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Zhiping Xiong

Yingying Qin

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INVARIANCE PROPERTIES OF AN OPERATOR PRODUCT INVOLVING GENERALIZED INVERSES*

ZHIPING XIONG[†] AND YINGYING QIN[‡]

Abstract. Given bounded linear operators T_1, T_2 and T_3 , this paper investigates certain invariance properties of the operator product $T_1 X T_3$ with respect to the choice of bounded linear operator X , where X is a generalized inverse of T_2 . Different types of generalized inverses are taken into account.

Key words. Bounded linear operators, Generalized inverse, Moore-Penrose inverse, Operator product, Invariance property.

AMS subject classifications. 15A09, 15A24, 47A05.

1. Introduction. Throughout this paper, \mathbb{H}, \mathbb{K} , and \mathbb{L} denote arbitrary Hilbert spaces. We use $L(\mathbb{H}, \mathbb{K})$ to denote the set of all bounded linear operators from \mathbb{H} to \mathbb{K} . Also, $L(\mathbb{H}) = L(\mathbb{H}, \mathbb{H})$. I denotes the identity operator on Hilbert spaces and O is the zero operator on Hilbert spaces. For $T \in L(\mathbb{H}, \mathbb{K})$, the symbols T^* , $R(T)$, and $N(T)$ will stand for the adjoint operator, the range, and the kernel of T , respectively.

Let $T \in L(\mathbb{H}, \mathbb{K})$. If there exists an operator $X \in L(\mathbb{K}, \mathbb{H})$ satisfying the following four operator equations:

$$(1) T X T = T, \quad (2) X T X = X, \quad (3) (T X)^* = T X, \quad (4) (X T)^* = X T,$$

then X is called a Moore-Penrose inverse of T and denoted by T^\dagger . As we know, T has a Moore-Penrose inverse if and only if $R(T)$ is closed and the Moore-Penrose inverse of T is unique (see, for example, [5, 7, 14, 17, 18, 19, 20, 21, 22, 23]). For a subset $\eta \subseteq \{1, 2, 3, 4\}$, the set of operators satisfying the equations contained in η is denoted by $T\eta$. An operator from $T\eta$ is called an η -inverse of T . For example, an operator X of the set $T\{1\}$ is called a $\{1\}$ -inverse of T and denoted by $T^{(1)}$ or T^- . One usually denotes any $\{1, 3\}$ -inverse of T as $T^{(1,3)}$ and any $\{1, 4\}$ -inverse of T is denoted by $T^{(1,4)}$. The unique $\{1, 2, 3, 4\}$ -inverse of T is the Moore-Penrose inverse of T . We

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[†]Department of Mathematics, Wuyi University, Jiangmen 529020, Guangdong, P.R. China (xzpwhere@163.com). The work was supported by the start-up fund of Wuyi University, Jiangmen 529020, Guangdong Province.

[‡]Department of Mathematics, Wuyi University, Jiangmen 529020, Guangdong, P.R. China (qinyy04@163.com).

refer the reader to [5, 6, 9, 10, 11, 15, 16, 23, 24] for basic results on the generalized inverses of operators.

Invariance properties of operator product involving generalized inverses are fundamental in the theory of operators. They have attracted considerable attention and many interesting results have been obtained (see, for example, [1, 2, 3, 4, 12, 13]). In this paper, given bounded linear operators $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$ and $T_3 \in L(\mathbb{H}, \mathbb{K})$, we investigate properties of the operator product $T_1 T_2^- T_3$ for various types of generalized inverses T_2^- of T_2 , where our interest is focused on invariance properties with respect to the choice of T_2^- concerning the value and range of $T_1 T_2^- T_3$.

We first mention the following three results, which will be used in this paper.

