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NONNEGATIVE REALIZATION OF COMPLEX SPECTRA*

RICARDO L. SOTO†, MARIO SALAS†, AND CRISTINA MANZANEDA†

Abstract. We consider a list of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and give a simple and efficient sufficient condition for the existence of an $n \times n$ nonnegative matrix with spectrum Λ. Our result extends a previous one for a list of real numbers given in [*Linear Algebra Appl.*, 416:844–856, 2006]. In particular, we show how to construct a nonnegative matrix with prescribed complex eigenvalues and diagonal entries. As a by-product, we also construct Hermitian matrices with prescribed spectrum, whose entries have nonnegative real parts.

Key words. Nonnegative inverse eigenvalue problem.

AMS subject classifications. 15A18.

1. Introduction.

The *nonnegative inverse eigenvalue problem* (*NIEP*) is the problem of characterizing all possible spectra of entrywise nonnegative matrices. This problem remains unsolved. A complete solution is known only for $n \leq 4$ [5, 11]. Sufficient conditions for the existence of a nonnegative matrix A with prescribed complex spectrum have been obtained in [1, 7, 8]. A list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers is said to be *realizable* if Λ is the spectrum of an entrywise nonnegative matrix A. In this case, A is said to be a *realizing matrix.* From the Perron-Frobenius Theory for nonnegative matrices, we have that if Λ is the spectrum of an $n \times n$ nonnegative matrix A then $\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|$ is an eigenvalue of A. This eigenvalue is called the Perron eigenvalue of A and we shall always assume, in this paper, that $\rho(A) = \lambda_1$. A matrix $A = (a_{ij})_{i,j=1}^{n}$ is said to have *constant row sums* if all its rows sum up to the same constant, say α, i.e.

$$\sum_{j=1}^{n} a_{ij} = \alpha, \quad i = 1, \ldots, n.$$

The set of all matrices with constant row sums equal to α will be denoted by $\mathcal{CS}_\alpha$. It is clear that any matrix in $\mathcal{CS}_\alpha$ has eigenvector $\mathbf{e} = (1, 1, \ldots, 1)^T$ corresponding to the eigenvalue α. We shall denote by $\mathbf{e}_k$ the n-dimensional vector with one in the k-th position and zero elsewhere.

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In this paper, we consider a list of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and give a sufficient condition for the existence of an $n \times n$ nonnegative matrix with spectrum Λ. In particular, we show how to construct an $r \times r$ nonnegative matrix with prescribed complex eigenvalues and diagonal entries. As a by-product, we also construct an Hermitian matrix with prescribed spectrum. Since our interest is the nonnegative realization of spectra (NIEP), we require that the constructed Hermitian matrix has entries with nonnegative real parts. To prove our results, we shall need the following theorems of Brauer [3] and Rado [6] (see also [9]). The Rado Theorem is an extension of another well known result of Brauer [3] (see also [9]). It shows how to change r eigenvalues, $r \leq n$, of a matrix A of order n, via a rank-r perturbation, without changing any of the remaining $n - r$ eigenvalues.

THEOREM 1.1. [3] *If $A = (a_{ij}) \in \mathcal{CS}_{\lambda_1}$ is an $n \times n$ matrix with $a_{kj} = b_j$, $j = 2, 3, \ldots, n$, $k < j$, then A has eigenvalues $\lambda_1, a_{22} - b_2, a_{33} - b_3, \ldots, a_{nn} - b_n$.*

THEOREM 1.2. [6] *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and let $X = [\mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_r]$ be such that $rank(X) = r$ and $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$, $i = 1, 2, \ldots, r$, $r \leq n$. Let also C be an $r \times n$ arbitrary matrix. Then the matrix $A + XC$ has eigenvalues $\mu_1, \mu_2, \ldots, \mu_r, \lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n$, where $\mu_1, \mu_2, \ldots, \mu_r$ are the eigenvalues of the matrix $\Omega + CX$ with $\Omega = diag\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$.*

From Theorem 1.2, the following lemmas are immediate:

LEMMA 1.3. *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}, a, a$ for some $a \in \mathbb{R}$, and let $X = [\mathbf{x}_1 \mid \mathbf{x}_2]$ be such that $rank(X) = 2$ and $A\mathbf{x}_i = a\mathbf{x}_i$, $i = 1, 2$. Let also C be the $2 \times n$ matrix such that*

$$CX = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}.$$

Then $A + XC$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}, a + bi, a - bi$.

LEMMA 1.4. *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}, a + bi, a - bi$ for some $a, b \in \mathbb{R}$, and let $X = [\mathbf{x}_1 \mid \mathbf{x}_2]$ be such that $rank(X) = 2$ and $A\mathbf{x}_1 = (a + bi)\mathbf{x}_1$, $A\mathbf{x}_2 = (a - bi)\mathbf{x}_2$. Let also C be the $2 \times n$ matrix such that*

$$CX = \begin{bmatrix} -bi & 0 \\ 0 & bi \end{bmatrix}.$$

Then $A + XC$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n-2}, a, a$.

