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THE REVERSE ORDER LAWS AND THE MIXED-TYPE REVERSE ORDER LAWS FOR GENERALIZED INVERSES OF MULTIPLE MATRIX PRODUCTS*

ZHIPING XIONG† AND BING ZHENG‡

Abstract. In this paper, necessary and sufficient conditions for a number of reverse order laws and mixed-type reverse order laws are derived by using the maximal ranks of the generalized Schur complements.

Key words. Reverse order law, Mixed-type reverse order law, Generalized inverse, Matrix product, Maximal rank, Generalized Schur complement.

AMS subject classifications. 15A03, 15A09.

1. Introduction. Let $\mathbb{C}^{m\times n}$ denote the set of $m \times n$ matrices with complex entries and $\mathbb{C}^m$ denote the set of m-dimensional vectors. I_k denotes the identity matrix of order k, $O_{m\times n}$ is the $m\times n$ matrix of all zero entries (if no confusion occurs, we will drop the subindex). For a matrix $A \in \mathbb{C}^{m\times n}$, A^*, $R(A)$ and $r(A)$ denote the conjugate transpose, the range space and the rank of the matrix A, respectively.

We recall that a generalized inverse $X \in \mathbb{C}^{n\times m}$ of a given matrix $A \in \mathbb{C}^{m\times n}$ is a matrix which satisfies some of the following four Penrose equations [2]:

$$(1.1)\quad (1)\ \ AXA = A,\quad (2)\ \ XAX = X,\quad (3)\ \ (AX)^* = AX,\quad (4)\ \ (XA)^* = XA.$$

For any matrix $A \in \mathbb{C}^{m\times n}$, let $A\{i, j, \ldots, k\}$ denote the set of matrices $X \in \mathbb{C}^{n\times m}$ which satisfy equations $(i), (j), \ldots, (k)$ from among equations $(1) - -(4)$ of (1.1). A matrix in $A\{i, j, \ldots, k\}$ is called an $\{i, j, \ldots, k\}$-inverse of A and denoted by $A^{(i,j,\ldots,k)}$. In particular, an $n \times m$ matrix X of the set $A\{1\}$ is called a $\{1\}$-inverse or a g-inverse of A. X is called a {1,3}-inverse or a least squares g-inverse of A if it is an element

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of $A\{1,3\}$; X is called a {1,4}-inverse or a minimum norm g-inverse of A if it is an element of $A\{1,4\}$. The Moore-Penrose inverse of A is an element of the set $A\{1,2,3,4\}$. Any matrix A admits a unique Moore-Penrose inverse, denoted by $A^{\dagger}$. We refer the readers to [2, 15] for basic results on the generalized inverses.

Let A_i, $i = 1, 2, \ldots, n$, be n matrices such that the product $A_1A_2 \cdots A_n$ exists. If each of the n matrices is nonsingular, then the product $A_1A_2 \cdots A_n$ is nonsingular too, and the inverse of $A_1A_2 \cdots A_n$ satisfies the reverse order law $(A_1A_2 \cdots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1} \cdots A_1^{-1}$. However, this reverse order law does not hold for generalized inverses. Hence the necessary and sufficient conditions for the reverse order laws of the generalized inverses of multiple matrix products to hold yield a class of interesting problems that are fundamental in the theory of generalized inverses of matrices and statistics. They have attracted considerable attention since the middle 1960s, and recently many interesting results have been obtained, see [1, 3, 4, 5, 6, 8, 9, 10, 13, 14, 16, 17, 18, 19, 20, 21].

In the paper [6], Liu and Wei studied the reverse order law for least squares g-inverses of multiple matrix products and derived some necessary and sufficient conditions for

$$A_n\{1,3\}A_{n-1}\{1,3\} \cdots A_1\{1,3\} \subseteq (A_1A_2 \cdots A_n)\{1,3\} \tag{1.2}$$

and

$$A_n\{1,4\}A_{n-1}\{1,4\} \cdots A_1\{1,4\} \subseteq (A_1A_2 \cdots A_n)\{1,4\} \tag{1.3}$$

by using P-SVD (Product Singular Value Decomposition). In this paper we revisit these two reverse order laws by using the maximal rank of the generalized Schur complements [12]. Some new simpler equivalent conditions for the inclusions (1.2) and (1.3) are obtained in terms of only the ranks of the known matrices. Compared with the conditions given in [6], our conditions can be easily checked and their proofs are very simple. Furthermore, with the same technique the necessary and sufficient conditions for the following mixed-type reverse order laws are derived:

$$A_n\{1,3\}A_{n-1}\{1,3\} \cdots A_1\{1,3\} \subseteq (A_1A_2 \cdots A_n)\{1\}, \tag{1.4}$$

$$A_n\{1,4\}A_{n-1}\{1,4\} \cdots A_1\{1,4\} \subseteq (A_1A_2 \cdots A_n)\{1\}, \tag{1.5}$$

$$A_n\{1\}A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1A_2 \cdots A_n)\{1,3\}, \tag{1.6}$$

and

$$A_n\{1\}A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1A_2 \cdots A_n)\{1,4\}. \tag{1.7}$$

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For the sake of the simplicity in the following discussion, we will adopt the following notations for the matrix products with $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$ and $X_i \in \mathbb{C}^{l_{i+1} \times l_i}$, $i = 1, 2, \ldots, n$:

$$(1.8) \qquad \mathscr{A}_i^j = A_i A_{i+1} \cdots A_j, \quad \mathscr{X}_i^j = X_i^* X_{i+1}^* \cdots X_j^*, \quad 1 \le i \le j \le n.$$

In particular, $\mathscr{A}_j^j = A_j$, $\mathscr{A}_1^j = A_1 A_2 \cdots A_j$, $\mathscr{X}_j^j = X_j^*$ and $\mathscr{X}_1^j = X_1^* X_2^* \cdots X_j^*$, $j = 1, 2, \ldots, n$, and define $\mathscr{X}_{n+1}^n = I_{l_{n+1}}$. Also, in order to present the necessary and sufficient conditions for the reverse order laws (1.2), (1.3) and the mixed-type reverse order laws (1.4)–(1.7), we define the following two matrix functions:

$$(1.9) \qquad \begin{aligned} &T_{A_1, A_2, \ldots, A_n}(X_1, X_2, \ldots, X_n) \\ &= A_n^* A_{n-1}^* \cdots A_1^* - A_n^* A_{n-1}^* \cdots A_1^* A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_1 \end{aligned}$$

and

$$(1.10) \qquad \begin{aligned} &P_{A_1, A_2, \ldots, A_n}(X_1, X_2, \ldots, X_n) \\ &= A_1 A_2 \cdots A_n - A_1 A_2 \cdots A_n X_n X_{n-1} \cdots X_1 A_1 A_2 \cdots A_n. \end{aligned}$$

The main tools in the later discussion are the following three lemmas. The first lemma gives the formulas of the maximal ranks of the generalized Schur complements related to the generalized inverses [11, 12], and the second shows the characterizations of some generalized inverses of a matrix.

