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ON THE CHARACTERIZATION OF GRAPHS WITH PENDENT VERTICES AND GIVEN NULLITY*

BOLIAN LIU†, YUFEI HUANG†, AND SIYUAN CHEN†

Abstract. Let G be a graph with n vertices. The nullity of G, denoted by $\eta(G)$, is the multiplicity of the eigenvalue zero in its spectrum. In this paper, we characterize the graphs (resp. bipartite graphs) with pendent vertices and nullity η, where $0 < \eta \leq n$. Moreover, the minimum (resp. maximum) number of edges for all (connected) graphs with pendent vertices and nullity η are determined, and the extremal graphs are characterized.

Key words. Eigenvalue, Nullity, Pendent vertex.

AMS subject classifications. 05C50.

1. Introduction. Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For any $v \in V(G)$, the degree and neighborhood of v are denoted by $d(v)$ and $N(v)$, respectively. If W is a nonempty subset of $V(G)$, then the subgraph induced by W is the subgraph of G obtained by taking the vertices in W and joining those pairs of vertices in W which are joined in G. We write $G - \{v_1,\ v_2,\ \ldots\ ,\ v_k\}$ for the graph obtained from G by removing the vertices $v_1,\ v_2,\ \ldots\ ,\ v_k$ and all edges incident to any of them.

The *disjoint union* of two graphs G_1 and G_2 is denoted by $G_1 \cup G_2$. The disjoint union of k copies of G is often written by kG. The *null graph* of order n is the graph with n vertices and no edges. As usual, the complete graph, the cycle, the path, and the star of order n are denoted by $K_n,\ C_n,\ P_n$ and S_n, respectively. An isolated vertex is sometimes denoted by K_1.

Let t (≥ 2) be an integer. A graph G is called *t-partite* if $V(G)$ admits a partition into t classes $X_1,\ X_2,\ \ldots\ ,\ X_t$ such that every edge has its ends in different classes; vertices in the same partition must not be adjacent. Such a partition $(X_1,\ X_2,\ \ldots\ ,\ X_t)$ is called a *t-partition* of G. A *complete t-partite graph* is a simple t-partite graph with partition $(X_1,\ X_2,\ \ldots\ ,\ X_t)$ in which each vertex of X_i is joined to each vertex of $G - X_i$ $(1 \leq i \leq t)$. If $|X_i| = n_i$ $(1 \leq i \leq t)$, such a graph is denoted by $K_{n_1,\ n_2,\ \ldots\ ,\ n_t}$.

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Instead of "2-partite" (resp. "3-partite") one usually says *bipartite (resp. tripartite).*

The *adjacency matrix* $A(G)$ of a graph G of order n, with vertex set $V(G) = \{v_1,\ v_2,\ \dots\ ,\ v_n\}$, is $n \times n$ symmetric matrix $[a_{ij}]$, such that $a_{ij} = 1$ if v_i and v_j are adjacent and 0, otherwise. A graph is said to be *singular (resp. nonsingular)* if its adjacency matrix is a singular (resp. nonsingular) matrix. The eigenvalues $\lambda_1,\ \lambda_2,\ \dots\ ,\ \lambda_n$ of $A(G)$ are said to be the *eigenvalues* of G, and to form the *spectrum* of this graph. The number of zero eigenvalues in the spectrum of a graph G is called its *nullity* and is denoted by $\eta(G)$. Let $r(A(G))$ be the rank of $A(G)$. Obviously, $\eta(G) = n - r(A(G))$. The rank of a graph G is the rank of its adjacency matrix $A(G)$, denoted by $r(G)$. Then $\eta(G) = n - r(G)$. Clearly, if G is a simple connected graph, then $0 \le r(G) \le |V(G)| \le |E(G)| + 1$.

The problem of characterizing all graphs G with $\eta(G) > 0$ was posed in [1] and [10]. This problem is relevant in many disciplines of science (see [2, 3]), and is very difficult. At present, only some particular cases are known (see [3-9,11-12]). On the other hand, this problem is of great interest in chemistry, because, for a bipartite graph G (corresponding to an alternant hydrocarbon), if $\eta(G) > 0$, then it indicates that the molecule which such a graph represents is unstable (see [8]). The nullity of a graph G is also meaningful in linear algebra, since it is related to the singularity and the rank of $A(G)$.

It is known that $0 \le \eta(G) \le n - 2$ if G is a simple graph on n vertices and G is not isomorphic to nK_1. In [4], B. Cheng and B. Liu characterized the extremal graphs attaining the upper bound $n - 2$ and the second upper bound $n - 3$.

Lemma 1.1. *([4]) Suppose that G is a simple graph of order n. Then*

(1) $\eta(G) = n - 2$ *if and only if* G *is isomorphic to* $K_{n_1,\ n_2} \cup kK_1$, *where* $n_1 + n_2 + k = n\ (\ge 2)$ *and* $n_1,\ n_2 > 0,\ \ k \ge 0$.

(2) $\eta(G) = n - 3$ *if and only if* G *is isomorphic to* $K_{n_1,\ n_2,\ n_3} \cup kK_1$, *where* $n_1 + n_2 + n_3 + k = n\ (\ge 3)$ *and* $n_1,\ n_2,\ n_3 > 0,\ \ k \ge 0$.

As a continuation, S. Li ([9]) determined the extremal graphs with pendent vertices which achieve the third upper bound $n - 4$ and fourth upper bound $n - 5$, respectively. Recently, Y. Fan and K. Qian ([6]) characterized all bipartite graphs of order n with nullity $n - 4$.

Definition 1.2. ([6]) Let $P_n = v_1 v_2 \cdots v_n$ $(n \ge 2)$ be a path. Replacing each vertex v_i by an empty graph O_{m_i} of order m_i for $i = 1,\ 2,\ \dots\ ,\ n$ and joining edges between each vertex of O_i and each vertex of O_{i+1} for $i = 1,\ 2,\ \dots\ ,\ n-1$, we get a graph G of order $(m_1 + m_2 + \cdots + m_n)$, denoted by $O_{m_1} O_{m_2} \cdots O_{m_n}$. Such graph is called an expanded path of length n, and the empty graph O_{m_i} is called an expanded

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vertex of order m_i for $i = 1,\ 2,\ \ldots\ ,\ n$.

LEMMA 1.3. ([6]) *Let G be a bipartite graph of order $n \geq 4$. Then $\eta(G) = n - 4$ if and only if G is isomorphic to a graph H possibly adding some isolated vertices, where H is one of the following graphs: a union of two disjoint expanded paths both of length 2, an expanded path of length 4 or 5.*

In Section 2 of this paper, we give a characterization of the graphs (resp. connected graphs) with pendent vertices and nullity η $(0 < \eta \leq n)$. As corollaries of this characterization, some results in [9] can be obtained immediately. Moreover, all bipartite graphs (resp. bipartite connected graphs) with pendent vertices and nullity $\eta = n - 2k$ are characterized. (It is known from [6] that the nullity set of all bipartite graphs of order n is $\{n - 2k \mid k = 0,\ 1,\ \ldots\ ,\ \lfloor n/2 \rfloor\}$.)

