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## REACHABILITY MATRICES AND CYCLIC MATRICES\*

AUGUSTO FERRANTE<sup>†</sup> AND HARALD K. WIMMER<sup>‡</sup>

**Abstract.** We study reachability matrices  $R(A, b) = [b, Ab, \dots, A^{n-1}b]$ , where  $A$  is an  $n \times n$  matrix over a field  $K$  and  $b$  is in  $K^n$ . We characterize those matrices that are reachability matrices for some pair  $(A, b)$ . In the case of a cyclic matrix  $A$  and an  $n$ -vector of indeterminates  $x$ , we derive a factorization of the polynomial  $\det(R(A, x))$ .

**Key words.** Reachability matrix, Krylow matrix, cyclic matrix, nonderogatory matrix, companion matrix, Vandermonde matrix, Hautus test.

**AMS subject classifications.** 15A03, 15A15, 93B05.

**1. Introduction.** Let  $K$  be a field, and  $A \in K^{n \times n}$ ,  $b \in K^n$ . The matrix

$$R(A, b) = [b, Ab, \dots, A^{n-1}b] \in K^{n \times n}$$

is the *reachability matrix* of the pair  $(A, b)$ . A matrix  $A$  is called *cyclic* (e.g. in [3], [4]) or *nonderogatory* (e.g. in [2], [9]), if there exists a vector  $b \in K^n$  such that

$$\text{span}\{b, Ab, A^2b, \dots, A^{n-1}b\} = K^n. \quad (1.1)$$

In that case the pair  $(A, b)$  is said to be *reachable*. Let

$$a(z) = z^n - (a_{n-1}z^{n-1} + \dots + a_1z + a_0) \quad (1.2)$$

be the characteristic polynomial of  $A$ . The matrix

$$F_a = \begin{bmatrix} 0 & \cdot & & & a_0 \\ 1 & 0 & \cdot & & a_1 \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ 0 & 0 & & 1 & a_{n-1} \end{bmatrix} \quad (1.3)$$

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is the *companion matrix of the second type* [1] associated with (1.2). It is well known (see e.g. [4, p.299]) that  $A$  is cyclic if and only if  $A$  is similar to the companion matrix  $F_a$ . Or equivalently, if  $x_0, \dots, x_{n-1}$  are indeterminates over  $K$  and  $x := [x_0, \dots, x_{n-1}]^\top$ , then  $A$  is cyclic if and only if the polynomial  $\det R(A, x)$  is not the zero polynomial.

In this note we are concerned with the following questions. When is a given matrix  $M \in K^{n \times n}$  a reachability matrix? How can one factorize the polynomial  $\det R(A, x)$ ?

**2. Companion and reachability matrices.** In this section we characterize those matrices that are reachability matrices for some pair  $(A, b)$ . We first show that each nonsingular matrix  $M$  is a reachability matrix. Let

$$e_0 = [1, 0, \dots, 0]^\top, \dots, e_{n-1} = [0, \dots, 0, 1]^\top,$$

be the unit vectors of  $K^n$ .

**THEOREM 2.1.** *Let  $M = [v_0, v_1, \dots, v_{n-1}] \in K^{n \times n}$  be nonsingular. Then  $M = R(A, b)$  if and only if  $b = v_0$  and  $A = MF_c M^{-1}$  for some nonsingular companion matrix  $F_c$ . In particular,  $M = R(A, v_0)$  with*

$$A = [v_1, \dots, v_{n-1}, v_0] M^{-1}. \tag{2.1}$$

*Proof.* We have  $e_0 = M^{-1}v_0$ . Hence, if  $b = v_0$  and  $A = MF_c M^{-1}$  then  $A^i b = MF_c^i e_0 = M e_i = v_i$ , and thus  $M = R(A, b)$ . We obtain (2.1) if we choose  $F_c = (e_1, e_2, \dots, e_{n-1}, e_0)$ . Conversely, if  $M = R(A, b)$ , then  $b = v_0$ , and  $AM = MF_a$  for some companion matrix  $F_a$ . Hence if  $M$  is nonsingular then  $A = MF_a M^{-1}$ .  $\square$

**THEOREM 2.2.** *Let  $M = [v_0, v_1, \dots, v_{n-1}] \in K^{n \times n}$  and  $\text{rank } M = r$ . The following statements are equivalent.*

- (i)  $M$  is a reachability matrix.
- (ii) Either  $\text{rank } M = n$ , i.e. the matrix  $M$  is nonsingular, or

$$\text{rank } M = \text{rank } [v_0, v_1, \dots, v_{r-1}] = r < n \tag{2.2}$$

and

$$\text{Ker}[v_{k-1}, v_k, \dots, v_{k-1+r}] \subseteq \text{Ker}[v_k, v_{k+1}, \dots, v_{k+r}], \tag{2.3}$$

$$k = 1, 2, \dots, n - 1 - r.$$

*Proof.* If (2.2) holds then (2.3) means that there exist  $c_i \in K$ ,  $i = 0, \dots, r-1$ , such that

$$v_{r+k} = \sum_{i=0}^{r-1} c_i v_{i+k}, \quad k = 0, 1, \dots, n-r-1. \quad (2.4)$$

(i)  $\Rightarrow$  (ii) If  $M$  is a reachability matrix and  $\text{rank } M = r < n$  then it is obvious that the conditions (2.2) and (2.4) are satisfied.