Lemma 1.1. [11, 12] *Let* $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times l}$, $C \in \mathbb{C}^{k \times n}$ *and* $D \in \mathbb{C}^{k \times l}$. *Then*

$$(1.11) \max_{A^{(1)} \in A\{1\}} r(D - CA^{(1)}B) = \min \left\{ r\,(C, \quad D\,), r\begin{pmatrix} B \\ D \end{pmatrix}, r\begin{pmatrix} A & B \\ C & D \end{pmatrix} - r(A) \right\};$$

$$(1.12) \max_{A^{(1,3)} \in A\{1,3\}} r(D - CA^{(1,3)}B) = \min \left\{ r\begin{pmatrix} A^*A & A^*B \\ C & D \end{pmatrix} - r(A), r\begin{pmatrix} B \\ D \end{pmatrix} \right\}.$$

Lemma 1.2. [10] *Let* $A \in \mathbb{C}^{m \times n}$ *and* $G \in \mathbb{C}^{n \times m}$. *Then*

$$(1.13) \qquad G \in A\{1\} \Leftrightarrow AGA = A;$$

$$(1.14) \qquad G \in A\{1,3\} \Leftrightarrow A^*AG = A^*;$$

$$(1.15) \qquad G \in A\{1,4\} \Leftrightarrow GAA^* = A^*.$$

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LEMMA 1.3. [20] Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \ldots, n$. Then

$$l_2 + l_3 + \cdots + l_n + r(A_1 \cdots A_n) \geq r(A_1) + r(A_2) + \cdots + r(A_n). \tag{1.16}$$

In addition, the following rank equalities [7] will be needed in the sequel:

$$r\,(\,A, \quad B\,) = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{1.17}$$

$$r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.18}$$

where $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$.

This paper is organized as follows. The necessary and sufficient conditions for the reverse order laws (1.2) and (1.3) to hold are investigated in Section 2. In Section 3 we study the necessary and sufficient conditions for the mixed-type reverse order laws (1.4) and (1.5). Finally, the necessary and sufficient conditions for the mixed-type reverse order laws (1.6) and (1.7) are discussed in Section 4.

2. The necessary and sufficient conditions for the inclusions (1.2) and (1.3). Let $T_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)$ be as in (1.9). It is easy to see from the characterization (1.14) of {1,3}-inverses that the reverse order law (1.2) holds if and only if the following rank identity

$$\max_{X_n, X_{n-1}, \ldots, X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) = 0 \tag{2.1}$$

holds for any $X_i \in A_i\{1,3\}$, $i = 1, 2, \ldots, n$. Hence, to give the necessary and sufficient conditions for the inclusion (1.2), we first state the following theorem:

THEOREM 2.1. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \ldots, n$, $\mathscr{A}_i^j$, $1 \leq i \leq j \leq n$, be as in* (1.8) *and $T_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)$ be as in* (1.9). *Then*

$$\begin{aligned} &\max_{X_n, X_{n-1}, \ldots, X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) \\ &= r\begin{pmatrix} A_n^* & O & \cdots & O \\ O & A_{n-1}^* & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^* \\ (\mathscr{A}_1^n)^* \mathscr{A}_1^{n-1} & (\mathscr{A}_1^n)^* \mathscr{A}_1^{n-2} & \cdots & (\mathscr{A}_1^n)^* \mathscr{A}_1^1 \end{pmatrix} - \sum_{i=2}^{n} r(A_i), \end{aligned} \tag{2.2}$$

where X_i varies over $A_i\{1,3\}$, $i = 1, 2, \ldots, n$.

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Proof. Let $\mathscr{A}_i^j$, $\mathscr{X}_i^j$, $1 \le i \le j \le n$, be as in (1.8). Then according to Lemma 1.1 (1.12) with $A = A_1$, $B = I_{l_1}$, $D = (\mathscr{A}_1^n)^*$ and $C = (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^*$, we have

$$(2.3)\qquad \begin{aligned} &\max_{X_1}\ r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n)) \\ &= \min\left\{ r\begin{pmatrix} A_1^*A_1 & A_1^* \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^* & (\mathscr{A}_1^n)^* \end{pmatrix} - r(A_1),\quad r\begin{pmatrix} I_{l_1} \\ (\mathscr{A}_1^n)^* \end{pmatrix} \right\} \\ &= \min\left\{ r((\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^* - (\mathscr{A}_1^n)^*A_1),\quad l_1 \right\} \\ &= r((\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^* - (\mathscr{A}_1^n)^*A_1), \end{aligned}$$

in which, by the row or column elementary block operations, from first equality to second one we use the rank identities

$$r\begin{pmatrix} I_{l_1} \\ (\mathscr{A}_1^n)^* \end{pmatrix} = l_1$$

and

$$\begin{aligned} r\begin{pmatrix} A_1^*A_1 & A_1^* \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^* & (\mathscr{A}_1^n)^* \end{pmatrix} &= r\begin{pmatrix} 0 & A_1^* \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^* - (\mathscr{A}_1^n)^*A_1 & 0 \end{pmatrix} \\ &= r((\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^* - (\mathscr{A}_1^n)^*A_1) - r(A_1). \end{aligned}$$

It's worthy mentioning that similar techniques will be repeatedly used in the sequel.

Again by Lemma 1.1 (1.12) with $A = A_2$, $B = I_{l_2}$, $C = (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_3^n)^*$ and $D = (\mathscr{A}_1^n)^*A_1$, we have

$$(2.4)\qquad \begin{aligned} &\max_{X_2,X_1}\ r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n)) \\ &= \max_{X_2}\ r((\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^* - (\mathscr{A}_1^n)^*A_1) \\ &= \min\left\{ r\begin{pmatrix} A_2^*A_2 & A_2^* \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_3^n)^* & (\mathscr{A}_1^n)^*A_1 \end{pmatrix} - r(A_2),\quad r\begin{pmatrix} I_{l_2} \\ (\mathscr{A}_1^n)^*A_1 \end{pmatrix} \right\} \\ &= \min\left\{ r\begin{pmatrix} O & A_2^* \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_3^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^2 & (\mathscr{A}_1^n)^*A_1 \end{pmatrix} - r(A_2),\quad l_2 \right\}. \end{aligned}$$

By the formula (1.18), we obtain

$$(2.5)\qquad \begin{aligned} &r\begin{pmatrix} O & A_2^* \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_3^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^2 & (\mathscr{A}_1^n)^*A_1 \end{pmatrix} \\ &= r\left(\, (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_3^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^2,\quad (\mathscr{A}_1^n)^*A_1F_{A_2^*} \,\right) + r(A_2) \\ &= r\left(\, (\mathscr{A}_1^n)^*\mathscr{A}_1^1(A_2\cdots A_n(\mathscr{X}_3^n)^* - A_2),\quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*} \,\right) + r(A_2) \\ &\le r(A_1) + r(A_2) \\ &\le l_2 + r(A_2). \end{aligned}$$

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Thus, from (2.4) and (2.5), we have

$$\begin{aligned}(2.6)\qquad &\max_{X_2,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n))\\ &= r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_3^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^2, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\,).\end{aligned}$$

Generally, for $2 \le i \le n$, we can prove the following fact:

$$\begin{aligned}(2.7)\qquad &\max_{X_i,X_{i-1},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n))\\ &= r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^i, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}F_{A_i^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\,).\end{aligned}$$

In fact, the identity (2.7) is true for $i = 2$ (see (2.6)). Now assume identity (2.7) is also true for $i-1 (i \ge 3)$, that is,

$$\begin{aligned}(2.8)\qquad &\max_{X_{i-1},X_{i-2},\ldots,X_1} r(T_{A_1,A_2,\cdots,A_n}(X_1,X_2,\ldots,X_n))\\ &= r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_i^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\,).\end{aligned}$$