In Section 2, we extend to a list of complex numbers, a previous result for lists of real numbers given in [9, Theorem 8]. Theorem 8 in [9] considers a partition of a real list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ into r sublists $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_r$, with $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $\lambda_{11} = \lambda_1$, $\lambda_{k1} \geq 0$, $\lambda_{k1} \geq \lambda_{k2} \geq \cdots \geq \lambda_{kp_k}$. The novelty

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of our results relies on the fact that some of the sublists Λ_k can be of the form $\Lambda_k = \{a \pm bi, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, that is, the first element in Λ_k can be complex nonreal. This kind of partitions will allow us to propose a simple and efficient sufficient condition for the complex nonnegative inverse eigenvalue problem. As a by-product we construct, in Section 3, an Hermitian matrix $H = (h_{ij})$ with $Reh_{ij} \geq 0$ and prescribed spectrum. We also give, in Section 4, some examples to illustrate our results.

2. Main results. In this section, we extend Theorem 8 in [9]. Now, the list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ can be partitioned as $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_r$, where some of the sublists Λ_k can have as a first element a complex nonreal number. Let $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $k = 1, \ldots, r$, where $\lambda_{11} = \lambda_1$. With each Λ_k, we associate a corresponding list $\Gamma_k = \{\omega_k, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $0 \leq \omega_k \leq \lambda_1$, which is realizable by a $p_k \times p_k$ nonnegative matrix $A_k \in \mathcal{CS}_{\omega_k}$. Let $A = diag\{A_1, A_2, \ldots, A_r\}$ and $\Omega = diag\{\omega_1, \omega_2, \ldots, \omega_r\}$. Let $X = [\mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_r]$, where $\mathbf{x}_k$ is an n-dimensional vector with p_k ones from the position $\sum_{j=1}^{k-1} p_j + 1$ to the position $\sum_{j=1}^{k} p_j$ and zeros elsewhere, with the first summation being zero for $k = 1$. Observe that $\mathbf{x}_k$ is an eigenvector of A corresponding to the eigenvalue ω_k. For example, if the matrices A_1, A_2 and A_3 are of order 3, 2 and 1, respectively, then

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix}.$$

Let $C = [C_1 \mid C_2 \mid \cdots \mid C_r]$ be a nonnegative matrix, where C_k is the $r \times p_k$ matrix whose first column is $(c_{1k}, c_{2k}, \ldots, c_{rk})^T$ and whose other columns are all zero. Observe that C is an $r \times (\sum_{j=1}^{r} p_j)$ matrix. For example, if $r = 3$, $p_1 = 3$, $p_2 = 2$, $p_3 = 1$, as above, then

$$C = \begin{bmatrix} c_{11} & 0 & 0 & c_{12} & 0 & c_{13} \\ c_{21} & 0 & 0 & c_{22} & 0 & c_{23} \\ c_{31} & 0 & 0 & c_{32} & 0 & c_{33} \end{bmatrix}, \quad CX = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$\text{and} \quad XC = \begin{bmatrix} c_{11} & 0 & 0 & c_{12} & 0 & c_{13} \\ c_{11} & 0 & 0 & c_{12} & 0 & c_{13} \\ c_{11} & 0 & 0 & c_{12} & 0 & c_{13} \\ c_{21} & 0 & 0 & c_{22} & 0 & c_{23} \\ c_{21} & 0 & 0 & c_{22} & 0 & c_{23} \\ c_{31} & 0 & 0 & c_{32} & 0 & c_{33} \end{bmatrix}.$$

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THEOREM 2.1. *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a list of complex numbers such that* $\Lambda = \overline{\Lambda}$, $\lambda_1 \geq \max_i |\lambda_i|$ *(*$i = 2, \ldots, n$*), and* $\sum_{i=1}^{n} \lambda_i \geq 0$. *Suppose that*

i) there exits a partition $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_r$, *where* $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $\lambda_{11} = \lambda_1$, *such that* $\Gamma_k = \{\omega_k, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$ *is realizable by a nonnegative matrix in* $\mathcal{CS}_{\omega_k}$, *and*

ii) there exists an $r \times r$ *nonnegative matrix* $B \in \mathcal{CS}_{\lambda_1}$, *with eigenvalues* $\lambda_1, \lambda_{21}, \ldots, \lambda_{r1}$ *and diagonal entries* $\omega_1, \omega_2, \ldots, \omega_r$.

Then Λ *is realizable.*

Proof. The proof is similar to the proof of Theorem 8 in [9]. We set down it here for the sake of completeness: Let $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_r$ with Λ_k and Γ_k, $k = 1, \ldots, r$, as defined above. Let $A_k \in \mathcal{CS}_{\omega_k}$ be a $p_k \times p_k$ nonnegative matrix realizing Γ_k, $k = 1, 2, \ldots, r$. Then $A = diag\{A_1, A_2, \ldots, A_r\}$ is an $n \times n$ nonnegative block diagonal matrix realizing $\Gamma_1 \cup \cdots \cup \Gamma_r$. Let $\Omega = diag\{\omega_1, \omega_2, \ldots, \omega_r\}$ and $X = [\mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_r]$, as defined above. Clearly, $rank(X) = r$ and $A\mathbf{x}_i = \lambda_{i1}\mathbf{x}_i$, $i = 1, \ldots, r$, $r \leq n$. Let $C = [C_1 \mid C_2 \mid \cdots \mid C_r]$ as defined above. Then

$$CX = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rr} \end{bmatrix} \quad \text{and} \quad XC = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ C_{21} & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rr} \end{bmatrix},$$

where C_{ik} is the $p_i \times p_k$ matrix whose first column is $(c_{ik}, c_{ik}, \ldots, c_{ik})^T$ and whose other columns are all zero. Now, we choose C with $c_{11} = c_{22} = \cdots = c_{rr} = 0$, in such a way that the nonnegative matrix $B = \Omega + CX \in \mathcal{CS}_{\lambda_1}$ has eigenvalues $\lambda_1, \lambda_{21}, \ldots, \lambda_{r1}$ and diagonal entries $\omega_1, \omega_2, \ldots, \omega_r$. Then, for this choice of C, from Theorem 1.2, we have that $A + XC$ is nonnegative with spectrum Λ. $\square$