Let $\Gamma(n,\ e)$ be the set of all simple graphs with n vertices and e edges. In [4], the maximum nullity number of graphs with n vertices and e edges, $M(n,\ e) = max\{\ \eta(A) \mid A \in \Gamma(n,\ e)\}$, was studied, where $n \geq 1$ and $0 \leq e \leq \binom{n}{2}$. Conversely, we shall study the number of edges for the graphs with pendent vertices and nullity η $(0 < \eta \leq n)$. Let $e_{min}^{(\eta)}$ and $e_{max}^{(\eta)}$ $(\widetilde{e}_{min}^{(\eta)}$ and $\widetilde{e}_{max}^{(\eta)})$ denote the minimum and maximum number of edges for all (connected) graphs with pendent vertices and nullity η. Let $G_{min}^{(\eta)}$ (resp. $\widetilde{G}_{min}^{(\eta)}$) denote the graphs (resp. connected graphs) of nullity η with pendent vertices and $e_{min}^{(\eta)}$ (resp. $\widetilde{e}_{min}^{(\eta)}$) edges. We call $G_{min}^{(\eta)}$ (resp. $\widetilde{G}_{min}^{(\eta)}$) the minimum graphs (resp. connected graphs) with pendent vertices and nullity η. Similarly, we can define $G_{max}^{(\eta)}$ (resp. $\widetilde{G}_{max}^{(\eta)}$), the maximum graphs (resp. connected graphs) with pendent vertices and nullity η. In Section 3, we determine the number $e_{min}^{(\eta)},\ e_{max}^{(\eta)}\ ,\ \widetilde{e}_{min}^{(\eta)},\ \widetilde{e}_{max}^{(\eta)}$ and characterize the graphs $G_{min}^{(\eta)},\ G_{max}^{(\eta)}\ ,\ \widetilde{G}_{min}^{(\eta)},\ \widetilde{G}_{max}^{(\eta)}$, respectively. Now we list some known results needed in this paper.

LEMMA 1.4. ([12]) *Let G be a simple graph of order n. Then*

(1) $\eta(G) = n$ if and only if G is a null graph.

(2) If $G = G_1 \cup G_2 \cup \cdots \cup G_t$, where $G_1,\ G_2,\ \ldots\ ,\ G_t$ are the connected components of G, then $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$.

LEMMA 1.5. ([9]) *Let v be a pendent vertex of a graph G and u be the vertex in G adjacent to v. Then $\eta(G) = \eta(G - \{u,\ v\})$.*

LEMMA 1.6. ([4])

$$r(P_n) = \begin{cases} n-1, & n \text{ is odd;} \\ n, & \text{otherwise.} \end{cases} \quad r(C_n) = \begin{cases} n-2, & n \equiv 0\ (mod\ 4); \\ n, & \text{otherwise.} \end{cases}$$

2. The graphs with pendent vertices and nullity η. Let η be an integer with $0 < \eta \leq n$. Now the graphs with pendent vertices and nullity η are characterized

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as follows, where $n-3 \le \eta \le n$.

LEMMA 2.1. *Let G be a simple graph of order n with pendent vertices. Then*

(1) There exists no such graph G with nullity $\eta(G) = n,\ n-1$ or $n-3$;

(2) $\eta(G) = n-2$ if and only if G is isomorphic to $S_{n-k} \cup kK_1$ $(0 \le k \le n-2)$.

Proof. (1) Obviously, there exists no such graph G with nullity $\eta(G) = n-1$. Moreover, by Lemmas 1.1 and 1.4, the graph G of nullity $\eta(G) = n$ (resp. $n-3$) contains no pendent vertices. This leads to the desired results.

(2) Since the graph G has pendent vertices, combining this with Lemma 1.1, $\eta(G) = n-2$ if and only if G is isomorphic to $K_{1,\ n_2} \cup kK_1$, where $1 + n_2 + k = n$ and $n_2 > 0$, $k \ge 0$. This completes the proof. □

Now we give a characterization of the graphs with pendent vertices and nullity η for $0 < \eta \le n-4$. Let $\widetilde{\Upsilon}_n^{(\eta)}$ be the set of all connected graphs of order n with nullity η $(0 \le \eta \le n)$. Then it follows from Lemmas 1.1 and 1.4 that $\widetilde{\Upsilon}_n^{(n)} = \widetilde{\Upsilon}_n^{(n-1)} = \emptyset$, $\widetilde{\Upsilon}_n^{(n-2)} = \{K_{n_1,\ n_2} \mid n_1 + n_2 = n,\ and\ n_1,\ n_2 > 0\}$, $\widetilde{\Upsilon}_n^{(n-3)} = \{K_{n_1,\ n_2,\ n_3} \mid n_1 + n_2 + n_3 = n,\ and\ n_1,\ n_2,\ n_3 > 0\}$.

Let $n,\ k,\ t$ be positive integers with $4 \le k < n$ and $1 \le t \le \lfloor \frac{k}{2} \rfloor - 1$, and let $p,\ n_j,\ p_j$ $(1 \le j \le t)$ be integers with $n_j \ge p_j > 1$ $(1 \le j \le t)$, $\sum_{j=1}^{t} p_j + 2 = k$, $\sum_{j=1}^{t} n_j + p + 2 = n$. Let $H_{n,\ k}$ be any graph of order n created from $H_j \in \widetilde{\Upsilon}_{n_j}^{(n_j - p_j)}$ $(j = 1,\ 2,\ \dots,\ t)$, pK_1 and K_2 (suppose $V(K_2) = \{u,\ v\}$) by connecting v to all vertices of pK_1 and H_j $(j = 1,\ 2,\ \dots,\ t)$ (see Figure 1.). Suppose that E^* is a subset of $E(G)$. Let $G\{E^*\}$ (resp. $\widetilde{G}\{E^*\}$) denote the (resp. connected) spanning subgraph of G which contains the edges in E^*.

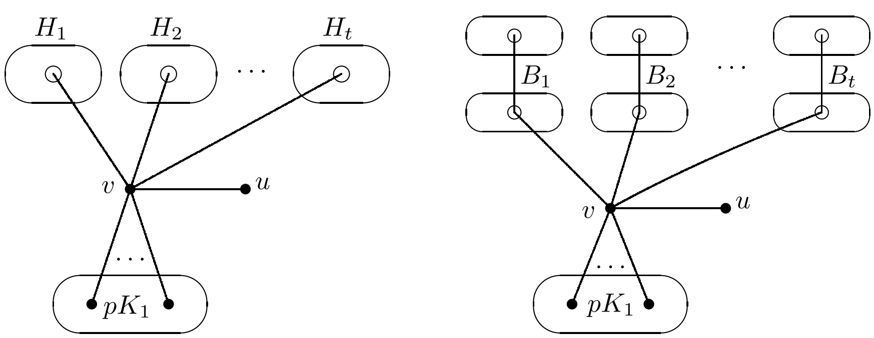

Figure 1. $H_{n,\ k}$ and $B_{n,\ k}$

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THEOREM 2.2. *Let G be a graph (resp. connected graph) of order n with pendent vertices. Then $\eta(G) = n-k$ $(4 \le k < n)$ if and only if G is isomorphic to $H_{n,\ k}\{E^*\}$ (resp. $\widetilde{H_{n,\ k}}\{E^*\}$), where $E^* = \cup_{j=1}^{t} E(H_j) \cup \{uv\}$.*

Proof. To begin with, we need to check that $\eta(H_{n,\ k}\{E^*\}) = \eta(\widetilde{H_{n,\ k}}\{E^*\}) = n-k$ $(4 \le k < n)$. Note that u is a pendent vertex of $H_{n,\ k}\{E^*\}$ (resp. $\widetilde{H_{n,\ k}}\{E^*\}$) and $N(u) = \{v\}$. Delete u, v from $H_{n,\ k}\{E^*\}$ (resp. $\widetilde{H_{n,\ k}}\{E^*\}$), then the resultant graph is $(\cup_{j=1}^{t} H_j) \cup pK_1$. Since $H_j \in \widetilde{\Upsilon}_{n_j}^{(n_j-p_j)}$, we have $\eta(H_j) = n_j - p_j$ $(j = 1,\ 2,\ \dots,\ t)$. Hence by Lemmas 1.4 and 1.5,

$$\eta(H_{n,\ k}\{E^*\}) = \eta(\widetilde{H_{n,\ k}}\{E^*\}) = \eta((\cup_{j=1}^{t} H_j) \cup pK_1) = \sum_{j=1}^{t} \eta(H_j) + p \cdot \eta(K_1)$$
$$= \sum_{j=1}^{t} (n_j - p_j) + p = (\sum_{j=1}^{t} n_j + p + 2) - (\sum_{j=1}^{t} p_j + 2) = n - k.$$