Next we shall prove that (2.7) is also true for i. Combining (2.8) with the formula (1.12) in Lemma 1.1 (with $A = A_i$, $B = (\,I_{l_i}, \quad O, \cdots, O\,)$, $C = (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*$ and $D = (\,(\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\,)$, we have

$$\begin{aligned}(2.9)\qquad &\max_{X_i,X_{i-1},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n))\\ &= \max_{X_i} r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_i^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*}, \ldots, (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\,)\\ &= \max_{X_i} r((\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_i^n)^*\,(\,I_{l_i}, \quad O, \quad \cdots, \quad O\,)\\ &\quad - (\,(\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\,))\\ &= \min\left\{ r\begin{pmatrix} A_i^*A_i & A_i^* & O & \cdots & O \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* & (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} & (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*} & \cdots & (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\end{pmatrix}\right.\\ &\quad \left. -r(A_i), \quad r\begin{pmatrix} I_{l_i} & O & \cdots & O \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} & (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*} & \cdots & (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\end{pmatrix}\right\}.\end{aligned}$$

According to the formula (1.18), we know

$$\begin{aligned}(2.10)\qquad &\\ &r\begin{pmatrix} A_i^*A_i & A_i^* & O & \cdots & O \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* & (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} & (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*} & \cdots & (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\end{pmatrix}\\ &= r\begin{pmatrix} O & A_i^* & \cdots & O \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^i & (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} & \cdots & (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\end{pmatrix}\\ &= r(A_i) + r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^i, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}F_{A_i^*}, \ldots, (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\,)\\ &\le r(A_i) + r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^i, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}F_{A_i^*}\,)\\ &\quad + r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1 F_{A_2^*}\,)\end{aligned}$$

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$$
\begin{aligned}
&\le r(A_i) + r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}) + r\left(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\,\right)\\
&\le r(A_i) + r(A_{i-1}) + r\left(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\,\right)\\
&\le r(A_i) + l_i + r\left(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\,\right).
\end{aligned}
$$

Hence, by (2.9), (2.10) and the formula (1.18) we have

$$
\begin{aligned}
&\max_{X_i, X_{i-1},\ldots,X_1} \; r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n))\\
&= \max_{X_i} \; r\left(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_i^n)^*, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*}, \ldots, (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\,\right)\\
&= r\begin{pmatrix} A_i^*A_i & A_i^* & O & \cdots & O\\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* & (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} & (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-2}F_{A_{i-1}^*} & \cdots & (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\end{pmatrix} - r(A_i)\\
&= r\left(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^i, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}F_{A_i^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\,\right).
\end{aligned}
$$

Specifically, in the case of $i = n$, we obtain

$$
\begin{aligned}
(2.11)\qquad &\max_{X_n, X_{n-1},\ldots,X_1} \; r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n))\\
&= r\left(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{n+1}^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^n, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}F_{A_n^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\,\right)\\
&= r\left(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n - (\mathscr{A}_1^n)^*\mathscr{A}_1^n, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}F_{A_n^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\,\right)\\
&= r\left(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}F_{A_n^*}, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-2}F_{A_{n-1}^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\,\right).
\end{aligned}
$$

We now repeatedly apply formula (1.18) to (2.11) and finally have

$$
\begin{aligned}
(2.12)\qquad &\max_{X_n, X_{n-1},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n))\\
&= r\left(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}F_{A_n^*}, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-2}F_{A_{n-1}^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\,\right)\\
&= r\begin{pmatrix} A_n^* & O & \cdots & O\\ O & A_{n-1}^* & \cdots & O\\ \vdots & \vdots & \ddots & \vdots\\ O & O & \cdots & A_2^*\\ (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1} & (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-2} & \cdots & (\mathscr{A}_1^n)^*\mathscr{A}_1^1\end{pmatrix} - \sum_{i=2}^n r(A_i). \quad \square
\end{aligned}
$$

From Theorem 2.1 and the identity (2.1), we can immediately get the necessary and sufficient conditions for the inclusion (1.2) to hold, which are stated in the following theorem.

THEOREM 2.2. *Let $A_i \in \mathbb{C}^{l_i\times l_{i+1}}$, $i = 1,2,\ldots,n$, and $\mathscr{A}_i^j$, $1 \le i \le j \le n$, be as in* (1.8). *Then the following statements are equivalent:*

(1) $A_n\{1,3\}A_{n-1}\{1,3\}\cdots A_1\{1,3\} \subseteq (A_1A_2\ldots A_n)\{1,3\}$;

(2) $r\left(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}F_{A_n^*}, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-2}F_{A_{n-1}^*}, \quad \ldots, \quad (\mathscr{A}_1^n)^*\mathscr{A}_1^1F_{A_2^*}\,\right) = 0$;

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(3) $R((\mathscr{A}_1^i)^*\mathscr{A}_1^n) \subseteq R(A_{i+1}), \quad i = 1, 2, \ldots, n-1;$

(4) $$r\begin{pmatrix} A_n^* & O & \cdots & O \\ O & A_{n-1}^* & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^* \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1} & (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-2} & \cdots & (\mathscr{A}_1^n)^*\mathscr{A}_1^1 \end{pmatrix} = \sum_{i=2}^{n} r(A_i).$$

Before presenting an example to confirm Theorem 2.2, we mention the following general expressions of $\{1,3\}-$ and $\{1,4\}-$inverse for any matrix $A \in \mathbb{C}^{m\times n}$:

$$A\{1,3\} = \{A^\dagger + (I_n - A^\dagger A)X \ : \ X \in \mathbb{C}^{n\times m}\},$$
$$A\{1,4\} = \{A^\dagger + Y(I_m - AA^\dagger) \ : \ Y \in \mathbb{C}^{n\times m}\}. \tag{2.13}$$

If we know the Moore-Penrose inverse $A^\dagger$ of matrix A, then the above formulas make very easy the computations of $A\{1,3\}$ and $A\{1,4\}$.

Example 2.3. Let

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then

$$r\begin{pmatrix} A_3^* & O \\ O & A_2^* \\ A_3^*A_2^*A_1^*A_1A_2 & A_3^*A_2^*A_1^*A_1 \end{pmatrix} = r(A_2) + r(A_3) = 4,$$

which means that the matrices A_1, A_2 and A_3 satisfy the condition (4) in Theorem 2.2. On the other hand, by the formula (2.13), we have

$$A_1\{1,3\} = \left\{\begin{pmatrix} 1/2 & 1/2 \\ a_1 & b_1 \end{pmatrix} \mid a_1, \ b_1 \in \mathbb{C}\right\}, \quad A_2\{1,3\} = \left\{\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ a_2 & b_2 \end{pmatrix} \mid a_2, \ b_2 \in \mathbb{C}\right\}$$

and

$$A_3\{1,3\} = \left\{\begin{pmatrix} 1 & 0 & 0 \\ a_3 & b_3 & c \\ -a_3 & 1/2 - b_3 & 1/2 - c \end{pmatrix} \mid a_3, \ b_3, \ c \in \mathbb{C}\right\}.$$