To make use use of Theorem 2.1, we need to know conditions under which there exists an $r \times r$ nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ and diagonal entries $\omega_1, \omega_2, \ldots, \omega_r$. For $r = 2$, it is necessary and sufficient that $0 \leq \omega_k \leq \lambda_1$, $k = 1, 2$, and $\omega_1 + \omega_2 = \lambda_1 + \lambda_2$. For $r = 3$, we slightly extend a result due to Perfect [6, Theorem 4]:

THEOREM 2.2. *The numbers* $\omega_1, \omega_2, \omega_3$ *and* $\lambda_1, \lambda_2, \lambda_3$ $(\lambda_1 \geq |\lambda_i|,\ i = 2, 3)$ *are, respectively, the diagonal entries and the eigenvalues of a* 3×3 *nonnegative matrix* $B \in \mathcal{CS}_{\lambda_1}$ *if and only if*

$$\begin{aligned} i)&\quad 0 \leq \omega_k \leq \lambda_1,\ \ k = 1, 2, 3, \\ ii)&\quad \omega_1 + \omega_2 + \omega_3 = \lambda_1 + \lambda_2 + \lambda_3, \\ iii)&\quad \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 \geq \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ iv)&\quad \max_k \omega_k \geq Re\lambda_2. \end{aligned}$$

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Proof. If λ_2, λ_3 are real, the result is [6, Theorem 4]. Let λ_2, λ_3 be complex ($\overline{\lambda_2} = \lambda_3$). The proof is like the proof in [6]. We include it here for the sake of completeness: The necessity of conditions $i)$, $ii)$, $iii)$ is clear. From a Brauer result [2, Theorem 11] about eigenvalues localization via ovals of Cassini, we have

$$|\lambda_3 - \omega_i|\,|\lambda_3 - \omega_j| \leq (\lambda_1 - \omega_i)(\lambda_1 - \omega_j), \quad i \neq j.$$

Then, for $i = 2$ and $j = 3$, it follows

$$(\lambda_1^2 - (Re\lambda_2)^2) - (\lambda_1 - \ Re\lambda_2)(\omega_2 + \omega_3) \geq 0$$

and

$$(\lambda_1 - Re\lambda_2)\,[\lambda_1 + Re\lambda_2 - (\omega_2 + \omega_3)] \geq 0.$$

Hence, $\omega_2 + \omega_3 \leq \lambda_1 + Re\lambda_2$, and from $ii)$, $\omega_1 \geq Re\lambda_2$. Thus, $iv)$ is established. Now, suppose that conditions $i) - iv)$ are satisfied. Then a straightforward calculation shows that the matrix $B \in \mathcal{CS}_{\lambda_1}$,

$$B = \begin{bmatrix} \omega_1 & 0 & \lambda_1 - \omega_1 \\ \lambda_1 - \omega_2 - p & \omega_2 & p \\ 0 & \lambda_1 - \omega_3 & \omega_3 \end{bmatrix}, \tag{2.1}$$

where

$$p = \frac{1}{\lambda_1 - \omega_3}[\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 - \lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3]$$

is nonnegative with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and diagonal entries $\omega_1, \omega_2, \omega_3$. □

The following corollary gives a simple sufficient condition to construct a 4×4 nonnegative matrix $M \in \mathcal{CS}_{\lambda_1}$ with prescribed complex spectrum.

Corollary 2.3. *Let $\Lambda = \{\lambda_1, \lambda_2, a + bi, a - bi\}$ be a list of complex numbers ($\lambda_1 \geq |a \pm bi|$). If there exist real numbers $\omega_1, \omega_2, \omega_3$, satisfying conditions*

$$\begin{aligned} &i) && 0 \leq \omega_k \leq \lambda_1, \ \ k = 1, 2, 3, \\ &ii) && \omega_1 + \omega_2 + \omega_3 = \lambda_1 + 2a, \\ &iii) && \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 \geq 2\lambda_1 a + a^2 + b^2, \\ &iv) && \max_k \omega_k \geq a, \ \ \max_k \omega_k \geq |\lambda_2|, \end{aligned} \tag{2.2}$$

then

$$M = \begin{bmatrix} \omega_1 & 0 & 0 & \lambda_1 - \omega_1 \\ \omega_1 - \lambda_2 & \lambda_2 & 0 & \lambda_1 - \omega_1 \\ \lambda_1 - \omega_2 - p & 0 & \omega_2 & p \\ 0 & 0 & \lambda_1 - \omega_3 & \omega_3 \end{bmatrix} \quad \textit{for} \ \ \lambda_2 \geq 0,$$

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and

$$M' = \begin{bmatrix} 0 & \omega_1 & 0 & \lambda_1-\omega_1 \\ -\lambda_2 & \omega_1+\lambda_2 & 0 & \lambda_1-\omega_1 \\ \lambda_1-\omega_2-p & 0 & \omega_2 & p \\ 0 & 0 & \lambda_1-\omega_3 & \omega_3 \end{bmatrix} \quad \textit{for } \lambda_2 \le 0,$$

where $p = \dfrac{1}{\lambda_1-\omega_3}[\omega_1\omega_2+\omega_1\omega_3+\omega_2\omega_3-(2\lambda_1 a+a^2+b^2)]$, *is a nonnegative matrix (in* $\mathcal{CS}_{\lambda_1}$*) with spectrum* Λ.

