{ij}]$ of (4.2) is

$$x_{ij} = \begin{cases} \frac{c_{ij}}{\lambda_i - \lambda_j} & \text{for } i \neq j \\ 0 & \text{for } i = j \end{cases}$$

Every matrix B that commutes with A is necessarily diagonal. If we delete from $X - B$ the odd columns and the even rows, we obtain the $\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$-submatrix

$$[x'_{ij}] = \left[\frac{1}{\lambda_{i^*} - \lambda_{j^*}} \right]$$

with $i^* = 2i - 1$, $j^* = 2j$. This matrix is nonsingular. Therefore, $rank(X - B) \geq \frac{n}{2}$. □

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REFERENCES

[1] Dennis S. Bernstein. *Matrix mathematics. Theory, facts, and formulas with application to linear systems theory.* Princeton University Press, Princeton, NJ, 2005.

[2] A. L. Cauchy. Sur l'éqution à l'aide de laquelle on détemine les inégalités séculaires. *Oeuvres Complètes de A.L.Cauchy, 2nd Ser., V IX*, Gauthier-Villars, Paris, 174–195, 1891.

[3] Man-Duen Choi. Almost commuting matrices need not be nearly commuting. *Proc. Amer. Math. Soc.*, 102:529–533, 1988.

[4] Man-Duen Choi, Pei Yuan Wu. Finite-rank perturbations of positive operators and isometries. *Studia Math.*, 173:73–79, 2006.

[5] R. Exel and T. Loring. Almost Commuting Unitary Matrices. *Proc. Amer. Math. Soc.*, 106:913–915, 1989.

[6] K. Fan and G. Pall. Imbedding conditions for Hermitian and normal matrix. *Canada J. Math.*, 9:298–304, 1957.

[7] Ilijas Farah. All automorphisms of the Calkin algebra are inner. Preprint ArXiv:0705.3085v3, 2007

[8] I. Gohberg, P. Lancaster, and L. Rodman. *Invariant Subspaces of Matrices with Applications.* John Wiley & Sons, New York, 1986.

[9] R. Horn, N. Rhee, and W. So. Eigenvalue inequalities and equalities. *Linear Algebra Appl.*, 270:29–44, 1998.

[10] Alexander Klyachko. Vector Bundles, Linear Representations, and Spectral Problems. Preprint ArXiv: math/0304325, 2003.

[11] Huaxin Lin. Almost commuting self-adjoint matrices and applications. *Operator algebras and their applications* (Waterloo, ON, 1994/1995), Fields Inst. Commun., *Amer. Math. Soc.*, Providence, RI, 13:193–233, 1997.

[12] J. Moro and F. M. Dopico. Low Rank Perturbation of Jordan Structure. *SIAM J. Matrix Anal. Appl.*, 25:495–506, 2003.

[13] S. V. Savchenko. On the change in the spectral properties of a matrix under a perturbation of a sufficiently low rank. (Russian) *Funktsional. Anal. i Prilozhen.*, 38:85–88, 2004; translation in *Funct. Anal. Appl.*, 38:69–71, 2004.

[14] S. V. Savchenko. On the Laurent series expansion of the determinant of the matrix of scalar resolvents. *Mat. Sb.* 196:121–144, 2005; translation in *Sb. Math.* 196:743–764, 2005.

[15] Helene Shapiro. The Weyr characteristic. *Amer. Math. Monthly*, 106:919–929, 1999.

[16] R. C. Thompson. Invariant factors under rank one perturbations. *Canada J. Math.*, 32:240–245, 1980.

[17] R. C. Thompson. The behavior of eigenvalues and singular values under perturbations of restricted rank. *Linear Algebra Appl.*, 13:69–78, 1976.

[18] Dan Voiculescu. Asymptotically commuting finite rank unitary operators without commuting approximants. *Acta Sci. Math. (Szeged)*, 45:429–431, 1983.

[19] Zhe-Xian Wan. *Geometry of Matrices: In Memory of Professor L. K. Hua (1910-1985).* World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[20] Eduard Weyr. Zur Theorie der bilinearen Formen. (German) *Monatsh. Math. Phys.*, 1:163–236, 1890.