LEMMA 1.1. [9]. *Let $T \in L(\mathbb{H}, \mathbb{K})$ have a closed range. Then*

$$T\{1\} = \{T^\dagger + Y - T^\dagger T Y T T^\dagger : Y \in L(\mathbb{K}, \mathbb{H})\}.$$

LEMMA 1.2. [8, 9]. *Let $T \in L(\mathbb{H}, \mathbb{K})$ have a closed range and $X \in L(\mathbb{K}, \mathbb{H})$. Then the following statements are equivalent:*

- (1) $TXT = T$ and $(TX)^* = TX$;
- (2) *there exists some $Y \in L(\mathbb{K}, \mathbb{H})$ such that $X = T^\dagger + (I - T^\dagger T)Y$.*

LEMMA 1.3. [8, 9]. *Let $T \in L(\mathbb{H}, \mathbb{K})$ have a closed range and $X \in L(\mathbb{K}, \mathbb{H})$. Then the following statements are equivalent:*

- (1) $TXT = T$ and $(XT)^* = XT$;
- (2) *there exists some $Y \in L(\mathbb{K}, \mathbb{H})$ such that $X = T^\dagger + Y(I - TT^\dagger)$.*

2. Invariance properties of operator product $T_1 T_2^{(1)} T_3$. Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$ and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1 , T_2 and T_3 have closed ranges. In this section, we will study several invariance properties of the operator product $T_1 T_2^{(1)} T_3$ with respect to the choice of $T_2^{(1)} \in T_2\{1\}$. The main result is the following theorem.

THEOREM 2.1. *Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$ and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1 , T_2 , T_3 have closed ranges. Suppose that T_1 , T_2 , and T_3 are not zero operators on Hilbert spaces. Then the following statements are equivalent:*

- (1) *The operator product $T_1 T_2^{(1)} T_3$ does not depend on the choice of $T_2^{(1)} \in T_2\{1\}$;*

(2) $R(T_1^*) \subseteq R(T_2^*)$ and $R(T_3) \subseteq R(T_2)$.

Proof. According to the technique of block operator matrices in [7], we know that the operator T_2 has the following matrix form with respect to the orthogonal sum of subspaces:

$$(2.1) \quad T_2 = \begin{pmatrix} T_2^{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix},$$

where T_2^{11} is invertible in $L(R(T_2^*), R(T_2))$, and

$$(2.2) \quad T_2^\dagger = \begin{pmatrix} (T_2^{11})^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}.$$

Also we have that the operator T_1 has the following form:

$$(2.3) \quad T_1 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix}$$

and

$$T_1^* = \begin{pmatrix} (T_1^{11})^* & O \\ (T_1^{12})^* & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}$$

and

$$T_1 T_1^* = \begin{pmatrix} D & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix},$$

where $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$ is positive and invertible in $L(R(T_1))$. In particular

$$(2.4) \quad T_1^\dagger = T_1^*(T_1 T_1^*)^\dagger = \begin{pmatrix} (T_1^{11})^* D^{-1} & O \\ (T_1^{12})^* D^{-1} & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}.$$

Furthermore, we obtain that the operator T_3 has the following matrix form with respect to the orthogonal sum of subspaces:

$$(2.5) \quad T_3 = \begin{pmatrix} T_3^{11} & O \\ T_3^{21} & O \end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}$$

and

$$T_3^* = \begin{pmatrix} (T_3^{11})^* & (T_3^{21})^* \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix}$$

and

$$T_3^* T_3 = \begin{pmatrix} S & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix},$$

where $S = (T_3^{11})^* T_3^{11} + (T_3^{21})^* T_3^{21}$ is positive and invertible in $L(R(T_3^*))$. Then

$$(2.6) \quad T_3^\dagger = (T_3^* T_3)^\dagger T_3^* = \begin{pmatrix} S^{-1}(T_3^{11})^* & S^{-1}(T_3^{21})^* \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix}.$$

Next, we will prove the facts that (1) and (2) in Theorem 2.1 are equivalent.