Hence, the matrix set $A_3\{1,3\}A_2\{1,3\}A_1\{1,3\}$ can be expressed as

$$A_3\{1,3\}A_2\{1,3\}A_1\{1,3\} = \{M_1 :$$
$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ a_3 & b_3 & c \\ -a_3 & 1/2 - b_3 & 1/2 - c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ a_1 & b_1 \end{pmatrix} \mid a_i, \ b_j, \ c \in \mathbb{C}\}.$$

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It is easy to verify that the identities

$$(A_1A_2A_3)M_1(A_1A_2A_3) = A_1A_2A_3 \quad \text{and} \quad (A_1A_2A_3M_1)^* = A_1A_2A_3M_1$$

hold for any matrix $M_1 \in A_3\{1,3\}A_2\{1,3\}A_1\{1,3\}$, that is

$$A_3\{1,3\}A_2\{1,3\}A_1\{1,3\} \subseteq (A_1A_2A_3)\{1,3\}.$$

In the remainder of this section, we will present the necessary and sufficient conditions for the inclusion (1.3) involved {1,4}-inverses. Notice that $GAA^* = A^*$ is equivalent to the equation $AA^*G^* = A$. This implies that, by Lemma 1.2, $G \in A\{1,4\}$ if and only if $G^* \in A^*\{1,3\}$. So from the results obtained in Theorem 2.2, we can immediately get the necessary and sufficient conditions for the inclusion (1.3), which are stated below without proofs.

Theorem 2.4. *Let* $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \ldots, n$, *and* $\mathscr{A}_i^j$, $1 \le i \le j \le n$, *be as in* (1.8). *Then the following statements are equivalent:*

(1) $A_n\{1,4\}A_{n-1}\{1,4\}\cdots A_1\{1,4\} \subseteq (A_1A_2\cdots A_n)\{1,4\}$;

(2) $r\begin{pmatrix} E_{A_1^*}\mathscr{A}_2^n(\mathscr{A}_1^n)^* \\ E_{A_2^*}\mathscr{A}_3^n(\mathscr{A}_1^n)^* \\ \vdots \\ E_{A_{n-1}^*}\mathscr{A}_n^n(\mathscr{A}_1^n)^* \end{pmatrix} = 0;$

(3) $R(\mathscr{A}_i^n(\mathscr{A}_1^n)^*) \subseteq R(A_{i-1}^*), \quad i = 2, 3, \ldots, n;$

(4) $r\begin{pmatrix} A_1^* & O & \cdots & O & \mathscr{A}_2^n(\mathscr{A}_1^n)^* \\ O & A_2^* & \cdots & O & \mathscr{A}_3^n(\mathscr{A}_1^n)^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_{n-1}^* & \mathscr{A}_n^n(\mathscr{A}_1^n)^* \end{pmatrix} = \sum_{i=1}^{n-1} r(A_i).$

Similar to Example 2.3, we can easily verify the following three matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

satisfy the inclusion relationship

$$A_3\{1,4\}A_2\{1,4\}A_1\{1,4\} \subseteq (A_1A_2A_3)\{1,4\}.$$

3. The necessary and sufficient conditions for the inclusions (1.4) and (1.5). Consider the matrix function $P_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)$ defined in (1.10).

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By Lemma 1.2 (1.13), we know that the mixed-type reverse order law (1.4) holds if and only if the rank identity

$$\max_{X_n, X_{n-1}, \ldots, X_1} r(P_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) = 0 \tag{3.1}$$

holds, where each X_i $(i = 1, 2, \ldots, n)$ varies over the set $A_i\{1, 3\}$ of all $\{1,3\}$-*inverses* of the matrix A_i. Hence, to give the necessary and sufficient conditions for the inclusion (1.4), we first study the concrete expression of the maximum rank of the matrix function $P_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)$ when $X_i \in A_i\{1, 3\}$, $i = 1, 2, \ldots, n$.

Theorem 3.1. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, \ldots, n$, and $P_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)$ be as in* (1.10). *Then*

$$\begin{aligned} &\max_{X_n, X_{n-1}, \ldots, X_1} r(P_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) \\ &= r\begin{pmatrix} A_n^* & O & \cdots & O \\ O & A_{n-1}^* & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^* \\ A_1A_2\cdots A_{n-1} & A_1A_2\cdots A_{n-2} & \cdots & A_1 \end{pmatrix} + r(A_1A_2\cdots A_n) - \sum_{i=1}^{n} r(A_i), \end{aligned} \tag{3.2}$$

where X_i varies over $A_i\{1, 3\}$, $i = 1, 2, \ldots, n$.

Proof. The basic idea for the proof of Theorem 3.1 is similar to that of Theorem 2.1. For the completeness of the paper, we still give the detailed proof here.

Let $\mathscr{A}_i^j$, $\mathscr{X}_i^j$, $1 \le i \le j \le n$, be as in (1.8). Then, by Lemma 1.1 (1.12) with $A = A_1$, $B = \mathscr{A}_1^n$, $D = \mathscr{A}_1^n$ and $C = \mathscr{A}_1^n(\mathscr{X}_2^n)^*$, we have

$$\begin{aligned} &\max_{X_1} r(P_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) \\ &= \min\left\{ r\begin{pmatrix} A_1^*A_1 & A_1^*\mathscr{A}_1^n \\ \mathscr{A}_1^n(\mathscr{X}_2^n)^* & \mathscr{A}_1^n \end{pmatrix} - r(A_1), r\begin{pmatrix} \mathscr{A}_1^n \\ \mathscr{A}_1^n \end{pmatrix} \right\} \\ &= \min\left\{ r\begin{pmatrix} A_1 & \mathscr{A}_1^n \\ \mathscr{A}_1^n(\mathscr{X}_2^n)^* & \mathscr{A}_1^n \end{pmatrix} - r(A_1), \quad r(\mathscr{A}_1^n) \right\} \\ &= \min\{ r(\mathscr{A}_1^n(\mathscr{X}_2^n)^* - A_1) + r(A_1A_2\cdots A_n) - r(A_1), \quad r(A_1A_2\cdots A_n) \} \\ &= r(\mathscr{A}_1^n(\mathscr{X}_2^n)^* - A_1) + r(A_1A_2\cdots A_n) - r(A_1). \end{aligned} \tag{3.3}$$

Again by Lemma 1.1 (1.12) with $A = A_2$, $B = I_{l_2}$, $C = \mathscr{A}_1^n(\mathscr{X}_3^n)^*$ and $D = A_1$, we know that

$$\begin{aligned} &\max_{X_2} r(\mathscr{A}_1^n(\mathscr{X}_2^n)^* - A_1) \\ &= \min\left\{ r\begin{pmatrix} A_2^*A_2 & A_2^* \\ \mathscr{A}_1^n(\mathscr{X}_3^n)^* & A_1 \end{pmatrix} - r(A_2), \quad r\begin{pmatrix} I_{l_2} \\ A_1 \end{pmatrix} \right\} \end{aligned} \tag{3.4}$$