Proof. Since $\omega_1,\omega_2,\omega_3$ satisfy conditions (2.2), by Theorem 2.2, we may construct the matrix

$$B = \begin{bmatrix} \omega_1 & 0 & \lambda_1-\omega_1 \\ \lambda_1-\omega_2-p & \omega_2 & p \\ 0 & \lambda_1-\omega_3 & \omega_3 \end{bmatrix}.$$

Let $\Lambda = \Lambda_1\cup\Lambda_2\cup\Lambda_3$ with $\Lambda_1=\{\lambda_1,\lambda_2\}$, $\Lambda_2=\{a+bi\}$ and $\Lambda_3=\{a-bi\}$, and let $\Gamma_1=\{\omega_1,\lambda_2\}$, $\Gamma_2=\{\omega_2\}$ and $\Gamma_3=\{\omega_3\}$ be the corresponding associated realizable lists (as in Theorem 2.1). Let also

$$A = \begin{bmatrix} \omega_1 & 0 & 0 & 0 \\ \omega_1-\lambda_2 & \lambda_2 & 0 & 0 \\ 0 & 0 & \omega_2 & 0 \\ 0 & 0 & 0 & \omega_3 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\lambda_2 & \omega_1+\lambda_2 & 0 & 0 \\ 0 & 0 & \omega_2 & 0 \\ 0 & 0 & 0 & \omega_3 \end{bmatrix},$$

for $\lambda_2 \ge 0$ and $\lambda_2 \le 0$, respectively. Then, from the proof of Theorem 2.1, $M = A+XC$ or $M' = A'+XC$ is the desired matrix. □

To compute a possible $r\times r$ nonnegative matrix B, $r\ge 4$, with diagonal entries $\omega_1,\ldots,\omega_r$ and eigenvalues $\lambda_1,\ldots,\lambda_{r-2}$, $a+bi$, $a-bi$, $b>0$, where λ_i $(i=1,\ldots,r-2)$ are real, we use Theorem 1.1. It is clear from Theorem 1.1 that if

$$\begin{aligned} &(i) \quad 0\le\omega_k\le\lambda_1,\ \ k=1,\ldots r, \\ &(ii) \quad \omega_1+\omega_2+\cdots+\omega_r = \lambda_1+\lambda_2+\cdots+\lambda_r, \\ &(iii) \quad \omega_k \ge Re\lambda_k,\ \text{ and }\ \omega_1\ge Re\lambda_k,\ \ k=2,\ldots,r, \end{aligned} \tag{2.3}$$

then

$$M = \begin{bmatrix} \omega_1 & \omega_2-Re\lambda_2 & \cdots & \omega_r-Re\lambda_r \\ \omega_1-Re\lambda_2 & \omega_2 & \cdots & \omega_r-Re\lambda_r \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1-Re\lambda_r & \omega_2-Re\lambda_2 & \cdots & \omega_r \end{bmatrix} \in \mathcal{CS}_{\lambda_1}$$

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is a nonnegative matrix with diagonal entries $\omega_1, \ldots, \omega_r$ and eigenvalues $Re\lambda_1, \ldots,$ $Re\lambda_r$, more precisely, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{r-2}, a, a$. It is easy to see that the eigenvectors of M associated with $a = Re\lambda_{r-1} = Re\lambda_r$ are of the form

$$\mathbf{x}_1 = (S_1, S_1, \ldots, S_1, 1, 0)^T \quad \text{and} \quad \mathbf{x}_2 = (S_2, S_2, \ldots, S_2, 0, 1)^T,$$

where

$$S_1 = -\frac{\omega_{r-1} - a}{\sum_{i=2}^{r-2}(\omega_i - \lambda_i) + (\omega_1 - a)} \quad \text{and} \quad S_2 = -\frac{\omega_r - a}{\sum_{i=2}^{r-2}(\omega_i - \lambda_i) + (\omega_1 - a)}$$

with

$$\sum_{i=2}^{r-2}(\omega_i - \lambda_i) + (\omega_1 - a) > 0.$$

The eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent. Let C be the $2 \times r$ matrix

$$C = \begin{bmatrix} 0 & \cdots\cdots & 0 & 0 & -b \\ 0 & \cdots\cdots & 0 & b & 0 \end{bmatrix}.$$

Then

$$CX = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}.$$

Now, from Rado Theorem 1.2, we have that the matrix $M + XC$ has eigenvalues $\lambda_1, \ldots, \lambda_{r-2}, a + bi, a - bi$, and diagonal entries $\omega_1, \ldots, \omega_r$. It only remains to show that $M + XC$ is nonnegative. To do this, we observe that

$$XC = \begin{bmatrix} 0 & \cdots & 0 & S_2 b & -S_1 b \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & S_2 b & -S_1 b \\ 0 & \cdots & 0 & 0 & -b \\ 0 & \cdots & 0 & b & 0 \end{bmatrix}.$$