(2) \Rightarrow (1): The inclusion $R(T_1^*) \subseteq R(T_2^*)$ is equivalent to $T_1 T_2^\dagger T_2 = T_1$. Then from (2.1)-(2.6), we have

$$(2.7) \quad T_1 T_2^\dagger T_2 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} \begin{pmatrix} (T_2^{11})^{-1} & O \\ O & O \end{pmatrix} \begin{pmatrix} T_2^{11} & O \\ O & O \end{pmatrix} = \begin{pmatrix} T_1^{11} & O \\ O & O \end{pmatrix}.$$

Hence, according to (2.3) and (2.7), the equality $T_1 T_2^\dagger T_2 = T_1$ is equivalent to $T_1^{12} = O$, that is

$$(2.8) \quad R(T_1^*) \subseteq R(T_2^*) \Leftrightarrow T_1^{12} = O.$$

In the same manner, we can prove that the inclusion $R(T_3) \subseteq R(T_2)$ is equivalent to $T_2 T_3^\dagger T_3 = T_3$ and

$$(2.9) \quad R(T_3) \subseteq R(T_2) \Leftrightarrow T_3^{21} = O.$$

On the other hand, from Lemma 1.1 it follows that arbitrary $T_2^{(1)} \in T_2\{1\}$ has the form

$$(2.10) \quad T_2^{(1)} = \begin{pmatrix} (T_2^{11})^{-1} & U \\ V & W \end{pmatrix},$$

where U, V and W are bounded linear operators on appropriate spaces. Hence, from (2.2), (2.3), (2.5), (2.8), (2.9), (2.10), we have that

$$(2.11) \quad T_1 T_2^{(1)} T_3 = \begin{pmatrix} T_1^{11} (T_2^{11})^{-1} T_3^{11} & O \\ O & O \end{pmatrix} = T_1 T_2^\dagger T_3.$$

Combining (2.10) with (2.11), we have the result (2) \Rightarrow (1).

(1) \Rightarrow (2): Since the Moore-Penrose inverse of a bounded linear operator is unique and belongs to the set of $\{1\}$ -inverse, it is clear that $T_1 T_2^{(1)} T_3$ doesn't depend on the choice of $T_2^{(1)} \in T_2\{1\}$ if and only if the equality $T_1 T_2^{(1)} T_3 = T_1 T_2^\dagger T_3$ holds for every $T_2^{(1)} \in T_2\{1\}$. By Lemma 1.1, it follows that both

$$(2.12) \quad M = \begin{pmatrix} (T_2^{11})^{-1} & (T_1^{11})^* \\ O & (T_1^{12})^* \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} (T_2^{11})^{-1} & O \\ (T_3^{11})^* & (T_3^{21})^* \end{pmatrix}$$

are $\{1\}$ -inverses of T_2 . Since

$$(2.13) \quad T_1 M T_3 = T_1 N T_3 = T_1 T_2^\dagger T_3 = \begin{pmatrix} T_1^{11} (T_2^{11})^{-1} T_3^{11} & O \\ O & O \end{pmatrix},$$

from (2.2), (2.3), (2.5), (2.12), (2.13), we get

$$(2.14) \quad T_1^{11}(T_1^{11})^*T_3^{21} + T_1^{12}(T_1^{12})^*T_3^{21} = O$$

and

$$(2.15) \quad T_1^{12}(T_3^{11})^*T_3^{11} + T_1^{12}(T_3^{21})^*(T_3^{21}) = O.$$

Combining (2.14), (2.15) with the equalities (2.3) and (2.5), we have

$$(2.16) \quad DT_3^{21} = O \quad \text{and} \quad T_1^{12}S = O.$$

Since D and S are invertible, from (2.16) we obtain

$$T_3^{21} = O \quad \text{and} \quad T_1^{12} = O,$$

which are respectively equivalent to $R(T_3) \subseteq R(T_2)$ and $R(T_1^*) \subseteq R(T_2^*)$. \square

COROLLARY 2.2. *Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2 , and T_3 are not zero operators on Hilbert spaces. Then the identity $T_1T_2^{(1)}T_3 = O$ holds for every $T_2^{(1)} \in T_2\{1\}$ if and only if $R(T_1^*) \subseteq R(T_2^*)$, $R(T_3) \subseteq R(T_2)$, and $R(T_3) \subseteq N(T_1T_2^\dagger)$.*

COROLLARY 2.3. *Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. If $R(T_1^*) \subseteq R(T_2^*)$ and $R(T_3) \subseteq R(T_2)$, then $R(T_1T_2^{(1)}T_3)$ is the same for every $T_2^{(1)} \in T_2\{1\}$.*

3. Invariance properties of products $T_1T_2^{(1,3)}T_3$ and $T_1T_2^{(1,4)}T_3$. Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$. In this section, we will investigate the invariance properties of the operator products $T_1T_2^{(1,3)}T_3$ and $T_1T_2^{(1,4)}T_3$ with respect to any $T_2^{(1,3)} \in T_2\{1, 3\}$ and $T_2^{(1,4)} \in T_2\{1, 4\}$.