On the other hand, assume that $\eta(G) = n - k$. Choose a pendent vertex, say x, in G. Let $N(x) = \{y\}$. Delete x, y from G, and let the resultant graph be $G_1 = G_{11} \cup G_{12} \cup \cdots \cup G_{1q}$, where $G_{11},\ G_{12},\ \dots,\ G_{1q}$ are connected components of G_1. Some of these components may be trivial, i.e. K_1. We conclude that there exist t nontrivial connected components, where $1 \le t \le \lfloor \frac{k}{2} \rfloor - 1$. Without loss of generality, assume that $G_{11},\ G_{12},\ \dots,\ G_{1t}$ be nontrivial. By contradiction, suppose that $t = 0$ or $t \ge \lfloor \frac{k}{2} \rfloor$.

Case 1. $t = 0$. Then all the connected components are trivial, adding x, y to G_1 gives a star with some isolated vertices, which contradicts to Lemma 2.1.

Case 2. $t \ge \lfloor \frac{k}{2} \rfloor$. By Lemmas 1.1, 1.4 and 1.5, $\eta(G) = \sum_{j=1}^{t} \eta(G_{1j}) + z\eta(K_1) \le \sum_{j=1}^{t} (|V(G_{1j}) - 2|) + z$, where z is the number of isolated vertices in G_1. The above equality holds iff $G_{11},\ \dots,\ G_{1t}$ are all complete bipartite graphs.

Therefore, $\eta(G) \le \sum_{j=1}^{t} |V(G_{1j})| - 2t + z = (n-2-z) - 2t + z = n - 2t - 2 < n - k$ for $t \ge \lfloor \frac{k}{2} \rfloor$, contradicting that $\eta(G) = n - k$.

Hence $1 \le t \le \lfloor \frac{k}{2} \rfloor - 1$. Let $|V(G_{1j})| = n_j$ $(j = 1,\ 2,\ \dots,\ t)$. Then $G_1 = (\cup_{j=1}^{t} G_{1j}) \cup (n - \sum_{j=1}^{t} n_j - 2)K_1$. It follows from Lemmas 1.4 and 1.5 that

$$n - k = \eta(G) = \eta(G_1) = \eta(\cup_{j=1}^{t} G_{1j}) + \eta((n - \sum_{j=1}^{t} n_j - 2)K_1).$$

Since G_{1j} $(j = 1,\ 2,\ \dots,\ t)$ are nontrivial connected components, suppose that $\eta(G_{1j}) = n_j - p_j$, where $1 < p_j \le n_j$ $(j = 1,\ 2,\ \dots,\ t)$. Thus we have

$$n - k = \sum_{j=1}^{t} (n_j - p_j) + (n - \sum_{j=1}^{t} n_j - 2).$$

Hence $\sum_{j=1}^{t} p_j + 2 = k$ and $G_{1j} \in \widetilde{\Upsilon}_{n_j}^{(n_j-p_j)}$ $(j = 1,\ 2,\ \dots,\ t)$.

Let $p = n - \sum_{j=1}^{t} n_j - 2$. In order to recover G, to add x, y to G_1, we need

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to insert edges from y to x and to some (maybe partial or all) vertices of pK_1 and G_{1j} $(j = 1, 2, \dots, t)$. Thus the graph (resp. connected graph) G is isomorphic to $H_{n,\,k}\{E^*\}$ (resp. $\widetilde{H_{n,\,k}}\{E^*\}$), where $E^* = \cup_{j=1}^{t} E(H_j) \cup \{uv\}$. □

Now we have the following corollaries of this characterization.

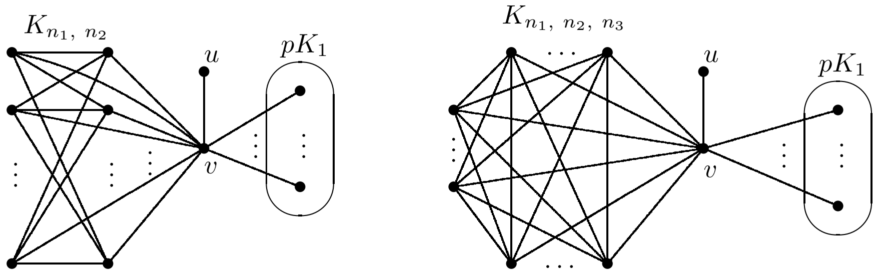

Figure 2. Q_1 and Q_2

Let Q_1 be a graph of order n created from $K_{n_1,\,n_2}$, pK_1 and K_2 (suppose $V(K_2) = \{u, v\}$) with $n_1+n_2+p+2 = n$ and $n_1, n_2 > 0$, $p \geq 0$ by connecting v to all vertices of pK_1 and $K_{n_1,\,n_2}$. Let Q_2 be a graph of order n created from $K_{n_1,\,n_2,\,n_3}$, pK_1 and K_2 $(V(K_2) = \{u, v\})$ with $n_1 + n_2 + n_3 + p + 2 = n$ and $n_1, n_2, n_3 > 0$, $p \geq 0$ by connecting v to all vertices of pK_1 and $K_{n_1,\,n_2,\,n_3}$ (see Figure 2.).

Corollary 2.3. *Let G be a graph (resp. connected graph) of order n with pendent vertices. Then*

(1) $\eta(G) = n - 4$ if and only if G is isomorphic to $Q_1\{E^*\}$ (resp. $\widetilde{Q_1}\{E^*\}$), where $E^* = E(K_{n_1,\,n_2}) \cup \{uv\}$.

(2) $\eta(G) = n - 5$ if and only if G is isomorphic to $Q_2\{E^*\}$ (resp. $\widetilde{Q_2}\{E^*\}$), where $E^* = E(K_{n_1,\,n_2,\,n_3}) \cup \{uv\}$.

Proof. By Theorem 2.2, $\eta(G) = n - k = n - 4$ implies $t = 1$, $p_1 = 2$, while $\eta(G) = n - k = n - 5$ implies $t = 1$, $p_1 = 3$. Besides, $\widetilde{\Upsilon}_n^{(n-2)} = \{K_{n_1,\,n_2} \mid n_1 + n_2 = n, \textit{ and } n_1, n_2 > 0\}$, $\widetilde{\Upsilon}_n^{(n-3)} = \{K_{n_1,\,n_2,\,n_3} \mid n_1 + n_2 + n_3 = n, \textit{ and } n_1, n_2, n_3 > 0\}$. Then we obtain the results as desired. □

Remark. If G is connected, the results of Corollary 2.3 are that in [9].

Now we shall determine all bipartite graphs with pendent vertices and nullity $\eta = n - 2k$ $(k = 0, 1, \dots, \lfloor n/2 \rfloor)$. Since $S_{n-k} \cup kK_1$ $(0 \leq k \leq n - 2)$ is a bipartite graph, combining Lemma 2.1, the following corollary is obvious.