Then the $r-2$ first columns of the matrix $M + XC$ are the same as the $r-2$ first columns of M and they are nonnegative. The $(r-1)$-th and r-th columns of $M + XC$ are of the form

$$(w_{r-1} - a + S_2 b, \ldots, w_{r-1} - a + S_2 b, \omega_{r-1}, \omega_{r-1} - a + b)^T$$

and

$$(w_r - a - S_1 b, \ldots, w_r - a - S_1 b, \omega_r - a - b, \omega_r)^T,$$

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respectively. From (2.3), $\omega_{r-1} \geq Re\lambda_{r-1} = a$, and hence, $\omega_{r-1} - a + b \geq 0$. If

$$\omega_r - a - b \geq 0, \tag{2.4}$$

then $w_r - a - S_1 b \geq b - S_1 b \geq 0$ since $S_1 \leq 0$. Finally,

$$w_{r-1} - a + bS_2 = w_{r-1} - a + b\left[-\frac{\omega_r - a}{\sum_{i=2}^{r-2}(\omega_i - \lambda_i) + (\omega_1 - a)}\right] \geq 0$$

if and only if

$$(w_{r-1} - a)\left[\sum_{i=2}^{r-2}(\omega_i - \lambda_i) + (\omega_1 - a)\right] - b(\omega_r - a) \geq 0,$$

and from (2.3) (ii), this is equivalent to

$$(w_{r-1} - a)\left[(\lambda_1 - \omega_r) - (\omega_{r-1} - a)\right] - b(\omega_r - a) \geq 0.$$

That is,

$$(w_{r-1} - a)^2 - (\omega_{r-1} - a)(\lambda_1 - \omega_r) + b(\omega_r - a) \leq 0. \tag{2.5}$$

Thus, we have that if (2.4) and (2.5) hold, then $B = M + XC$ is nonnegative. We have proved the following result:

THEOREM 2.4. *Let $r \geq 4$. The numbers $\omega_1, \omega_2, \ldots, \omega_r$ and $\lambda_1, \lambda_2, \ldots, \lambda_{r-2}, a + bi, a - bi$, with $a, b, \lambda_i \in \mathbb{R}$, $i = 1, \ldots, r-2$, are the diagonal entries and the eigenvalues of an $r \times r$ nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$, respectively, if*

$$\begin{aligned}
&(i) && 0 \leq \omega_k \leq \lambda_1, \ \ k = 1, \ldots r,\\
&(ii) && \omega_1 + \omega_2 + \cdots + \omega_r = \lambda_1 + \lambda_2 + \cdots + \lambda_r,\\
&(iii) && \omega_1 \geq Re\lambda_k, \ \ \textit{and} \ \ \omega_k \geq Re\lambda_k, \ \ k = 2, \ldots, r,\\
&(iv) && \omega_r \geq a + b,\\
&(v) && (w_{r-1} - a)^2 - (\omega_{r-1} - a)(\lambda_1 - \omega_r) + b(\omega_r - a) \leq 0,\\
&(vi) && \sum_{i=2}^{r-2}(\omega_i - \lambda_i) + (\omega_1 - a) > 0.
\end{aligned}$$

3. Constructing Hermitian matrices with prescribed spectrum. Since we are interested in the nonnegative realization of spectra, in this section, we show how to construct an $n \times n$ Hermitian matrix with prescribed spectrum, whose entries have nonnegative real part. In particular, we generalize a result in [10] for symmetric nonnegative matrices. The technique is the same as in Section 2. We start with the

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following result, which is an Hermitian version of the Rado Theorem 1.2 . The proof is similar to the proof of Theorem 2.6 in [10], which gives a symmetric version of Rado Theorem. We include it here for the sake of completeness.

THEOREM 3.1. *Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$, and, for some $r \leq n$, let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ be an orthonormal set of eigenvectors of A associated with $\lambda_1, \ldots, \lambda_r$, $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$. Let X be the $n \times r$ matrix with i-th column $\mathbf{x}_i$, let $\Omega = diag\{\lambda_1, \ldots, \lambda_r\}$, and let C be any $r \times r$ Hermitian matrix. Then the Hermitian matrix $A + XCX^T$ has eigenvalues $\mu_1, \mu_2, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_n$, where $\mu_1, \mu_2, \ldots, \mu_r$ are the eigenvalues of the matrix $\Omega + C$.*

Proof. Since the columns of X are an orthonormal set, we may complete X to an orthogonal matrix $W = [\,X \ Y\,]$, i.e., $X^TX = I_r$, $Y^TY = I_{n-r}$, $X^TY = 0$, $Y^TX = 0$. Then

$$W^{-1}AW = \begin{bmatrix} X^T \\ Y^T \end{bmatrix} A \begin{bmatrix} X & Y \end{bmatrix} = \begin{bmatrix} \Omega & X^TAY \\ 0 & Y^TAY \end{bmatrix}$$

and

$$W^{-1}(XCX^T)W = \begin{bmatrix} I_r \\ 0 \end{bmatrix} C \begin{bmatrix} I_r & 0 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore,

$$W^{-1}(A + XCX^T)W = \begin{bmatrix} \Omega + C & X^TAY \\ 0 & Y^TAY \end{bmatrix},$$

and $A + XCX^T$ is an Hermitian matrix with eigenvalues $\mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_n$. □

It is clear that Theorem 3.1 also holds for A Hermitian.