THEOREM 3.1. *Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2 , and T_3 are not zero operators on Hilbert spaces. Then the following statements are equivalent:*

- (1) *The equality $T_1T_2^{(1,3)}T_3 = T_1T_2^\dagger T_3$ holds for every $T_2^{(1,3)} \in T_2\{1, 3\}$;*
- (2) $R(T_1^*) \subseteq R(T_2^*)$.

Proof. By Lemma 1.2, we have that arbitrary $T_2^{(1,3)} \in T_2\{1, 3\}$ has the form

$$(3.1) \quad T_2^{(1,3)} = \begin{pmatrix} (T_2^{11})^{-1} & O \\ W_{21} & W_{22} \end{pmatrix},$$

where W_{21} and W_{22} are bounded linear operators on appropriate subspaces. Then from the equalities (2.2), (2.3), (2.5) and (3.1), we know that for any $T_2^{(1,3)} \in T_2\{1, 3\}$

$$T_1T_2^{(1,3)}T_3 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} \begin{pmatrix} (T_2^{11})^{-1} & O \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} T_3^{11} & O \\ T_3^{21} & O \end{pmatrix}$$

$$= \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}W_{21}T_3^{11} + T_1^{12}W_{22}T_3^{21} & O \\ O & O \end{pmatrix}$$

and

$$(3.2) \quad T_1T_2^\dagger T_3 = \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} & O \\ O & O \end{pmatrix}.$$

We now prove that (1) and (2) in Theorem 3.1 are equivalent.

(2) \Rightarrow (1): Since

$$R(T_1^*) \subseteq R(T_2^*) \Leftrightarrow T_1^{12} = O,$$

we have that for arbitrary $T_2^{(1,3)} \in T_2\{1, 3\}$,

$$\begin{aligned} T_1T_2^{(1,3)}T_3 &= \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}W_{21}T_3^{11} + T_1^{12}W_{22}T_3^{21} & O \\ O & O \end{pmatrix} \\ &= \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} & O \\ O & O \end{pmatrix} \\ &= T_1T_2^\dagger T_3. \end{aligned}$$

(1) \Rightarrow (2): From Lemma 1.2, it follows that

$$M_1 = \begin{pmatrix} (T_2^{11})^{-1} & O \\ (T_3^{11})^* & (T_3^{21})^* \end{pmatrix}$$

is a $\{1, 3\}$ -inverses of T_2 . Then

$$(3.3) \quad \begin{aligned} T_1M_1T_3 &= \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} \begin{pmatrix} (T_2^{11})^{-1} & O \\ (T_3^{11})^* & (T_3^{21})^* \end{pmatrix} \begin{pmatrix} T_3^{11} & O \\ T_3^{21} & O \end{pmatrix} \\ &= \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} + T_1^{12}(T_3^{11})^*T_3^{11} + T_1^{12}(T_3^{21})^*T_3^{21} & O \\ O & O \end{pmatrix} \end{aligned}$$

and

$$(3.4) \quad T_1M_1T_3 = T_1T_2^\dagger T_3 = \begin{pmatrix} T_1^{11}(T_2^{11})^{-1}T_3^{11} & O \\ O & O \end{pmatrix}.$$

Hence, from (3.3) and (3.4), we have

$$(3.5) \quad T_1^{12}(T_3^{11})^*T_3^{11} + T_1^{12}(T_3^{21})^*T_3^{21} = O.$$

Combining (3.5) with the equality (2.5), we have $T_1^{12}S = O$. Since S is invertible, $T_1^{12} = O$, that is $R(T_1^*) \subseteq R(T_2^*)$. \square

COROLLARY 3.2. Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2 , and T_3 are not zero operators on Hilbert spaces. Then the identity $T_1 T_2^{(1,3)} T_3 = O$ holds for every $T_2^{(1,3)} \in T_2\{1, 3\}$ if and only if $R(T_1^*) \subseteq R(T_2^*)$ and $R(T_3) \subseteq N(T_1 T_2^\dagger)$.