Corollary 2.4. *Let G be a bipartite graph of order n with pendent vertices. Then*

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(1) There exists no such graphs G with nullity $\eta(G) = n$;

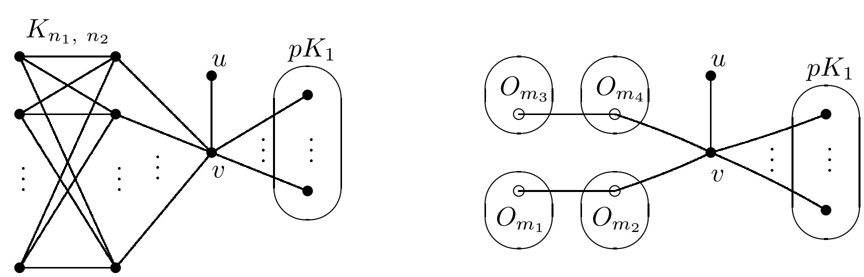
(2) $\eta(G) = n - 2$ if and only if G is isomorphic to $S_{n-k} \cup kK_1$ $(0 \leq k \leq n-2)$.

Let $\widetilde{\Phi}_n^{(\eta)}$ be the set of all connected bipartite graphs of order n with nullity $\eta = n - 2k$ $(k = 0,\ 1,\ \dots,\ \lfloor n/2 \rfloor)$. It is easy to see that $\widetilde{\Phi}_n^{(n)} = \emptyset$, $\widetilde{\Phi}_n^{(n-2)} = \{K_{n_1,\ n_2} \mid n_1 + n_2 = n,\ n_1,\ n_2 > 0\}$. Let $n,\ k,\ t$ be positive integers such that k is even, $4 \leq k < n$, and $1 \leq t \leq \frac{k}{2} - 1$. Let $p,\ n_j,\ p_j$ $(1 \leq j \leq t)$ be integers such that p_j is even, $n_j \geq p_j > 1$ $(1 \leq j \leq t)$, $\sum_{j=1}^{t} p_j + 2 = k$, $\sum_{j=1}^{t} n_j + p + 2 = n$. Let $B_{n,\ k}$ be a graph of order n created from $B_j \in \widetilde{\Phi}_{n_j}^{(n_j - p_j)}$ $(j = 1,\ 2,\ \dots,\ t)$, pK_1 and K_2 (suppose $V(K_2) = \{u,\ v\}$) by connecting v to all vertices of pK_1 and to all vertices in one partite set of B_j $(j = 1,\ 2,\ \dots,\ t)$ (also see Figure 1.).

Theorem 2.5. *Let G be a bipartite graph (resp. connected graph) of order n with pendent vertices. Then $\eta(G) = n - k$ (k is even and $4 \leq k < n$) if and only if G is isomorphic to $B_{n,\ k}\{E^*\}$ (resp. $\widetilde{B_{n,\ k}}\{E^*\}$), where $E^* = \cup_{j=1}^{t} E(B_j) \cup \{uv\}$.*

Proof. Note that $B_{n,\ k}\{E^*\}$ (resp. $\widetilde{B_{n,\ k}}\{E^*\}$) is a bipartite graph. The proof is now analogous to that of Theorem 2.2. ∎

Let Q_3 be a graph of order n created from $K_{n_1,\ n_2}$, pK_1 and K_2 (suppose $V(K_2) = \{u,\ v\}$) with $n_1 + n_2 + p + 2 = n$ and $n_1,\ n_2 > 0$, $p \geq 0$ by connecting v to all vertices of pK_1 and all vertices in one partite set of $K_{n_1,\ n_2}$. Let Q_4 be a graph of order n created from $O_{m_1}O_{m_2}$, $O_{m_3}O_{m_4}$, pK_1 and K_2 ($V(K_2) = \{u,\ v\}$) with $m_i > 0$ $(i = 1,\ \dots,\ 4)$, $p \geq 0$ and $\sum_{i=1}^{4} m_i + p + 2 = n$ by connecting v to all vertices of O_{m_1} (*or* O_{m_2}), O_{m_3} (*or* O_{m_4}) and pK_1. Let Q_5 be a graph of order n created from $O_{m_1}O_{m_2}O_{m_3}O_{m_4}$, pK_1 and K_2 ($V(K_2) = \{u,\ v\}$) with $m_i > 0$ $(i = 1,\ \dots,\ 4)$, $p \geq 0$ and $\sum_{i=1}^{4} m_i + p + 2 = n$ by connecting v to all vertices of pK_1, O_{m_1}, O_{m_3} (*or* pK_1, O_{m_2}, O_{m_4}). Let Q_6 be a graph of order n created from $O_{m_1}O_{m_2}O_{m_3}O_{m_4}O_{m_5}$, pK_1 and K_2 ($V(K_2) = \{u,\ v\}$) with $m_i > 0$ $(i = 1,\ \dots,\ 5)$, $p \geq 0$ and $\sum_{i=1}^{5} m_i + p + 2 = n$ by connecting v to all vertices of pK_1, O_{m_1}, O_{m_3}, O_{m_5} (*or* pK_1, O_{m_2}, O_{m_4}) (see Figure 3.).



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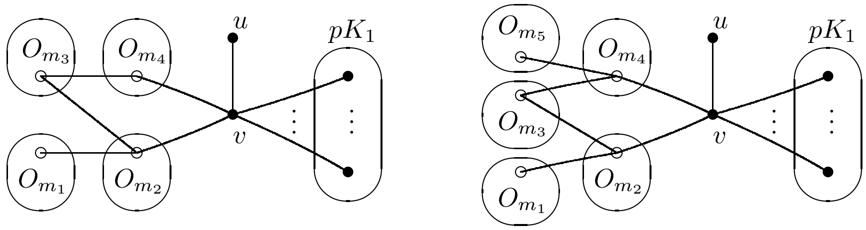

Figure 3. Q_3, Q_4, Q_5 and Q_6

COROLLARY 2.6. *Let G be a bipartite graph (resp. connected graph) of order n with pendent vertices. Then*

(1) $\eta(G) = n - 4$ if and only if G is isomorphic to $Q_3\{E^*\}$ (resp. $\widetilde{Q_3}\{E^*\}$), where $E^* = E(K_{n_1,\ n_2}) \cup \{uv\}$.

(2) $\eta(G) = n-6$ if and only if G is isomorphic to $Q_4\{E_1^*\}$, $Q_5\{E_2^*\}$ or $Q_6\{E_3^*\}$ (resp. $\widetilde{Q_4}\{E_1^*\}$, $\widetilde{Q_5}\{E_2^*\}$ or $\widetilde{Q_6}\{E_3^*\}$), where $E_1^* = E(O_{m_1}O_{m_2}) \cup E(O_{m_3}O_{m_4}) \cup \{uv\}$, $E_2^* = E(O_{m_1}O_{m_2}O_{m_3}\ O_{m_4}) \cup \{uv\}$, $E_3^* = E(O_{m_1}O_{m_2}O_{m_3}O_{m_4}O_{m_5}) \cup \{uv\}$.

Proof. (1) Note that $\eta(G) = n-4$ implies $t = 1$, $p_1 = 2$. Since $\widetilde{\Phi}_n^{(n-2)} = \{K_{n_1,\ n_2} \mid n_1 + n_2 = n,\ and\ n_1,\ n_2 > 0\}$, by Theorem 2.5, the result follows.

(2) Notice that $\eta(G) = n - 6$ implies the following two cases: Case 1. $t = 1$, $p_1 = 4$; Case 2. $t = 2$, $p_1 = 2$, $p_2 = 2$. By Lemma 1.3, we have $\widetilde{\Phi}_n^{(n-4)} = \{O_{m_1}O_{m_2}O_{m_3}O_{m_4},\ O_{m_1}O_{m_2}O_{m_3}\ O_{m_4}O_{m_5}\}$, $\widetilde{\Phi}_n^{(n-2)} = \{O_{m_1}O_{m_2}\}$ (Here $\sum m_i = n$). Thus the results are obtained by applying Theorem 2.5 to Cases 1 and 2. ∎

3. The minimum and maximum (connected) graphs with pendent vertices and nullity η. In this section, we shall determine the number $e_{min}^{(\eta)}$, $e_{max}^{(\eta)}$, $\widetilde{e}_{min}^{(\eta)}$, $\widetilde{e}_{max}^{(\eta)}$ and characterize $G_{min}^{(\eta)}$, $G_{max}^{(\eta)}$, $\widetilde{G}_{min}^{(\eta)}$, $\widetilde{G}_{max}^{(\eta)}$ for $0 < \eta \le n$.