THEOREM 3.2. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of real numbers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and, for some $t \leq n$, let $\omega_1, \omega_2, \ldots, \omega_t$ be real numbers satisfying $0 \leq \omega_k \leq \lambda_1$, $k = 1, \ldots t$. Suppose there exist*

$i)$ a partition $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_t$, with $\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$, $\lambda_{11} = \lambda_1$, $\lambda_{k1} \geq 0$, $\lambda_{k1} \geq \lambda_{k2} \geq \cdots \geq \lambda_{kp_k}$, such that for each $k = 1, \ldots, t$, the list $\Gamma_k = \{\omega_k, \lambda_{k2}, \ldots, \lambda_{kp_k}\}$ is realizable by a symmetric nonnegative matrix, and

$ii)$ a $t \times t$ Hermitian matrix, whose entries have nonnegative real parts, with eigenvalues $\lambda_{11}, \lambda_{21}, \ldots, \lambda_{t1}$ and diagonal entries $\omega_1, \omega_2, \ldots, \omega_t$.

Then Λ is the spectrum of some $n \times n$ Hermitian matrix, whose entries have nonnegative real part.

Proof. For each $k = 1, \ldots, t$ denote by A_k the symmetric nonnegative $p_k \times p_k$ matrix realizing Γ_k. Then the $n \times n$ matrix $A = diag\{A_1, A_2, \ldots, A_t\}$ is symmetric

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nonnegative with spectrum $\Gamma_1 \cup \cdots \cup \Gamma_t$. Let $\{\mathbf{x}_1, \ldots \mathbf{x}_t\}$ be an orthonormal set of eigenvectors of A associated, respectively, with $\omega_1, \ldots, \omega_t$. If $\Omega = diag\{\omega_1, \ldots, \omega_t\}$, then the $n \times t$ matrix X with i-th column $\mathbf{x}_i$ satisfies $AX = X\Omega$. Moreover, X is entrywise nonnegative since the nonzero entries of each $\mathbf{x}_i$ constitute a Perron vector of A_i. Now, from ii) let B the $t \times t$ Hermitian matrix with entries with nonnegative real parts, spectrum $\{\lambda_{11}, \lambda_{21}, \ldots, \lambda_{t1}\}$ and diagonal entries $\omega_1, \ldots, \omega_t$. If we set $C = B - \Omega$, then the matrix C is Hermitian with its entries having nonnegative real parts, and $\Omega + C$ has eigenvalues $\lambda_{11}, \lambda_{21}, \ldots, \lambda_{t1}$. Therefore, by Theorem 3.1, the Hermitian matrix $A + XCX^T$ has spectrum Λ. $\square$

Theorem 3.2 not only ensures the existence of an Hermitian realizing matrix, but it also allows to construct the realizing matrix. Of course, the key step is knowing under which conditions does there exist a $t \times t$ Hermitian matrix B with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_t$ and diagonal entries $\omega_1, \omega_2, \ldots, \omega_t$, whose entries have nonnegative real parts. Schur proved that the vector of eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ of an $n \times n$ Hermitian matrix A majorizes the vector of diagonal entries $\omega_1 \geq \cdots \geq \omega_n$ of A.

For $t = 2$, the Schur conditions become

$$\lambda_1 \geq \omega_1 \quad \text{and} \quad \lambda_1 + \lambda_2 = \omega_1 + \omega_2,$$

and they are also sufficient for the existence of a 2×2 Hermitian matrix

$$B = \begin{bmatrix} \omega_1 & a + bi \\ a - bi & \omega_2 \end{bmatrix},$$

$a, b \in \mathbb{R}$, with eigenvalues $\lambda_1 \geq \lambda_2$ and diagonal entries $\omega_1 \geq \omega_2 \geq 0$, namely,

$$B = \begin{bmatrix} \omega_1 & a + \sqrt{(\lambda_1 - \omega_1)(\lambda_1 - \omega_2) - a^2}\, i \\ a - \sqrt{(\lambda_1 - \omega_1)(\lambda_1 - \omega_2) - a^2}\, i & \omega_2 \end{bmatrix}.$$

For general t, we have an algorithmic procedure to construct the required $t \times t$ Hermitian matrix B. A converse to the result of Schur was proved by Horn [4]:

Theorem 3.3. [4] *Let $n \geq 1$, and let $\lambda_1 \geq \cdots \geq \lambda_n$ and $\omega_1 \geq \cdots \geq \omega_n$ be given real numbers. If the vector $\lambda = [\lambda_1, \ldots, \lambda_n]$ majorizes the vector $\omega = [\omega_1, \ldots, \omega_n]$, then there exists an $n \times n$ real symmetric matrix $A = [a_{ij}]$ with eigenvalues $\{\lambda_i\}$ and diagonal entries $a_{ii} = \omega_i$, $i = 1, 2, \ldots, n$.*

The steps to construct the required $t \times t$ Hermitian matrix B with eigenvalues $\lambda_1, \ldots, \lambda_t$ and diagonal entries $\omega_1, \ldots, \omega_t$ are as follows:

i) We construct a $(t-1) \times (t-1)$ symmetric nonnegative matrix M with eigenvalues $\gamma_1, \ldots, \gamma_{t-1}$ and diagonal entries $\omega_2, \ldots, \omega_t$, where

$$\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \cdots \geq \gamma_{t-1} \geq \lambda_t,$$

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and the vector $\gamma = [\gamma_1, \ldots, \gamma_{t-1}]$ majorizes the vector $\omega = [\omega_2, \ldots, \omega_t]$. The existence of the γ_i is ensured by Lemma 4.3.28 in [4]. The construction of M is based on the results in [10, Section 3].