Next, we will investigate the invariance of the range of the operator product $T_1 T_2^{(1,3)} T_3$ with respect to the choices of $T_2^{(1,3)} \in T_2\{1, 3\}$.

THEOREM 3.3. Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2 , and T_3 are not zero operators on Hilbert spaces and $T_1 T_2^\dagger T_3 \neq O$. Then the following statements are equivalent:

- (1) $R(T_1 T_2^{(1,3)} T_3)$ is the same for every $T_2^{(1,3)} \in T_2\{1, 3\}$;
- (2) $R(T_1^*) \subseteq R(T_2^*)$.

Proof. According to the above proof in Theorem 3.1, it follows that for any $T_2^{(1,3)} \in T_2\{1, 3\}$

$$(3.6) \quad T_1 T_2^{(1,3)} T_3 = \begin{pmatrix} T_1^{11} (T_2^{11})^{-1} T_3^{11} + T_1^{12} W_{21} T_3^{11} + T_1^{12} W_{22} T_3^{21} & O \\ O & O \end{pmatrix}$$

and

$$(3.7) \quad (T_1 T_2^{(1,3)} T_3)^\dagger = \begin{pmatrix} (T_1^{11} (T_2^{11})^{-1} T_3^{11} + T_1^{12} W_{21} T_3^{11} + T_1^{12} W_{22} T_3^{21})^\dagger & O \\ O & O \end{pmatrix},$$

where W_{21} and W_{22} are bounded linear operators on appropriate subspaces. Furthermore, from (3.2) we have

$$(3.8) \quad (T_1 T_2^\dagger T_3)^\dagger = \begin{pmatrix} (T_1^{11} (T_2^{11})^{-1} T_3^{11})^\dagger & O \\ O & O \end{pmatrix}.$$

(2) \Rightarrow (1): Clearly the invariance of the product $T_1 T_2^{(1,3)} T_3$ with respect to $T_2^{(1,3)} \in T_2\{1, 3\}$ is sufficient for the invariance of $R(T_1 T_2^{(1,3)} T_3)$ with respect to the choices of $T_2^{(1,3)} \in T_2\{1, 3\}$. Then from Theorem 3.1, we have the result “(2) \Rightarrow (1)” in Theorem 3.3.

(1) \Rightarrow (2): From (1) in Theorem 3.3, we know that the equality $R(T_1 T_2^{(1,3)} T_3) = R(T_1 T_2^\dagger T_3)$ holds for any $T_2^{(1,3)} \in T_2\{1, 3\}$. Under the definition of the range of operators, it follows that the equality $R(T_1 T_2^{(1,3)} T_3) = R(T_1 T_2^\dagger T_3)$ holds for any $T_2^{(1,3)}$ if and only if the following two inclusions:

$$R(T_1 T_2^{(1,3)} T_3) \subseteq R(T_1 T_2^\dagger T_3)$$

and

$$R(T_1 T_2^\dagger T_3) \subseteq R(T_1 T_2^{(1,3)} T_3)$$

hold for any $T_2^{(1,3)} \in T_2\{1, 3\}$, which are respectively equivalent to the following two identities:

$$(3.9) \quad T_1 T_2^\dagger T_3 (T_1 T_2^\dagger T_3)^\dagger T_1 T_2^{(1,3)} T_3 = T_1 T_2^{(1,3)} T_3$$

and

$$(3.10) \quad T_1 T_2^{(1,3)} T_3 (T_1 T_2^{(1,3)} T_3)^\dagger T_1 T_2^\dagger T_3 = T_1 T_2^\dagger T_3$$

valid for any $T_2^{(1,3)} \in T_2\{1, 3\}$. Hence, from the equalities (3.2), (3.6), (3.7), (3.8) (3.9), (3.10), we have