Note that there exists no graph G of order n with pendent vertices and nullity $\eta(G) = n,\ n-1,\ n-3$ by Lemma 2.1, so we exclude these three cases.

THEOREM 3.1. *$G_{min}^{(n-2k)} \cong kK_2 \cup (n-2k)K_1$, $e_{min}^{(n-2k)} = k$, where $k = 1,\ 2,\ \dots,\ \lfloor \frac{n}{2} \rfloor$.*

Proof. Suppose $|E(G_{min}^{(n-2k)})| = i$ and there are j nontrivial connected components $G_{11},\ G_{12},\ \dots,\ G_{1j}$ of $G_{min}^{(n-2k)}$. Then $j \le i$.

Claim 1. $|E(G_{min}^{(n-2k)})| = k$. By contradiction, suppose $i \le k-1$.

Note that $|V(G_{1t})| \le |E(G_{1t})| + 1$ $(t = 1,\ 2,\ \dots,\ j)$. It follows that

$$r(G_{min}^{(n-2k)}) = \sum_{t=1}^{j} r(G_{1t}) \le \sum_{t=1}^{j} |V(G_{1t})| \le \sum_{t=1}^{j} |E(G_{1t})| + j = i + j \le 2i \le 2k-2.$$

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Hence $\eta(G_{min}^{(n-2k)}) = n - r(G_{min}^{(n-2k)}) \geq n - 2k + 2$, a contradiction.

Hence $i \geq k$. Note that $\eta(kK_2 \cup (n-2k)K_1) = n-2k$, and $|E(kK_2 \cup (n-2k)K_1)| = k$, then we have $|E(G_{min}^{(n-2k)})| = k$.

Claim 2. There are k nontrivial connected components of $G_{min}^{(n-2k)}$.

Since $|E(G_{min}^{(n-2k)})| = k$, we have $j \leq k$. Assume that $j \leq k-1$.

Notice that $|V(G_{1t})| \leq |E(G_{1t})| + 1$ $(t = 1, 2, \dots, j)$, hence

$$r(G_{min}^{(n-2k)}) = \sum_{t=1}^{j} r(G_{1t}) \leq \sum_{t=1}^{j} |E(G_{1t})| + j = k + j \leq 2k - 1.$$

It is a contradiction that $n - 2k = \eta(G_{min}^{(n-2k)}) = n - r(G_{min}^{(n-2k)}) \geq n - 2k + 1$.

Hence $j = k$. Combining Claims 1 and 2, $G_{min}^{(n-2k)}$ is isomorphic to a graph with k edges and k nontrivial connected components. Clearly, $G_{min}^{(n-2k)} \cong kK_2 \cup (n-2k)K_1$, and $e_{min}^{(n-2k)} = |E(G_{min}^{(n-2k)})| = k$, where $k = 1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. $\square$

Theorem 3.2. *$G_{min}^{(n-2k-1)} \cong K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1$, and $e_{min}^{(n-2k-1)} = k+2$, where $k = 2, 3, \dots, \lfloor \frac{n-1}{2} \rfloor$.*

Proof. Suppose that $|E(G_{min}^{(n-2k-1)})| = i$ and there are j nontrivial connected components $G_{11}, G_{12}, \dots, G_{1j}$ of $G_{min}^{(n-2k-1)}$.

Claim 1. There are at most k nontrivial connected components of $G_{min}^{(n-2k-1)}$.

By contradiction, suppose $j \geq k+1$. By Lemma 1.4, $\eta(G_{1t}) \leq |V(G_{1t})| - 2$ $(t = 1, 2, \dots, j)$ and $\eta(G_{min}^{(n-2k-1)}) = \sum_{t=1}^{j} \eta(G_{1t}) + z$, where z is the number of isolated vertices of $G_{min}^{(n-2k-1)}$. Hence $n-2k-1 = \eta(G_{min}^{(n-2k-1)}) = \sum_{t=1}^{j} \eta(G_{1t}) + z \leq \sum_{t=1}^{j} (|V(G_{1t})| - 2) + z \leq n - 2j \leq n - 2k - 2$, a contradiction.

Claim 2. $|E(G_{min}^{(n-2k-1)})| = k + 2$.

Note that $|V(G_{1t})| \leq |E(G_{1t})| + 1$ $(t = 1, 2, \cdots, j)$. Thus

$$r(G_{min}^{(n-2k-1)}) = \sum_{t=1}^{j} r(G_{1t}) \leq \sum_{t=1}^{j} |V(G_{1t})| \leq \sum_{t=1}^{j} |E(G_{1t})| + j = i + j.$$

It follows that

$$n - 2k - 1 = \eta(G_{min}^{(n-2k-1)}) = n - r(G_{min}^{(n-2k-1)}) \geq n - i - j.$$

Hence $i + j \geq 2k + 1$. Since $j \leq k$ by Claim 1, we have $i \geq k + 1$.

If $i = k+1$, then $j = k$. Thus $G_{min}^{(n-2k-1)} \cong K_{1,\,2} \cup (k-1)K_2 \cup (n-2k-1)K_1$. However, $\eta(K_{1,\,2} \cup (k-1)K_2 \cup (n-2k-1)K_1) = n - 2k \neq n - 2k - 1$.

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Thus $i \geq k+2$. Note that $\eta(K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1) = n-2k-1$, and $|E(K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1)| = k+2$. Then $|E(G_{min}^{(n-2k-1)})| = k+2$.

By Claim 2, $|E(G_{min}^{(n-2k-1)})| = i = k+2$, and it follows that $i + j = (k+2) + j \geq 2k+1$. Combining this with Claim 1, we have $j = k-1$ or k.

Case 1. $j = k-1$. First we show that there is no nontrivial connected components which are isomorphic to P_3. Suppose to the contrary that $G_{11} \cong P_3$.

Note that $r(P_3) = 2$ by Lemma 1.6 and $\sum_{t=2}^{j} |E(G_{1t})| = k$. Hence

$$\begin{aligned} r(G_{min}^{(n-2k-1)}) &= r(P_3) + \sum_{t=2}^{j} r(G_{1t}) \\ &\leq r(P_3) + \sum_{t=2}^{j} |V(G_{1t})| \leq r(P_3) + \sum_{t=2}^{j} |E(G_{1t})| + (j-1) = 2k. \end{aligned}$$

Thus $n-2k-1 = \eta(G_{min}^{(n-2k-1)}) = n - r(G_{min}^{(n-2k-1)}) \geq n-2k$, a contradiction.

Therefore, $G_{min}^{(n-2k-1)}$ may be isomorphic to one of the following:

(1) $T_1 = C_4 \cup (k-2)K_2 \cup (n-2k)K_1$;

(2) $T_2 = P_4 \cup (k-2)K_2 \cup (n-2k-1)K_1$;

(3) $T_3 = T^* \cup (k-2)K_2 \cup (n-2k)K_1$, where T^* is a graph of order 4 created from C_3 and K_2 by identifying a vertex of C_3 with a vertex of K_2;

(4) $T_4 = T^{**} \cup (k-2)K_2 \cup (n-2k-1)K_1$, where T^{**} is a graph of order 5 created from K_2 and S_3 by connecting the center of S_3 to a vertex of K_2;

(5) $T_5 = S_5 \cup (k-2)K_2 \cup (n-2k-1)K_1$.