(ii)  $\Rightarrow$  (i) If  $\text{rank } M = n$  then it follows from Theorem 2.1 that  $M$  is a reachability matrix. Now assume (2.2) and (2.4). Let  $Q \in K^{n \times n}$  be nonsingular such that

$$Q[v_0, \dots, v_{r-1}] = \begin{bmatrix} I_r \\ 0 \end{bmatrix}.$$

Let  $\hat{e}_0, \dots, \hat{e}_{r-1}$  be the canonical unit vectors of  $K^r$ . Then (2.4) implies

$$QM = \begin{bmatrix} w_0 & \dots & w_{r-1} & w_r & \dots & w_{n-1} \\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} \hat{e}_0 & \dots & \hat{e}_{r-1} & w_r & \dots & w_{n-1} \\ 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}, \quad (2.5)$$

and the vectors  $w_i$  satisfy

$$w_{r+k} = \sum_{i=0}^{r-1} c_i w_{i+k}, \quad k = 0, 1, \dots, n-r-1.$$

Set

$$\hat{A} = \begin{bmatrix} 0 & 0 & & c_0 \\ 1 & 0 & & c_1 \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ 0 & 0 & 1 & c_{r-1} \end{bmatrix}. \quad (2.6)$$

Then

$$w_r = [c_0, c_1, \dots, c_{r-1}]^T = \hat{A} \hat{e}_{r-1} = \hat{A} \hat{A}^{r-1} \hat{e}_0 = \hat{A}^r \hat{e}_0,$$

and

$$[\hat{e}_0, \hat{A} \hat{e}_0, \dots, \hat{A}^{r-1} \hat{e}_0, \hat{A}^r \hat{e}_0, \dots, \hat{A}^{n-1} \hat{e}_0] = [\hat{e}_0, \dots, \hat{e}_{r-1}, w_r, \dots, w_{n-1}].$$

Hence the matrix  $QM$  in (2.5) can be written as

$$QM = R(A, b) \quad \text{with} \quad A = \begin{bmatrix} \hat{A} & 0 \\ 0 & 0 \end{bmatrix} \in K^{n \times n}, \quad b = \begin{bmatrix} \hat{e}_0 \\ 0 \end{bmatrix} \in K^n,$$

and we obtain  $M = R(Q^{-1}AQ, Q^{-1}b)$ .  $\square$

**3. A factorization theorem.** We first characterize companion matrices in terms of reachability matrices. Let  $b = (b_0, b_1, \dots, b_{n-1})^\top \in K^n$ . We call

$$b(z) = (1, z, \dots, z^{n-1})b = b_0 + b_1z + \dots + b_{n-1}z^{n-1} \quad (3.1)$$

the polynomial associated to  $b$ .

PROPOSITION 3.1. *Let  $a(z) = z^n - \sum a_i z^i$  be the characteristic polynomial of  $A \in K^{n \times n}$ , and let  $F_a$  be the companion matrix in (1.3). Then  $A = F_a$  if and only if*

$$R(A, b) = b(A) \quad \text{for all } b \in K^n. \quad (3.2)$$

*Proof.* It is obvious that  $A$  is a companion matrix of the form (1.3) if and only if

$$A[e_0, e_1, \dots, e_{n-2}] = [e_1, e_2, \dots, e_{n-1}]. \quad (3.3)$$

Assume now that (3.2) is satisfied. Choose  $b = e_0$ . Then  $b(z) = 1$  and  $b(A) = I$ . Therefore

$$R(A, e_0) = [e_0, Ae_0, \dots, A^{n-1}e_0] = I = [e_0, e_1, \dots, e_{n-1}].$$

Hence we obtain (3.3), and we conclude that  $A = F_a$ . To prove the converse we have to show that

$$R(F_a, b) = b(F_a) \quad (3.4)$$

holds for all  $b = \sum_{i=0}^{n-1} b_i e_i$ . We have  $e_i = F_a^i e_0$ ,  $i = 0, \dots, n-1$ . From  $R(F_a, e_0) = I$  and  $R(F_a, e_i) = F_a^i R(F_a, e_0)$  follows  $R(F_a, e_i) = F_a^i$ . Therefore

$$R(F_a, b) = \sum_{i=0}^{n-1} b_i R(F_a, e_i) = \sum_{i=0}^{n-1} b_i F_a^i = b(F_a). \quad \square$$