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$$
\begin{aligned}
&= \min\left\{ r\begin{pmatrix} O & A_2^* \\ \mathscr{A}_1^n(\mathscr{X}_3^n)^* - \mathscr{A}_1^2 & \mathscr{A}_1^1 \end{pmatrix} - r(A_2), \quad l_2 \right\} \\
&= \min\{ r\,(\,\mathscr{A}_1^n(\mathscr{X}_3^n)^* - \mathscr{A}_1^2, \quad \mathscr{A}_1^1 F_{A_2^*}\,) + r(A_2) - r(A_2), \quad l_2 \} \\
&= \min\{ r(A_1\,(\,\mathscr{A}_2^n(\mathscr{X}_3^n)^* - A_2, \quad F_{A_2^*}\,)), \quad l_2 \} \\
&= r\,(\,\mathscr{A}_1^n(\mathscr{X}_3^n)^* - \mathscr{A}_1^2, \quad \mathscr{A}_1^1 F_{A_2^*}\,),
\end{aligned}
$$

where the rank identity (1.18) is used in the third equality. Hence, from (3.3) and (3.4), we have

$$
\begin{aligned}
&\max_{X_2, X_1} r(P_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) \\
&= \max_{X_2} r(\mathscr{A}_1^n(\mathscr{X}_2^n)^* - A_1) + r(A_1A_2\cdots A_n) - r(A_1) \\
&= r\,(\,\mathscr{A}_1^n(\mathscr{X}_3^n)^* - \mathscr{A}_1^2, \quad \mathscr{A}_1^1 F_{A_2^*}\,) + r(A_1A_2\cdots A_n) - r(A_1).
\end{aligned} \tag{3.5}
$$

We assert that, for $2 \le i \le n$,

$$
\begin{aligned}
&\max_{X_i, X_{i-1}, \ldots, X_1} r(P_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) \\
&= r\,(\,\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - \mathscr{A}_1^i, \quad \mathscr{A}_1^{i-1} F_{A_i^*}, \quad \mathscr{A}_1^{i-2} F_{A_{i-1}^*}, \quad \cdots, \quad \mathscr{A}_1^1 F_{A_2^*}\,) \\
&\quad + r(A_1A_2\cdots A_n) - r(A_1).
\end{aligned} \tag{3.6}
$$

This can be proved by induction on i. In fact, the identity (3.5) has shown the truth of the equality relation (3.6) for $i = 2$. Assume that (3.6) is true for $i-1$ ($i \ge 3$), that is

$$
\begin{aligned}
&\max_{X_{i-1}, X_{i-2}, \ldots, X_1} r(P_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) \\
&= r\,(\,\mathscr{A}_1^n(\mathscr{X}_i^n)^* - \mathscr{A}_1^{i-1}, \quad \mathscr{A}_1^{i-2} F_{A_{i-1}^*}, \quad \mathscr{A}_1^{i-3} F_{A_{i-2}^*}, \quad \cdots, \quad \mathscr{A}_1^1 F_{A_2^*}\,) \\
&\quad + r(A_1A_2\cdots A_n) - r(A_1).
\end{aligned} \tag{3.7}
$$

We now prove that (3.6) is also true for i. Combining (3.7) with the formula (1.12) in Lemma 1.1 (with $A = A_i$, $B = (\,I_{l_i}, \quad O, \quad O, \quad \cdots, \quad O\,)$, $C = \mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*$ and $D = (\,\mathscr{A}_1^{i-1}, \quad \mathscr{A}_1^{i-2} F_{A_{i-1}^*}, \quad \mathscr{A}_1^{i-3} F_{A_{i-2}^*}, \quad \cdots, \quad \mathscr{A}_1^1 F_{A_2^*}\,))$, we have

$$
\begin{aligned}
&\max_{X_i, X_{i-1}, \ldots, X_1} r(P_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) \\
&= \max_{X_i} r\,(\,\mathscr{A}_1^n(\mathscr{X}_i^n)^* - \mathscr{A}_1^{i-1}, \quad \mathscr{A}_1^{i-2} F_{A_{i-1}^*}, \quad \mathscr{A}_1^{i-3} F_{A_{i-2}^*}, \quad \cdots, \quad \mathscr{A}_1^1 F_{A_2^*}\,) \\
&\quad + r(A_1A_2\cdots A_n) - r(A_1) \\
&= \max_{X_i} r(\mathscr{A}_1^n(\mathscr{X}_i^n)^*\,(\,I_{l_i}, \quad O, \quad O, \quad \cdots, \quad O\,) \\
&\quad - (\,\mathscr{A}_1^{i-1}, \quad \mathscr{A}_1^{i-2} F_{A_{i-1}^*}, \quad \mathscr{A}_1^{i-3} F_{A_{i-2}^*}, \quad \cdots, \quad \mathscr{A}_1^1 F_{A_2^*}\,)) + r(A_1\cdots A_n) - r(A_1) \\
&= \min\{ r\begin{pmatrix} A_i^*A_i & A_i^* & O & O & \cdots & O \\ \mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* & \mathscr{A}_1^{i-1} & \mathscr{A}_1^{i-2} F_{A_{i-1}^*} & \mathscr{A}_1^{i-3} F_{A_{i-2}^*} & \cdots & \mathscr{A}_1^1 F_{A_2^*} \end{pmatrix} - r(A_i),
\end{aligned} \tag{3.8}
$$

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$$r\begin{pmatrix} I_{l_i} & O & O & \cdots & O \\ \mathscr{A}_1^{i-1} & \mathscr{A}_1^{i-2}F_{A_{i-1}^*} & \mathscr{A}_1^{i-3}F_{A_{i-2}^*} & \cdots & \mathscr{A}_1^1F_{A_2^*} \end{pmatrix}\} + r(A_1A_2\cdots A_n) - r(A_1)$$
$$= \min\{r\begin{pmatrix} O & A_i^* & O & \cdots & O \\ \mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - \mathscr{A}_1^i & \mathscr{A}_1^{i-1} & \mathscr{A}_1^{i-2}F_{A_{i-1}^*} & \cdots & \mathscr{A}_1^1F_{A_2^*} \end{pmatrix} - r(A_i),$$
$$l_i + r\,(\,\mathscr{A}_1^{i-2}F_{A_{i-1}^*},\quad \mathscr{A}_1^{i-3}F_{A_{i-2}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,)\} + r(A_1A_2\cdots A_n) - r(A_1).$$

Since

$$(3.9)\quad r\begin{pmatrix} O & A_i^* & O & O & \cdots & O \\ \mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - \mathscr{A}_1^i & \mathscr{A}_1^{i-1} & \mathscr{A}_1^{i-2}F_{A_{i-1}^*} & \mathscr{A}_1^{i-3}F_{A_{i-2}^*} & \cdots & \mathscr{A}_1^1F_{A_2^*} \end{pmatrix}$$
$$= r(A_i) + r\,(\,\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - \mathscr{A}_1^i,\quad \mathscr{A}_1^{i-1}F_{A_i^*},\quad \mathscr{A}_1^{i-2}F_{A_{i-1}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,)$$
$$\le r(A_i) + r\,(\,\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - \mathscr{A}_1^i,\quad \mathscr{A}_1^{i-1}F_{A_i^*}\,)$$
$$+ r\,(\,\mathscr{A}_1^{i-2}F_{A_{i-1}^*},\quad \mathscr{A}_1^{i-3}F_{A_{i-2}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,)$$
$$\le r(A_i) + r(\mathscr{A}_1^{i-1}) + r\,(\,\mathscr{A}_1^{i-2}F_{A_{i-1}^*},\quad \mathscr{A}_1^{i-3}F_{A_{i-2}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,)$$
$$\le r(A_i) + r(A_{i-1}) + r\,(\,\mathscr{A}_1^{i-2}F_{A_{i-1}^*},\quad \mathscr{A}_1^{i-3}F_{A_{i-2}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,)$$
$$\le r(A_i) + l_i + r\,(\,\mathscr{A}_1^{i-2}F_{A_{i-1}^*},\quad \mathscr{A}_1^{i-3}F_{A_{i-2}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,)$$

in which we have used the identity (1.18) in the first equality, so from (3.8) and (3.9) we have

$$\max_{X_i,X_{i-1},\ldots,X_1} r(P_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n))$$
$$= r\,(\,\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - \mathscr{A}_1^i,\quad \mathscr{A}_1^{i-1}F_{A_i^*},\quad \mathscr{A}_1^{i-2}F_{A_{i-1}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,)$$
$$+ r(A_1A_2\cdots A_n) - r(A_1).$$