By Lemmas 1.4 and 1.6, we get $\eta(T_1) = \eta(T_5) = n-2k+2 \neq n-2k-1$, $\eta(T_2) = \eta(T_3) = \eta(T_4) = n-2k \neq n-2k-1$. Hence $j \neq k-1$.

Case 2. $j = k$. $G_{min}^{(n-2k-1)}$ may be isomorphic to one of the following:

(1) $U_1 = K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1$;

(2) $U_2 = K_{1,\,3} \cup (k-1)K_2 \cup (n-2k-2)K_1$;

(3) $U_3 = P_4 \cup (k-1)K_2 \cup (n-2k-2)K_1$;

(4) $U_4 = 2K_{1,\,2} \cup (k-2)K_2 \cup (n-2k-2)K_1$.

It is not difficult to check that $\eta(U_1) = n-2k-1$, $\eta(U_2) = \eta(U_4) = n-2k \neq n-2k-1$, $\eta(U_3) = n-2k-2 \neq n-2k-1$.

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All in all, $G_{min}^{(n-2k-1)} \cong U_1 = K_3 \cup (k-1)K_2 \cup (n-2k-1)K_1$, and $e_{min}^{(n-2k-1)} = k+2$, where $k = 2,\ 3,\ \dots,\ \lfloor \frac{n-1}{2} \rfloor$. □

Let S_{n_j} be a star of order n_j, where $j = 1,\ 2,\ \dots,\ k$ and $\sum_{j=1}^{k} n_j = n$. Let $S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}$ denote a tree of order n created from S_{n_j} $(j = 1,\ 2,\ \dots,\ k)$ by adding $k-1$ edges to connect these stars, but the connection of two non-center vertices (not the center of a star) is not permitted. It is easy to see that $S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_p}$ $(2 \le p \le k)$ can be constructed recurrently by connecting the center of S_{n_p} to one vertex of $S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{p-1}}$.

Now $\widetilde{G}_{min}^{(n-2k)}$ can be characterized for $k = 1,\ 2,\ \dots,\ \lfloor \frac{n}{2} \rfloor$ as follows.

THEOREM 3.3. *$\widetilde{G}_{min}^{(n-2k)} \cong S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}$, $\widetilde{e}_{min}^{(n-2k)} = n-1$, where $\sum_{j=1}^{k} n_j = n$ and $k = 1,\ 2,\ \dots,\ \lfloor \frac{n}{2} \rfloor$.*

Proof. On one hand, by the definition of $S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}$, there is a pendent vertex u_{n_k} which is adjacent to the center of S_{n_k}. Then

$$\begin{aligned}
\eta(S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}) &= \eta(S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-1}}) + \eta((n_k-2)K_1) \\
&= \eta(S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-1}}) + (n_k-2) \\
&= \cdots = \eta(S_{n_1}) + \sum_{i=2}^{k} (n_i - 2) = n - 2k.
\end{aligned}$$

On the other hand we prove that $\widetilde{G}_{min}^{(n-2k)}$ is isomorphic to $S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}$ by induction on k, where $\sum_{j=1}^{k} n_j = n$ and $k = 1,\ 2,\ \dots,\ \lfloor \frac{n}{2} \rfloor$.

For $k = 1$, by Lemma 2.1, $\widetilde{G}_{min}^{(n-2)} \cong S_n$. Thus, the statement holds in this case. Suppose the statement holds for $k \le p-1$. Now we consider the case of $k = p$, where $2 \le p \le \lfloor \frac{n}{2} \rfloor$.

Claim 1. It's obvious that for any connected graph of order n, the minimum connected graph is a tree which has $n-1$ edges.

Claim 2. If T is a tree of order n with $\eta(T) = n - l$, then l is even.

Note that a tree T could be decomposed into t (with possibly $t = 0$) isolated vertices by deleting a pendent vertex and its adjacent vertex from T (and its resultant graph, suppose s times) recurrently. Hence $r(T) = r(tK_1) + 2s = 2s$, and then $\eta(T) = n - r(T) = n - 2s$. Therefore, $l = 2s$ is even.

Notice that $\widetilde{G}_{min}^{(n-2p)}$ has pendent vertices and $\eta(\widetilde{G}_{min}^{(n-2p)}) = n - 2p$. Choose a pendent vertex, say x, in $\widetilde{G}_{min}^{(n-2p)}$. Let $N(x) = \{y\}$. Delete x, y from $\widetilde{G}_{min}^{(n-2p)}$, and

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let the resultant graph be $\widetilde{G}_1 = \widetilde{G}_{11} \cup \widetilde{G}_{12} \cup \cdots \cup \widetilde{G}_{1q} \cup zK_1$, where $\widetilde{G}_{1j}$ are nontrivial connected components of order n_j^* $(j = 1,\ 2,\ \ldots,\ q)$, and $\sum_{j=1}^{q} n_j^* + z + 2 = n$.

By the definition of $\widetilde{G}_{min}^{(n-2p)}$ and Claim 1, each nontrivial connected component $\widetilde{G}_{1j}$ should be a tree with $n_j^* - 1$ edges $(j = 1,\ 2,\ \ldots,\ q)$. Moreover, it follows from Claim 2 that we suppose $\eta(\widetilde{G}_{1j}) = n_j^* - p_j$, where p_j is even and $0 < p_j \le n_j^*$ $(1 \le j \le q)$. By Theorem 2.2, we have $\sum_{j=1}^{q} p_j + 2 = 2p$.

Let $p_j = 2k_j$, and then $k_j = \frac{p_j}{2} \le p - 1$ $(j = 1,\ 2,\ \ldots,\ q)$. According to the inductive assumption, since $\eta(\widetilde{G}_{1j}) = n_j^* - 2k_j$, each $\widetilde{G}_{1j}$ is isomorphic to $S_{n_{j_1}^*} \oplus S_{n_{j_2}^*} \oplus \cdots \oplus S_{n_{j_{k_j}}^*}$, where $\sum_{i=1}^{k_j} n_{j_i}^* = n_j^*$ $(1 \le j \le q)$.

In order to recover the connected graph $\widetilde{G}_{min}^{(n-2p)}$, to add $x,\ y$ to $\widetilde{G}_1$, we need to insert edges from y to each of z isolated vertices of $\widetilde{G}_1$ and x. This gives a star $K_{1,\ z+1} = S_{z+2}$. Moreover, we shall connect the vertex y (namely, the center of S_{z+2}) to one vertex of each $\widetilde{G}_{1j}$ $(j = 1,\ 2,\ \ldots,\ q)$. So $\widetilde{G}_{min}^{(n-2p)}$ is a tree of order n created from $S_{n_{j_i}^*}$ ($i = 1,\ 2,\ \ldots,\ k_j;\ j = 1,\ 2,\ \ldots,\ p$) and S_{z+2} by adding $\sum_{j=1}^{q} k_j = p - 1$ edges to connect these stars, and any two non-center vertices are not connected since y is the center of S_{z+2}.

All in all, it follows from the induction that $\widetilde{G}_{min}^{(n-2k)} \cong S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_k}$, and then $\widetilde{e}_{min}^{(n-2k)} = n - 1$, where $\sum_{j=1}^{k} n_j = n$ and $k = 1,\ 2,\ \ldots,\ \lfloor \frac{n}{2} \rfloor$. □

Let C_{2h+1} be a $(2h+1)$-cycle and let S_{n_j} be a star of order n_j, where $1 \le h < k$, $1 \le j \le k - h$ and $(2h + 1) + \sum_{j=1}^{k-h} n_j = n$. Let $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}}$ denote a unicyclic connected graph of order n created from C_{2h+1} $(1 \le h < k)$ and S_{n_j} $(j = 1,\ 2,\ \cdots,\ k-h)$ by adding $k-h$ edges to connect them, but the connection of two non-center vertices is not permitted. It is easy to see that $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_p}$ $(1 \le p \le k - h)$ can be constructed recurrently by connecting the center of S_{n_p} to one vertex of $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{p-1}}$.