Suppose  $A$  is cyclic. Let  $S$  be nonsingular such that  $SAS^{-1} = F_a$ , and let the polynomial  $(Sb)(z)$  be defined in analogy to (3.1). Then  $S R(A, b) = R(F_a, Sb)$ . From (3.2) we obtain

$$R(A, b) = S^{-1} (Sb)(F_a). \quad (3.5)$$

Note that for all  $b \in K^n$  we have  $A R(A, b) = R(A, b) F_a$ . Hence, if the pair  $(A, b)$  is reachable then the matrix  $S = R(A, b)^{-1}$  satisfies

$$SAS^{-1} = F_a. \quad (3.6)$$

The identity (3.6) can be found in [6, Section 6.1].

For the following observation we are indebted to a referee. Suppose  $A$  is a matrix with distinct eigenvalues, and  $X^{-1}AX = D$  is a Jordan form. Then the corresponding companion matrix  $F_a$  is similar to the diagonal matrix  $D$ . The similarity transformation  $VF_aV^{-1} = D$  is accomplished by a Vandermonde matrix  $V$  whose nodes are the eigenvalues of  $A$  (see e.g. [12, Section 1.11]). One can write  $V$  as a reachability matrix, that is,  $V = R(D, e)$ , where  $e = (1, 1, \dots, 1)^T$ .

Let  $a_j(z)$ ,  $j = 1, \dots, r$ , be the monic irreducible factors of the polynomial  $a(z)$  in (1.2). Suppose  $\deg a_j(z) = \ell_j$  and

$$a(z) = a_1(z)^{m_1} \cdots a_r(z)^{m_r}, \quad (3.7)$$

such that  $\sum_{j=1}^r m_j \ell_j = n$ . Let  $F_{a_j} \in K^{\ell_j \times \ell_j}$  be the corresponding companion matrices. The main result of this section is the following.

**THEOREM 3.2.** *Let  $A$  be cyclic with characteristic polynomial  $a(z)$ , and let (3.7) be the prime factorization of  $a(z)$ . Suppose*

$$SAS^{-1} = F_a \quad \text{and} \quad \det S = 1. \quad (3.8)$$

Set  $g_j(x) = \det(Sx)(F_{a_j})$ ,  $j = 1, \dots, r$ . Then

$$\det R(A, x) = (g_1(x))^{m_1} \cdots (g_r(x))^{m_r}. \quad (3.9)$$

The polynomials  $g_1(x), \dots, g_r(x)$  are irreducible, and homogeneous of degree  $\ell_1, \dots, \ell_r$ , respectively.

*Proof.* Define

$$C(a_j, m_j) = \begin{pmatrix} F_{a_j} & I_{\ell_j} & 0 & \cdot & 0 \\ 0 & F_{a_j} & I_{\ell_j} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & F_{a_j} & I_{\ell_j} \\ \cdot & \cdot & \cdot & \cdot & F_{a_j} \end{pmatrix}_{m_j \ell_j \times m_j \ell_j}.$$

Then

$$TF_aT^{-1} = \text{diag} \left( C(a_1, m_1), \dots, C(a_r, m_r) \right) \quad (3.10)$$

for some  $T \in K^{n \times n}$ . The right hand side of (3.10) is the rational canonical form (the Frobenius canonical form) of  $F_a$ . Let  $\hat{b} \in K^n$ . From (3.4) we obtain

$$R(F_a, \hat{b}) = \hat{b}(F_a) = T^{-1} \hat{b} \left[ \text{diag} \left( C(a_1, m_1), \dots, C(a_r, m_r) \right) \right] T.$$

Hence

$$\begin{aligned} \det R(F_a, \hat{b}) &= \det \hat{b}(C(a_1, m_1)) \cdots \det \hat{b}(C(a_r, m_r)) = \\ &= (\det \hat{b}(F_{a_1}))^{m_1} \cdots (\det \hat{b}(F_{a_r}))^{m_r}. \end{aligned} \quad (3.11)$$

Suppose  $SAS^{-1} = F_a$  and  $\det S = 1$ . Then (3.5) implies  $\det R(A, x) = \det R(F_a, Sx)$ . Taking  $\hat{b} = Sx$  in (3.11) we obtain (3.9).

Suppose one of the polynomials  $g_j(x)$  is reducible. E.g. let  $g_1(x) = p(x)q(x)$  and  $\deg p \geq 1, \deg q \geq 1$ . Then  $x = S^{-1}(-1, z, 0, \dots, 0)^\top$  yields  $(Sx)(F_{a_1}) = -F_{a_1} + zI$ . Hence  $g_1(x) = \det(-F_{a_1} + zI) = a_1(z)$ . On the other hand  $g_1(x) = \tilde{p}(z)\tilde{q}(z)$ , and  $\deg \tilde{p} \geq 1, \deg \tilde{q} \geq 1$ . This is a contradiction to the irreducibility of  $a_1(z)$ .