In particular, when $i = n$, we have

$$(3.10)\quad \max_{X_n,X_{n-1},\ldots,X_1} r(P_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n))$$
$$= r\,(\,\mathscr{A}_1^n(\mathscr{X}_{n+1}^n)^* - \mathscr{A}_1^n,\quad \mathscr{A}_1^{n-1}F_{A_n^*},\quad \mathscr{A}_1^{n-2}F_{A_{n-1}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,)$$
$$+ r(A_1A_2\cdots A_n) - r(A_1)$$
$$= r\,(\,\mathscr{A}_1^n - \mathscr{A}_1^n,\quad \mathscr{A}_1^{n-1}F_{A_n^*},\quad \mathscr{A}_1^{n-2}F_{A_{n-1}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,)$$
$$+ r(A_1A_2\cdots A_n) - r(A_1)$$
$$= r\,(\,\mathscr{A}_1^{n-1}F_{A_n^*},\quad \mathscr{A}_1^{n-2}F_{A_{n-1}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,) + r(A_1A_2\cdots A_n) - r(A_1).$$

By repeatedly applying the formula (1.18) to (3.10), we can finally have

$$\max_{X_n,X_{n-1},\ldots,X_1} r(P_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n))$$
$$= r\,(\,\mathscr{A}_1^{n-1}F_{A_n^*},\quad \mathscr{A}_1^{n-2}F_{A_{n-1}^*},\quad \cdots,\quad \mathscr{A}_1^1F_{A_2^*}\,) + r(A_1A_2\cdots A_n) - r(A_1)$$
$$= r\begin{pmatrix} A_n^* & O & \cdots & O \\ O & A_{n-1}^* & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^* \\ A_1\cdots A_{n-1} & A_1\cdots A_{n-2} & \cdots & A_1 \end{pmatrix} - \sum_{i=2}^{n} r(A_i) + r(A_1A_2\cdots A_n) - r(A_1)$$

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$$= r\begin{pmatrix} A_n^* & O & \cdots & O \\ O & A_{n-1}^* & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^* \\ A_1A_2\cdots A_{n-1} & A_1A_2\cdots A_{n-2} & \cdots & A_1 \end{pmatrix} + r(A_1A_2\cdots A_n) - \sum_{i=1}^{n} r(A_i). \quad \square$$

The next theorem is a direct result from Theorem 3.1 and the identity (3.1). It provides some equivalently necessary and sufficient conditions for the inclusion (1.4) to hold.

THEOREM 3.2. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \ldots, n$. Then the following statements are equivalent:*

(1). $A_n\{1,3\}A_{n-1}\{1,3\}\cdots A_1\{1,3\} \subseteq (A_1A_2\cdots A_n)\{1\}$;

(2). $r\,(\,A_1A_2\cdots A_{n-1}F_{A_n^*},\ A_1A_2\cdots A_{n-2}F_{A_{n-1}^*},\ \ldots,\ A_1F_{A_2^*}\,)$
$= r(A_1) - r(A_1\cdots A_n)$;

(3). $$r\begin{pmatrix} A_n^* & O & \cdots & O \\ O & A_{n-1}^* & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & A_2^* \\ A_1A_2\cdots A_{n-1} & A_1A_2\cdots A_{n-2} & \cdots & A_1 \end{pmatrix} = \sum_{i=1}^{n} r(A_i) - r(A_1A_2\cdots A_n).$$

EXAMPLE 3.3. Take

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We easily get

$$r\begin{pmatrix} A_3^* & O \\ O & A_2^* \\ A_1A_2 & A_1 \end{pmatrix} + r(A_1A_2A_3) = r(A_1) + r(A_2) + r(A_3) = 5.$$

This implies that the matrices A_1, A_2 and A_3 satisfy the condition (3) in Theorem 3.2. On the other hand, since (also by formula (2.13))

$$A_1\{1,3\} = \left\{ \begin{pmatrix} 1 & 0 \\ a_1 & b_1 \end{pmatrix} \mid a_1,\ b_1 \in \mathbb{C} \right\},$$

$$A_2\{1,3\} = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ a_2 & b_2 \end{pmatrix} \mid a_2,\ b_2 \in \mathbb{C} \right\}$$

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and

$$A_3\{1,3\} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a_3 & b_3 & c \\ -a_3 & 1/2-b_3 & 1/2-c \end{pmatrix} \mid a_3,\ b_3,\ c \in \mathbb{C} \right\},$$

we have

$$\begin{aligned} &A_3\{1,3\}A_2\{1,3\}A_1\{1,3\} = \{M_2 : \\ &M_2 = \begin{pmatrix} 1 & 0 & 0 \\ a_3 & b_3 & c \\ -a_3 & 1/2-b_3 & 1/2-c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_1 & b_1 \end{pmatrix} \mid a_i,\ b_j,\ c \in \mathbb{C}\}. \end{aligned}$$

Similar to Example 2.3, we can easily check that the identity

$$(A_1A_2A_3)M_2(A_1A_2A_3) = A_1A_2A_3$$

holds for any matrix $M_2 \in A_3\{1,3\}A_2\{1,3\}A_1\{1,3\}$. Hence,

$$A_3\{1,3\}A_2\{1,3\}A_1\{1,3\} \subseteq (A_1A_2A_3)\{1\}.$$

Again by the assertion that $G \in A\{1,4\}$ if and only if $G^* \in A^*\{1,3\}$, we can get the necessary and sufficient conditions for the inclusion (1.5) from the results obtained in Theorem 3.2.

THEOREM 3.4. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \ldots, n$. Then the following statements are equivalent:*

(1). $A_n\{1,4\}A_{n-1}\{1,4\} \cdots A_1\{1,4\} \subseteq (A_1A_2 \cdots A_n)\{1\}$;

(2). $r\begin{pmatrix} E_{A_1^*}A_2A_3 \cdots A_n \\ E_{A_2^*}A_3A_4 \cdots A_n \\ \vdots \\ E_{A_{n-1}^*}A_n \end{pmatrix} + r(A_1A_2 \cdots A_n) = r(A_n)$;

(3). $r\begin{pmatrix} A_1^* & O & \cdots & O & A_2A_3 \cdots A_n \\ O & A_2^* & \cdots & O & A_3A_4 \cdots A_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & A_{n-1}^* & A_n \end{pmatrix} + r(A_1A_2 \cdots A_n) = \sum_{i=1}^{n} r(A_i)$.