ii) Let $D = diag\{\gamma_1, \ldots, \gamma_{t-1}\}$, and let Q be the real orthogonal matrix such that $M = QDQ^T$. Then we construct an Hermitian matrix

$$H = \begin{bmatrix} D & \mathbf{z} \\ \mathbf{z}^* & \alpha \end{bmatrix}, \quad \alpha \in \mathbb{R},$$

with eigenvalues $\lambda_1, \ldots, \lambda_t$. The entries of the complex vectors $\mathbf{z}$ and $\mathbf{z}^*$ come from the use of Theorem 4.3.10 in [4]. In fact, this result establishes that if

$$\lambda_1 \geq \gamma_1 \geq \lambda_2 \geq \gamma_2 \geq \cdots \geq \gamma_{t-1} \geq \lambda_t$$

and $D = diag\{\gamma_1, \ldots, \gamma_{t-1}\}$, then there exist a scalar $\alpha \in \mathbb{R}$ and a vector $\mathbf{y} \in \mathbb{R}^{t-1}$ such that $\{\lambda_1, \ldots, \lambda_t\}$ is the spectrum of the real symmetric matrix

$$A = \begin{bmatrix} D & \mathbf{y} \\ \mathbf{y}^T & \alpha \end{bmatrix}.$$

The proof can be easily modified for $\mathbf{y} = \mathbf{z} = (z_1, \ldots, z_{t-1}) \in \mathbb{C}^{t-1}$. In this case, the $t-1$ real numbers $|z_i|$ can be found from

$$\left[(\lambda_k - \alpha) - \sum_{i=1}^{t-1} |z_i|^2 \frac{1}{\lambda_k - \gamma_i}\right] \prod_{i=1}^{t-1} (\lambda_k - \gamma_i) = 0, \quad k = 1, \ldots, t,$$

and then we may choose z_i with nonnegative real part or z_i being pure imaginary.

iii) Now we set

$$B = \begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} H \begin{bmatrix} Q^T & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} M & Q\mathbf{z} \\ (Q\mathbf{z})^* & \alpha \end{bmatrix},$$

which is an Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_t$ and diagonal entries $\omega_1, \ldots, \omega_t$.

4. Examples.

Example 4.1. Let $\Lambda = \{7, 1, -2, -2, -2 + 4i, -2 - 4i\}$. Consider the partition

$$\Lambda_1 = \{7, 1, -2, -2\}, \quad \Lambda_2 = \{-2 + 4i\}, \quad \Lambda_3 = \{-2 - 4i\}$$

with

$$\Gamma_1 = \{3, 1, -2, -2\}, \quad \Gamma_2 = \{0\}, \quad \Gamma_3 = \{0\}.$$

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Then we look for a nonnegative matrix $B \in \mathcal{CS}_7$ with eigenvalues $7, -2+4i, -2-4i$ and diagonal entries $3, 0, 0$. From (2.1),

$$B = \begin{bmatrix} 3 & 0 & 4 \\ \frac{41}{7} & 0 & \frac{8}{7} \\ 0 & 7 & 0 \end{bmatrix}.$$

It is clear that

$$A_1 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

has the spectrum Γ_1. Then from Theorem 2.1, we have that

$$A = \begin{bmatrix} A_1 & & \\ & 0 & \\ & & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 4 \\ \frac{41}{7} & 0 & 0 & 0 & 0 & \frac{8}{7} \\ 0 & 0 & 0 & 0 & 7 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 4 \\ 2 & 0 & 0 & 1 & 0 & 4 \\ 0 & 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 2 & 0 & 0 & 4 \\ \frac{41}{7} & 0 & 0 & 0 & 0 & \frac{8}{7} \\ 0 & 0 & 0 & 0 & 7 & 0 \end{bmatrix}$$

has the spectrum Λ.

Example 4.2. Let $\Lambda = \{7, -3, 1+3i, 1-3i, 1+i, 1-i\}$. Consider the partition

$$\Lambda_1 = \{7, -3\}, \;\; \Lambda_2 = \{1+3i, 1+i, 1-i\}, \;\; \Lambda_3 = \{1-3i\},$$

$$\Gamma_1 = \{3, -3\}, \;\; \Gamma_2 = \{3, 1+i, 1-i\}, \;\; \Gamma_3 = \{3\},$$

$$\lambda_i : \; 7, 1+3i, 1-3i, \;\; \omega_i : \; 3, 3, 3, \;\; p = \frac{3}{4}.$$

Then

$$B = \begin{bmatrix} 3 & 0 & 4 \\ \frac{13}{4} & 3 & \frac{3}{4} \\ 0 & 4 & 3 \end{bmatrix},$$

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and

$$A+XC=\begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 4 \\ 3 & 0 & 0 & 0 & 0 & 4 \\ \frac{13}{4} & 0 & 2 & 0 & 1 & \frac{3}{4} \\ \frac{13}{4} & 0 & 2 & 1 & 0 & \frac{3}{4} \\ \frac{13}{4} & 0 & 0 & 1 & 2 & \frac{3}{4} \\ 0 & 0 & 4 & 0 & 0 & 3 \end{bmatrix}$$

has the spectrum Λ.