Since  $F_{a_j}$  is of size  $\ell_j \times \ell_j$  we obtain  $g_j(\lambda x) = \lambda^{\ell_j} g_j(x)$ . Thus,  $g_j(x)$  is homogeneous of degree  $\ell_j$ .  $\square$

We now assume that the characteristic polynomial  $a(z)$  of  $A$  splits over  $K$ . If  $\lambda_1, \dots, \lambda_r$  are the different eigenvalues of  $A$  then

$$a(z) = (z - \lambda_1)^{m_1} \dots (z - \lambda_r)^{m_r}. \quad (3.12)$$

In that case  $a_j(z) = (z - \lambda_j)$  and  $F_{a_j} = (\lambda_j)$ , and

$$g_j(x) = [1, \lambda_j, \lambda_j^2, \dots, \lambda_j^{n-1}] S \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}, \quad j = 1, \dots, r. \quad (3.13)$$

The factors  $g_j(x)$  in (3.13) are related to the Popov-Belevitch-Hautus controllability test (see e.g. [11, p.93]). It is known that - up multiplicative constants - the row vectors

$$w_j^\top = [1, \lambda_j, \dots, \lambda_j^{n-1}], \quad j = 1, \dots, r,$$

are the left eigenvectors of  $F_a$ . Then  $SAS^{-1} = F_a$  implies that  $v_j^\top = w_j^\top S$  are the left eigenvectors of  $A$ . Hence

$$v_j^\top b = g_j(b), \quad j = 1, \dots, r. \quad (3.14)$$

Therefore we obtain the PBH criterion in the special case of cyclic matrices.

**COROLLARY 3.3.** *Let  $A$  be cyclic. The following statements are equivalent. (i) The pair  $(A, b)$  is reachable. (ii) If*

$$v^\top (A - \lambda I) = 0, \quad v \in K^n, \quad v \neq 0,$$

then  $v^\top b \neq 0$ .

*Proof.* Because of (3.14) we can rewrite (ii) in the form

$$g_j(b) \neq 0, \quad j = 1, \dots, r. \quad (3.15)$$

From (3.9) follows that (3.15) is equivalent to  $\det R(A, b) \neq 0$ .  $\square$

We illustrate Theorem 3.2 with an example. Consider

$$A = \begin{bmatrix} -6 & -38 & 6 & -4 & 281 \\ -11 & -131 & 10 & -5 & 928 \\ 11 & -155 & -6 & -16 & 1191 \\ 1 & -170 & 1 & -11 & 1253 \\ -1 & -21 & 1 & -1 & 151 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}.$$

Then

$$R(A, b) = \begin{bmatrix} 3 & 6 & 5 & 2 & 6 \\ 0 & 7 & 0 & 0 & 8 \\ 4 & 9 & 6 & 3 & 4 \\ 0 & 7 & 1 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \det R(A, b) = 1,$$

and the pair  $(A, b)$  is reachable. Set  $S = R(A, b)^{-1}$ . Then

$$S = \begin{bmatrix} 3 & -16 & -2 & -3 & 133 \\ 0 & -1 & 0 & 0 & 8 \\ 0 & 2 & 0 & 1 & -21 \\ -4 & 19 & 3 & 2 & -150 \\ 0 & 1 & 0 & 0 & -7 \end{bmatrix},$$

and  $SAS^{-1} = F_a$ . We have

$$\det(zI - A) = a(z) = z^5 + 3z^4 - 6z^3 - 10z^2 - 21z - 9 = (z - 1)^3(z + 3)^2 = (a_1(z))^3(a_2(z))^2.$$

Hence  $\det R(A, x) = g_1(x)^3 g_2(x)^2$  with

$$g_1(x) = [1, 1, 1, 1, 1]Sx = -x_0 + 5x_1 + x_2 - 37x_4$$

and

$$g_2(x) = [1, -3, 9, -27, 81]Sx = 111x_0 - 427x_1 - 83x_2 - 48x_3 + 3403x_4.$$

We conclude with some remarks which place our study into a larger context. Matrices of the form  $\mathcal{K}_r(A, b) = [b, Ab, \dots, A^{r-1}b]$ ,  $1 \leq r \leq n$ , are known as *Krylov matrices* (see e.g. [9, p.646]). Thus  $R(A, b) = \mathcal{K}_n(A, b)$ . We refer to [8] for an investigation of numerical aspects of Krylov and reachability matrices. The concept of *Faddeev reachability matrix* was introduced in [5] and further elaborated in [10]. A “spectral factorization” of  $R(A, b)$  is due to [7] (see also [13]).



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