4. The necessary and sufficient conditions for the inclusions (1.6) and (1.7). In order to present the necessary and sufficient condition for the inclusion (1.6),

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we first give the maximum rank of matrix function $T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n)$ when each X_i $(i=1,2,\ldots,n)$ varies over the set $A_i\{1\}$ of all g-inverses of the matrix A_i.

THEOREM 4.1. *Let* $A_i \in \mathbb{C}^{l_i\times l_{i+1}}$, $i=1,\ldots,n$ *and* $T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n)$ *be as in* (1.9). *Then*

$$\begin{aligned}(4.1)\qquad &\max_{X_n,X_{n-1},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n))\\ &= \min\left\{ r(A_1A_2\cdots A_n), \sum_{m=1}^{n} l_m - \sum_{m=1}^{n} r(A_m)\right\},\end{aligned}$$

where X_i *varies over* $A_i\{1\}$ *for* $i=1,2,\ldots,n$.

Proof. Let $\mathscr{A}_i^j$ and $\mathscr{X}_i^j$, $1\le i\le j\le n$, be as in (1.8). Then for $2\le i\le n-1$ and $X_j\in A_j\{1\}$, $j=i,i+1,\ldots,n$, we first prove

$$\begin{aligned}(4.2)\qquad &\max_{X_i} r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_i^n)^*)\\ &= \min\{r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}),\quad r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*) + l_i - r(A_i)\}.\end{aligned}$$

By the formula (1.11) in Lemma 1.1 (with $A=A_i$, $B=I_{l_i}$, $C=(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*$, $D=(\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}$), we have

$$\begin{aligned}&\max_{X_i} r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_i^n)^*)\\ &= \min\left\{ r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*,\quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}\,),\ r\begin{pmatrix} I_{l_i}\\ (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}\end{pmatrix},\right.\\ &\qquad \left. r\begin{pmatrix} A_i & I_{l_i}\\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* & (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}\end{pmatrix} - r(A_i)\right\}\\ &= \min\left\{ r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*,\quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}\,),\ r\begin{pmatrix} I_{l_i}\\ (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}\end{pmatrix},\right.\\ &\qquad \left. r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*) + l_i - r(A_i)\right\}\\ &= \min\{r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}),\quad r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*) + l_i - r(A_i)\},\end{aligned}$$

i.e., (4.2) holds, where the second equality holds as

$$\begin{aligned}&r\begin{pmatrix} A_i & I_{l_i}\\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* & (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}\end{pmatrix}\\ &\qquad = r\begin{pmatrix} O & I_{l_i}\\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* - (\mathscr{A}_1^n)^*\mathscr{A}_1^{i} & O\end{pmatrix}.\end{aligned}$$

The last equality holds as

$$r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*,\quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}\,) = r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1})$$

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and

$$r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}) \leq r(A_{i-1}) \leq l_i = r\begin{pmatrix} I_{l_i} \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} \end{pmatrix}.$$

When $i = n$, again by Lemma 1.1 (1.11) with $A = A_n$, $B = I_{l_n}$, $D = (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}$ and $C = (\mathscr{A}_1^n)^*\mathscr{A}_1^n$, we have

$$\begin{aligned}(4.3)\qquad &\max_{X_n} r((\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_n^n)^*) \\ &= \min\left\{ r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n,\quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}\,),\quad r\begin{pmatrix} I_{l_n} \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1} \end{pmatrix},\right. \\ &\left. r\begin{pmatrix} A_n & I_{l_n} \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n & (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1} \end{pmatrix} - r(A_n)\right\} \\ &= \min\{r((\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}),\quad l_n - r(A_n)\}\end{aligned}$$

in which the last equality holds since

$$r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n,\quad (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}\,) = r((\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}),$$

$$r((\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}) \leq r(A_{n-1}) \leq l_n = r\begin{pmatrix} I_{l_n} \\ (\mathscr{A}_1^n)^*A\mathscr{A}_1^{n-1} \end{pmatrix}$$

and

$$r\begin{pmatrix} A_n & I_{l_n} \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n & (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1} \end{pmatrix} = r\begin{pmatrix} I_{l_n} \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1} \end{pmatrix}.$$

We now prove (4.1). According to Lemma 1.1 (1.11) with $A = A_1$, $B = I_{l_1}$, $C = (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^*$ and $D = (\mathscr{A}_1^n)^*$, we have

$$\begin{aligned}(4.4)\qquad &\max_{X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) \\ &= \min\left\{ r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^*,\quad (\mathscr{A}_1^n)^*\,), r\begin{pmatrix} I_{l_1} \\ (\mathscr{A}_1^n)^* \end{pmatrix},\right. \\ &\left. r\begin{pmatrix} A_1 & I_{l_1} \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^* & (\mathscr{A}_1^n)^* \end{pmatrix} - r(A_1)\right\} \\ &= \min\{r(\mathscr{A}_1^n),\quad r((\mathscr{A}_1^n)^*\mathscr{A}_1^1 - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^*) + l_1 - r(A_1)\},\end{aligned}$$

where the last equality holds as

$$r\,(\,(\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^*,\quad (\mathscr{A}_1^n)^*\,) = r(\mathscr{A}_1^n) \leq r\begin{pmatrix} I_{l_1} \\ (\mathscr{A}_1^n)^* \end{pmatrix}$$

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and

$$r\begin{pmatrix} A_1 & I_{l_1} \\ (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^* & (\mathscr{A}_1^n)^* \end{pmatrix} = r((\mathscr{A}_1^n)^*\mathscr{A}_1^1 - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^*) + l_1.$$

Combining (4.2) and (4.4), we have

$$\begin{aligned}
&\max_{X_2,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n)) \\
&= \min\{r(\mathscr{A}_1^n),\ \ \max_{X_2} r((\mathscr{A}_1^n)^*\mathscr{A}_1^1 - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_2^n)^*) + l_1 - r(A_1)\} \\
&= \min\{r(\mathscr{A}_1^n),\ \ r((\mathscr{A}_1^n)^*\mathscr{A}_1^1) + l_1 - r(A_1),\ r((\mathscr{A}_1^n)^*\mathscr{A}_1^2 - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_3^n)^*) \\
&\quad + l_2 + l_1 - r(A_2) - r(A_1)\} \\
&= \min\{r(\mathscr{A}_1^n),\ \ r((\mathscr{A}_1^n)^*\mathscr{A}_1^2 - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_3^n)^*) + l_2 + l_1 - r(A_2) - r(A_1)\},
\end{aligned}$$

in which the last equality holds since from Lemma 1.3, we have

$$r((\mathscr{A}_1^n)^*\mathscr{A}_1^1) + l_1 - r(A_1) \geq r((\mathscr{A}_1^n)^*) + r(A_1) - r(A_1) = r(\mathscr{A}_1^n).$$

We contend that, for $2 \leq i \leq n-1$,

$$\begin{aligned}
(4.5)\quad &\max_{X_i,X_{i-1},\ldots,X_1} r(S_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n)) \\
&= \min\left\{r(\mathscr{A}_1^n),\ \ r\left((\mathscr{A}_1^n)^*\mathscr{A}_1^i - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*\right) + \sum_{m=1}^{i} l_m - \sum_{m=1}^{i} r(A_m)\right\}.
\end{aligned}$$