EXAMPLE 4.3. Let $\Lambda = \{7, 5, 1, -3, -4, -6\}$. We shall construct an Hermitian matrix, whose entries have nonnegative real parts, with spectrum Λ. Consider the partition $\Lambda_1 = \{7, -6\}$, $\Lambda_2 = \{5, -4\}$, $\Lambda_1 = \{1, -3\}$, with the symmetrically realizable list $\Gamma_1 = \{6, -6\}$, $\Gamma_2 = \{4, -4\}$, $\Gamma_3 = \{3, -3\}$. Then we look for an Hermitian matrix with eigenvalues $7, 5, 1$ and diagonal entries $6, 4, 3$. We choose $\gamma_1 = 6$ and $\gamma_2 = 1$, and construct the matrix

$$M=\begin{bmatrix} 4 & \sqrt{6} \\ \sqrt{6} & 3 \end{bmatrix}$$

with eigenvalues 6 and 1. Let

$$D=\begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q=\begin{bmatrix} \sqrt{\frac{3}{5}} & -\sqrt{\frac{2}{5}} \\ \frac{\sqrt{10}}{5} & \frac{\sqrt{15}}{5} \end{bmatrix}.$$

Then $Q^T M Q = D$. We construct an Hermitian matrix

$$H=\begin{bmatrix} 6 & 0 & z_1 \\ 0 & 1 & z_2 \\ z_1^* & z_2^* & 6 \end{bmatrix}$$

with eigenvalues $7, 5, 1$. Then we solve the system

$$\begin{aligned} |z_1|^2 + |z_2|^2 &= 1 \\ |z_1|^2 - 6\,|z_2|^2 &= 1, \end{aligned}$$

which allows us to choose $z_1 = i$ and $z_2 = 0$. Thus,

$$H=\begin{bmatrix} 6 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 6 \end{bmatrix}$$

and the required Hermitian matrix is

$$B=\begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} H \begin{bmatrix} Q^T & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} C & Q\mathbf{z} \\ (Q\mathbf{z})^* & \alpha \end{bmatrix} = \begin{bmatrix} 4 & \sqrt{6} & \sqrt{\frac{3}{5}}i \\ \sqrt{6} & 3 & \frac{\sqrt{10}}{5}i \\ -\sqrt{\frac{3}{5}}i & -\frac{\sqrt{10}}{5}i & 6 \end{bmatrix}.$$

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Now, we apply Theorem 3.1 to obtain the Hermitian matrix A with spectrum Λ

$$A = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 6 & 0 \end{bmatrix} + XCX^T$$

$$= \begin{bmatrix} 0 & 4 & \frac{1}{2}\sqrt{6} & \frac{1}{2}\sqrt{6} & \frac{\sqrt{15}}{10}i & \frac{\sqrt{15}}{10}i \\ 4 & 0 & \frac{1}{2}\sqrt{6} & \frac{1}{2}\sqrt{6} & \frac{\sqrt{15}}{10}i & \frac{\sqrt{15}}{10}i \\ \frac{1}{2}\sqrt{6} & \frac{1}{2}\sqrt{6} & 0 & 3 & \frac{\sqrt{10}}{10}i & \frac{\sqrt{10}}{10}i \\ \frac{1}{2}\sqrt{6} & \frac{1}{2}\sqrt{6} & 3 & 0 & \frac{\sqrt{10}}{10}i & \frac{\sqrt{10}}{10}i \\ -\frac{\sqrt{15}}{10}i & -\frac{\sqrt{15}}{10}i & -\frac{\sqrt{10}}{10}i & -\frac{\sqrt{10}}{10}i & 0 & 6 \\ -\frac{\sqrt{15}}{10}i & -\frac{\sqrt{15}}{10}i & -\frac{\sqrt{10}}{10}i & -\frac{\sqrt{10}}{10}i & 6 & 0 \end{bmatrix},$$

where

$$X^T = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & \sqrt{6} & \sqrt{\frac{3}{5}}i \\ \sqrt{6} & 0 & \frac{\sqrt{10}}{5}i \\ -\sqrt{\frac{3}{5}}i & -\frac{\sqrt{10}}{5}i & 0 \end{bmatrix}.$$

REFERENCES

[1] A. Borobia, J. Moro, and R.L. Soto. Negativity compensation in the nonnegative inverse eigenvalue problem. *Linear Algebra Appl.*, 393:73–89, 2004.

[2] A. Brauer. Limits for the characteristic roots of a matrix II: Applications to stochastic matrices. *Duke Math. J.*, 14:21–26, 1947.

[3] A. Brauer. Limits for the characteristic roots of a matrix IV: Applications to stochastic matrices. *Duke Math. J.*, 19:75–91, 1952.

[4] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, 1991.

[5] M.E. Meehan. *Some Results on Matrix Spectra*. Ph.D. Thesis, National University of Ireland, Dublin, 1998.

[6] H. Perfect. Methods of constructing certain stochastic matrices II. *Duke Math. J.*, 22:305–311, 1955.

[7] N. Radwan. An inverse eigenvalue problem for symmetric and normal matrices. *Linear Algebra Appl.*, 248:101–109, 1996.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 20, pp. 595-609, September 2010

ELA
http://math.technion.ac.il/iic/ela

[8] O. Rojo and R.L. Soto. Existence and construction of nonnegative matrices with complex spectrum. *Linear Algebra Appl.*, 368:53–69, 2003.

[9] R.L. Soto and O. Rojo. Applications of a Brauer theorem in the nonnegative inverse eigenvalue problem. *Linear Algebra Appl.*, 416:844–856, 2006.

[10] R.L. Soto, O. Rojo, J. Moro, and A. Borobia. Symmetric nonnegative realization of spectra. *Electron. J. Linear Algebra*, 16:1–18, 2007.

[11] J. Torre-Mayo, M.R. Abril-Raymundo, E. Alarcia-Estévez, C. Marijuán, and M. Pisonero. The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial. EBL digraphs. *Linear Algebra Appl.*, 426:729–773, 2007.