$$(3.11) \quad (T_1^{11} (T_2^{11})^{-1} T_3^{11}) (T_1^{11} (T_2^{11})^{-1} T_3^{11})^\dagger \mu = \mu$$

and

$$(3.12) \quad \mu \mu^\dagger T_1^{11} (T_2^{11})^{-1} T_3^{11} = T_1^{11} (T_2^{11})^{-1} T_3^{11},$$

where W_{21} and W_{22} are bounded linear operators on appropriate subspaces and

$$(3.13) \quad \mu = T_1^{11} (T_2^{11})^{-1} T_3^{11} + T_1^{12} W_{21} T_3^{11} + T_1^{12} W_{22} T_3^{21}.$$

Combining (3.11), (3.12), with (3.13), we get the following equality:

$$(3.14) \quad (T_1^{11} (T_2^{11})^{-1} T_3^{11}) (T_1^{11} (T_2^{11})^{-1} T_3^{11})^\dagger = \mu \mu^\dagger$$

is valid for arbitrary bounded linear operators W_{21} and W_{22} .

On the other hand, since $T_1 T_2^\dagger T_3 \neq O$, it follows that

$$T_1 T_2^\dagger T_3 = \begin{pmatrix} T_1^{11} (T_2^{11})^{-1} T_3^{11} & O \\ O & O \end{pmatrix} \neq O,$$

that is $T_1^{11} (T_2^{11})^{-1} T_3^{11} \neq O$, which also implies that $T_1^{11} \neq O$, $T_2^{11} \neq O$ and $T_3^{11} \neq O$. Let $W_{22} = O$, then from (3.14), we obtain that the following equality:

$$\begin{aligned} & (T_1^{11} (T_2^{11})^{-1} T_3^{11}) (T_1^{11} (T_2^{11})^{-1} T_3^{11})^\dagger \\ &= (T_1^{11} (T_2^{11})^{-1} T_3^{11} + T_1^{12} W_{21} T_3^{11}) (T_1^{11} (T_2^{11})^{-1} T_3^{11} + T_1^{12} W_{21} T_3^{11})^\dagger \end{aligned}$$

holds for any W_{21} . This implies $T_1^{12} = O$. According to the fact: $R(T_1^*) \subseteq R(T_2^*) \Leftrightarrow T_1^{12} = O$, it follows that (1) \Rightarrow (2). \square

By Lemma 1.2 and Lemma 1.3, we know that for a bounded linear operator T_2 , $X \in T_2\{1, 4\}$ if and only $X^* \in T_2^*\{1, 3\}$. So results for the operator product $T_1 T_2^{(1,4)} T_3$ involving $\{1, 4\}$ -inverses of T_2 follow from the previous theorems in this section.

THEOREM 3.4. *Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2 , and T_3 are not zero operators on Hilbert spaces. Then the following statements are equivalent:*

- (1) *The equality $T_1 T_2^{(1,4)} T_3 = T_1 T_2^\dagger T_3$ holds for every $T_2^{(1,4)} \in T_2\{1, 4\}$;*
- (2) *$R(T_3) \subseteq R(T_2)$.*

COROLLARY 3.5. *Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2 , and T_3 are not zero operators on Hilbert spaces. Then the identity $T_1 T_2^{(1,4)} T_3 = O$ holds for every $T_2^{(1,4)} \in T_2\{1, 4\}$ if and only if $R(T_3) \subseteq R(T_2)$ and $R(T_3) \subseteq N(T_1 T_2^\dagger)$.*

THEOREM 3.6. *Let $T_1 \in L(\mathbb{L}, \mathbb{H})$, $T_2 \in L(\mathbb{L}, \mathbb{K})$, and $T_3 \in L(\mathbb{H}, \mathbb{K})$ be such that T_1, T_2, T_3 have closed ranges. Suppose that T_1, T_2 , and T_3 are not zero operators on Hilbert spaces and $T_1 T_2^\dagger T_3 \neq O$. Then the following statements are equivalent.*

- (1) *$R(T_1 T_2^{(1,4)} T_3)$ is the same for every $T_2^{(1,4)} \in B\{1, 4\}$;*
- (2) *$R(T_3) \subseteq R(T_2)$.*

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