THEOREM 3.4. $\widetilde{G}_{min}^{(n-2k-1)} \cong C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}}$, $\widetilde{e}_{min}^{(n-2k-1)} = n$, *where* $1 \le h < k$, $(2h + 1) + \sum_{j=1}^{k-h} n_j = n$ *and* $k = 2,\ 3,\ \ldots,\ \lfloor \frac{n-1}{2} \rfloor$.

Proof. By the definition of $C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}}$,

$$\eta(C_{2h+1} \oplus S_{n_1} \oplus \cdots \oplus S_{n_{k-h}}) = \eta(C_{2h+1} \oplus S_{n_1} \oplus \cdots \oplus S_{n_{k-h-1}}) + \eta((n_{k-h} - 2)K_1)$$
$$= \cdots = \eta(C_{2h+1}) + \textstyle\sum_{i=1}^{k-h}(n_i - 2) = 0 + (\sum_{i=1}^{k-h} n_i - 2k + 2h) = n - 2k - 1.$$

On the other hand, we show that $\widetilde{G}_{min}^{(n-2k-1)}$ is isomorphic to $C_{2h+1} \oplus S_{n_1} \oplus \cdots \oplus S_{n_{k-h}}$ by induction on k, where $1 \le h < k$ and $(2h + 1) + \sum_{j=1}^{k-h} n_j = n$.

For $k = 2$, we have $h = 1$, and it follows from Corollary 2.3 (2) that $\widetilde{G}_{min}^{(n-5)} \cong$

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$C_3 \oplus S_{n-3}$. Therefore, the statement holds in this case. Suppose the statement holds for $k \le p-1$. We consider the case of $k = p$, where $3 \le p \le \lfloor \frac{n-1}{2} \rfloor$.

Note that $\widetilde{G}_{min}^{(n-2p-1)}$ has pendent vertices and $\eta(\widetilde{G}_{min}^{(n-2p-1)}) = n-2p-1$. Choose a pendent vertex, say x, in $\widetilde{G}_{min}^{(n-2p-1)}$. Let $N(x) = \{y\}$. Delete $x,\ y$ from $\widetilde{G}_{min}^{(n-2p-1)}$, and let the resultant graph be $\widetilde{G}_1 = \widetilde{G}_{11} \cup \cdots \cup \widetilde{G}_{1q} \cup zK_1$, where $\widetilde{G}_{1j}$ are nontrivial connected components of order n_j^* $(j = 1,\ 2,\ \dots,\ q)$, and $\sum_{j=1}^{q} n_j^* + z + 2 = n$.

Assume that $\eta(\widetilde{G}_{1j}) = n_j^* - l_j^*$ $(0 < l_j^* \le n_j^*)$ for $j = 1,\ 2,\ \dots,\ q$.

Claim 1. One of the nontrivial connected components (suppose $\widetilde{G}_{11}$) is an unicyclic connected graph, and others are trees.

If all $\widetilde{G}_{1j}$ are trees, then l_j^* $(j = 1,\ 2,\ \dots,\ q)$ is even by Theorem 3.3 Claim 2, and

$$2p+1 = n - \eta(\widetilde{G}_{min}^{(n-2p-1)}) = n - [\sum_{j=1}^{q} \eta(\widetilde{G}_{1j}) + z] = 2 + \sum_{j=1}^{q} l_j^*,$$

a contradiction. Since the number of edges for $\widetilde{G}_{min}^{(n-2p-1)}$ should be as least as possible, and $C_{2h+1} \oplus S_{n_1} \oplus \cdots \oplus S_{n_{p-h}}$ with nullity $n-2p-1$ which satisfies this claim, it follows that Claim 1 holds.

Claim 2. l_1^* is odd. Otherwise, we get a similar contradiction as Claim 1.

Claim 3. Let $l_1^* = 2t^*+1$. Then $\widetilde{G}_{11} \cong C_{2t^*+1}$ $(n_1^* = 2t^*+1)$, or $\widetilde{G}_{11} \cong C_{2h_1+1} \oplus S_{n_{1,\ 1}^*} \oplus \cdots \oplus S_{n_{1,\ t^*-h_1}^*}$, where $1 \le h_1 < t^*$, $(2h_1+1) + \sum_{j=1}^{t^*-h_1} n_{1j}^* = n_1^*$.

Case 1. If $\widetilde{G}_{11}$ has pendent vertices, since $t^* = \frac{l_1^*-1}{2} \le p-1$ (note that $\sum_{j=1}^{q} l_j^* = 2p-1$) and $\eta(\widetilde{G}_{11}) = n_1^* - 2t^* - 1$, according to the inductive assumption, $\widetilde{G}_{11} \cong C_{2h_1+1} \oplus S_{n_{1,\ 1}^*} \oplus \cdots \oplus S_{n_{1,\ t^*-h_1}^*}$, where $1 \le h_1 < t^*$, $(2h_1+1) + \sum_{j=1}^{t^*-h_1} n_{1j}^* = n_1^*$.

Case 2. If $\widetilde{G}_{11}$ has no pendent vertex, since $\widetilde{G}_{11}$ is an unicyclic connected graph, $\widetilde{G}_{11}$ is an odd cycle of order n_1^*. Hence $\widetilde{G}_{11} \cong C_{2t^*+1}$ and $l_1^* = 2t^*+1 = n_1^*$.

Claim 4. Combining Claim 1 with Theorem 3.3, each $\widetilde{G}_{1j}$ $(2 \le j \le q)$ is isomorphic to $S_{n_{j,\ 1}^*} \oplus S_{n_{j,\ 2}^*} \oplus \cdots \oplus S_{n_{j,\ k_j}^*}$, where $\sum_{i=1}^{k_j} n_{j,\ i}^* = n_j^*$ and $l_j^* = 2k_j$.

In order to recover the connected graph $\widetilde{G}_{min}^{(n-2p-1)}$, to add $x,\ y$ to $\widetilde{G}_1$, we insert edges from y to each of z isolated vertices of $\widetilde{G}_1$ and x. This gives a star $K_{1,\ z+1} = S_{z+2}$. Moreover, we shall connect the vertex y (namely, the center of S_{z+2}) to one vertex of each $\widetilde{G}_{1j}$ $(j = 1,\ 2,\ \dots,\ q)$. Let $t^* - h_1 = k_1$. Then $\widetilde{G}_{min}^{(n-2p-1)}$ is an unicyclic connected graph of order n created from C_{2h_1+1}, $S_{n_{j,\ i}^*}$ $(i = 1,\ 2,\ \dots,\ k_j;\ j = 1,\ 2,\ \dots,\ p)$ and S_{z+2} by adding $\sum_{j=1}^{q} k_j + 1 = p - h_1$

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$(1 \leq h_1 < p)$ edges to connect these graphs, and any two non-center vertices are not connected since y is the center of S_{z+2}.

In conclusion,

$$\widetilde{G}_{min}^{(n-2k-1)} \cong C_{2h+1} \oplus S_{n_1} \oplus S_{n_2} \oplus \cdots \oplus S_{n_{k-h}},$$

and then $\widetilde{e}_{min}^{(n-2k-1)} = n$, where $1 \leq h < k$, $(2h+1) + \sum_{j=1}^{k-h} n_j = n$ and $k = 2,\ 3,\ \ldots,\ \lfloor \frac{n-1}{2} \rfloor$. □

The following lemma describes the relationship between $G_{max}^{(\eta)}$ and $\widetilde{G}_{max}^{(\eta)}$.

Lemma 3.5. *$G_{max}^{(\eta)} \cong \widetilde{G}_{max}^{(\eta)}$, $e_{max}^{(\eta)} = \widetilde{e}_{max}^{(\eta)}$, where $0 < \eta \leq n$.*

Proof. Since we want to insert edges as many as possible, by Lemma 2.1 and Theorem 2.2, this lemma is proved. □

Now $G_{max}^{(\eta)}$ (*namely*, $\widetilde{G}_{max}^{(\eta)}$) is characterized for $\eta = n-2,\ n-4,\ n-5$.

Theorem 3.6. *$G_{max}^{(n-2)} \cong \widetilde{G}_{max}^{(n-2)} \cong S_n$, $e_{max}^{(n-2)} = \widetilde{e}_{max}^{(n-2)} = n-1$.*

Proof. By Lemma 2.1 (2), we obtain the results as desired. □

Let $U_{max}^{(n-4)}$ be a graph of order n created from $K_{\lceil \frac{n}{2} \rceil -1,\ \lfloor \frac{n}{2} \rfloor -1}$ and K_2 by connecting a vertex v of K_2 to all vertices of $K_{\lceil \frac{n}{2} \rceil -1,\ \lfloor \frac{n}{2} \rfloor -1}$.

Theorem 3.7. *$G_{max}^{(n-4)} \cong \widetilde{G}_{max}^{(n-4)} \cong U_{max}^{(n-4)}$, $e_{max}^{(n-4)} = \widetilde{e}_{max}^{(n-4)} = \lfloor \frac{n^2}{4} \rfloor$.*

Proof. By Corollary 2.3 (1), $G_{max}^{(n-4)}$ should be a graph Q_{max} of order n created from $K_{n_1,\ n_2}$, pK_1 and K_2 such that $n_1 + n_2 + p + 2 = n$ and $n_1,\ n_2 > 0,\ p \geq 0$ by connecting a vertex v of K_2 to all vertices of pK_1 and $K_{n_1,\ n_2}$.

Since $n_2 = n - n_1 - p - 2$ and $n_1,\ n_2 > 0,\ p \geq 0$, we have

$$\begin{aligned} |E(Q_{max})| &= n_1 n_2 + n - 1 = -n_1^2 + (n-p-2)n_1 + (n-1) \\ &\leq -n_1^2 + (n-2)n_1 + (n-1) \\ &= -(n_1 - \frac{n}{2} + 1)^2 + \frac{n^2}{4} \\ &\leq \begin{cases} \frac{n^2}{4}, & n \text{ is even}; \\ \frac{n^2-1}{4}, & n \text{ is odd}. \end{cases} \end{aligned}$$

where the first equality holds if and only if $p = 0$, and the second equality holds if and only if $n_1 = \frac{n}{2} - 1$ (n is even); $n_1 = \frac{n-1}{2} - 1$ or $\frac{n+1}{2} - 1$ (n is odd), which implies that $n_2 = \frac{n}{2} - 1$ (n is even); $n_2 = \frac{n+1}{2} - 1$ or $\frac{n-1}{2} - 1$ (n is odd).

Combining Lemma 3.5, it follows that $G_{max}^{(n-4)} \cong \widetilde{G}_{max}^{(n-4)} \cong U_{max}^{(n-4)}$.

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Moreover, $e_{max}^{(n-4)} = \widetilde{e}_{max}^{(n-4)} = \begin{cases} \frac{n^2}{4}, & n \text{ is even;} \\ \frac{n^2-1}{4}, & n \text{ is odd.} \end{cases}$ □

Let $U_{max}^{(n-5)}$ be a graph of order n created from

$$U^* = \begin{cases} K_{\frac{n-2}{3},\ \frac{n-2}{3},\ \frac{n-2}{3}}, & n \equiv 2\ (mod\ 3) \\ K_{\frac{n}{3},\ \frac{n-3}{3},\ \frac{n-3}{3}}, & n \equiv 0\ (mod\ 3) \\ K_{\frac{n-4}{3},\ \frac{n-1}{3},\ \frac{n-1}{3}}, & n \equiv 1\ (mod\ 3) \end{cases}$$

and K_2 by connecting a vertex v of K_2 to all vertices of U^*.

THEOREM 3.8. $G_{max}^{(n-5)} \cong \widetilde{G}_{max}^{(n-5)} \cong U_{max}^{(n-5)}$, $e_{max}^{(n-5)} = \widetilde{e}_{max}^{(n-5)} = \lfloor \frac{n^2-n+1}{3} \rfloor$.

Proof. By Corollary 2.3 (2), $G_{max}^{(n-5)}$ is isomorphic to a graph C_{max} of order n created from $K_{n_1,\ n_2,\ n_3}$, pK_1 and K_2 satisfying $n_1 + n_2 + n_3 + p + 2 = n$ and $n_1,\ n_2,\ n_3 > 0,\ p \geq 0$ by connecting a vertex v of K_2 to all vertices of pK_1 and $K_{n_1,\ n_2,\ n_3}$.

Since $n_3 = n - n_1 - n_2 - p - 2$ and $n_1,\ n_2,\ n_3 > 0,\ p \geq 0$, we have

$$\begin{aligned} |E(C_{max})| &= n_1n_2 + n_2n_3 + n_3n_1 + n - 1 \\ &= -(n_1+n_2)^2 + (n-2-p)(n_1+n_2) + (n-1) + n_1n_2 \\ &\leq -(n_1+n_2)^2 + (n-2-p)(n_1+n_2) + (n-1) + \frac{(n_1+n_2)^2}{4} \\ &= -\frac{3}{4}(n-n_3-p-2)^2 + (n-2-p)(n-n_3-p-2) + (n-1) \\ &= \frac{1}{4}[-3n_3^2 + 2(n-p-2)n_3 + (n-p-2)^2] + (n-1) \\ &\leq \frac{1}{4}[-3n_3^2 + 2(n-2)n_3 + (n-2)^2] + (n-1) \\ &= -\frac{3}{4}(n_3 - \frac{n-2}{3})^2 + \frac{n^2-n+1}{3} \leq \begin{cases} \frac{n^2-n+1}{3}, & n-2 \equiv 0\ (mod\ 3); \\ \frac{n^2-n}{3}, & n-2 \not\equiv 0\ (mod\ 3), \end{cases} \end{aligned}$$

where the first equality holds if and only if $n_1 = n_2$, the second equality holds if and only if $p = 0$, and the third equality holds if and only if

$$n_3 = \begin{cases} \frac{n-2}{3}, & n-2 \equiv 0\ (mod\ 3); \\ \frac{n}{3}, & n-2 \equiv 1\ (mod\ 3); \\ \frac{n-4}{3}, & n-2 \equiv 2\ (mod\ 3). \end{cases}$$

Thus $n_1 = n_2 = \begin{cases} \frac{n-2}{3}, & n-2 \equiv 0\ (mod\ 3); \\ \frac{n-3}{3}, & n-2 \equiv 1\ (mod\ 3); \\ \frac{n-1}{3}, & n-2 \equiv 2\ (mod\ 3). \end{cases}$

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Hence $G_{max}^{(n-5)} \cong U_{max}^{(n-5)}$ and then $e_{max}^{(n-5)} = \begin{cases} \frac{n^2-n+1}{3}, & n-2 \equiv 0 \ (mod\ 3); \\ \frac{n^2-n}{3}, & n-2 \not\equiv 0 \ (mod\ 3). \end{cases}$

Combining this with Lemma 3.5 gives the desired results. □

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