We proceed by induction on i. For $i = 2$, the equality relation (4.5) has been proved. Assume that (4.5) is true for $i-1$ ($i \geq 3$), that is

$$\begin{aligned}
(4.6)\quad &\max_{X_{i-1},X_{i-2},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n)) \\
&= \min\left\{r(\mathscr{A}_1^n),\ \ r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_i^n)^*) + \sum_{m=1}^{i-1} l_m - \sum_{m=1}^{i-1} r(A_m)\right\}.
\end{aligned}$$

We now prove that (4.5) is also true for i. By (4.2) and (4.6), we have

$$\begin{aligned}
&\max_{X_i,X_{i-1},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1,X_2,\ldots,X_n)) \\
&= \min\{r(\mathscr{A}_1^n),\ \ \max_{X_i} r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_i^n)^*) + \sum_{m=1}^{i-1} l_m - \sum_{m=1}^{i-1} r(A_m)\} \\
&= \min\{r(\mathscr{A}_1^n),\ \ r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}) + \sum_{m=1}^{i-1} l_m - \sum_{m=1}^{i-1} r(A_m), \\
&\quad r((\mathscr{A}_1^n)^*\mathscr{A}_1^i - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^*) + l_i - r(A_i) + \sum_{m=1}^{i-1} l_m - \sum_{m=1}^{i-1} r(A_m)\}.
\end{aligned}$$

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From Lemma 1.3, we know

$$r((\mathscr{A}_1^n)^*\mathscr{A}_1^{i-1}) + \sum_{m=1}^{i-1} l_m \geq r(\mathscr{A}_1^n) + \sum_{m=1}^{i-1} r(A_m),$$

thus

$$\max_{X_i, X_{i-1},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n))$$
$$= \min\{r(\mathscr{A}_1^n),\ \ r\left((\mathscr{A}_1^n)^*\mathscr{A}_1^i - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_{i+1}^n)^* \right) + \sum_{m=1}^{i} l_m - \sum_{m=1}^{i} r(A_m)\}.$$

In particular, when $i = n-1$, we have

$$\max_{X_{n-1}, X_{n-2},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) \tag{4.7}$$
$$= \min\{r(\mathscr{A}_1^n),\ \ r((\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_n^n)^*) + \sum_{m=1}^{n-1} l_m - \sum_{m=1}^{n-1} r(A_m)\}.$$

On account of (4.3) and (4.7), it is seen that

$$\max_{X_n, X_{n-1},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n))$$
$$= \min\{r(\mathscr{A}_1^n),\ \ \max_{X_n} r((\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1} - (\mathscr{A}_1^n)^*\mathscr{A}_1^n(\mathscr{X}_n^n)^*) + \sum_{m=1}^{n-1} l_m - \sum_{m=1}^{n-1} r(A_m)\}$$
$$= \min\{r(\mathscr{A}_1^n),\ \ r((\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}) + \sum_{m=1}^{n-1} l_m - \sum_{m=1}^{n-1} r(A_m),$$
$$l_n - r(A_n) + \sum_{m=1}^{n-1} l_m - \sum_{m=1}^{n-1} r(A_m)\}.$$

Noting that

$$r((\mathscr{A}_1^n)^*\mathscr{A}_1^{n-1}) + \sum_{m=1}^{n-1} l_m \geq r(\mathscr{A}_1^n) + \sum_{m=1}^{n-1} r(A_m),$$

we finally have

$$\max_{X_n, X_{n-1},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n))$$
$$= \min\{r(A_1A_2\cdots A_n), \sum_{m=1}^{n} l_m - \sum_{m=1}^{n} r(A_m)\}. \quad \square$$

Since the inclusion (1.6) holds if and only if

$$\max_{X_n, X_{n-1},\ldots,X_1} r(T_{A_1,A_2,\ldots,A_n}(X_1, X_2, \ldots, X_n)) = 0,$$

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by Theorem 4.1 and Lemma 1.2 (1.14), we can immediately obtain the following result:

Theorem 4.2. *Let $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \ldots, n$. Then the inclusion* (1.6) *holds if and only if*

$$\min\{r(A_1A_2\cdots A_n), \sum_{m=1}^{n} l_m - \sum_{m=1}^{n} r(A_m)\} = 0,$$

that is,

$$A_1A_2\cdots A_n = O \quad or \quad \sum_{m=1}^{n} l_m = \sum_{m=1}^{n} r(A_m), \;\; i.e., \;\; r(A_i) = l_i, \quad i = 1, 2, \ldots, n.$$

Example 4.3. Let

$$A_1 = (1, 0), \; A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } A_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then $r(A_1) = 1$, $r(A_2) = 2$ and $r(A_3) = 3$, and the ranks of these matrices satisfy the conditions in Theorem 4.2.

On the other hand, by the definition of $\{1\}$-inverse, we have

$$A_3\{1\} = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 - a_1 & -a_2 & -a_3 \end{pmatrix} \mid a_1, \; a_2, \; a_3 \in \mathbb{C} \right\},$$

$$A_2\{1\} = \left\{ \begin{pmatrix} b_1 & b_2 \\ 0 & 1 \\ 1 - b_1 & -1 - b_2 \end{pmatrix} \mid b_1, \; b_2 \in \mathbb{C} \right\} \;\; \text{and} \;\; A_1\{1\} = \left\{ \begin{pmatrix} 1 \\ c_1 \end{pmatrix} \mid c_1 \in \mathbb{C} \right\}.$$

Hence

$$A_3\{1\}A_2\{1\}A\{1\} = \{M_3 :$$
$$M_3 = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 - a_1 & -a_2 & -a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & 1 \\ 1 - b_1 & -1 - b_2 \end{pmatrix} \begin{pmatrix} 1 \\ c_1 \end{pmatrix} \mid a_i, \; b_j, \; c_k \in \mathbb{C}\}.$$

It is easy to verify that if $M_3 \in A_3\{1\}A_2\{1\}A\{1\}$, then

$$(A_1A_2A_3)M_3(A_1A_2A_3) = A_1A_2A_3 \;\; \text{and} \;\; (A_1A_2A_3M_3)^* = A_1A_2A_3M_3.$$

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Hence

$$A_3\{1\}A_2\{1\}A_1\{1\} \subseteq (A_1A_2A_3)\{1,3\}.$$

By Lemma 1.2 (1.14) and (1.15), $G \in A\{1,4\}$ if and only if $G^* \in A^*\{1,3\}$. So from the results obtained in Theorem 4.2, we can get the necessary and sufficient conditions for the inclusion (1.7).

THEOREM 4.4. *Let* $A_i \in \mathbb{C}^{l_i \times l_{i+1}}$, $i = 1, 2, \ldots, n$. *Then the inclusion* (1.7) *holds if and only if*

$$\min\{r(A_1A_2\cdots A_n), \sum_{m=2}^{n+1} l_m - \sum_{m=1}^{n} r(A_m)\} = 0,$$

that is,

$$A_1A_2\cdots A_n = O \quad or \quad \sum_{m=2}^{n+1} l_m = \sum_{m=1}^{n} r(A_m), \; i.e., \; \; r(A_i) = l_{i+1}, \quad i = 1, 2, \ldots, n.$$

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REFERENCES

[1] D.T. Barwick and J.D. Gilbert. On generalizations of the reverse order law. *SIAM J. Appl. Math.*, 27:326–330, 1974.
[2] A. Ben-Israel and T.N.E. Greville. *Generalized Inverses: Theory and Applications*. Wiley-Interscience, 1974; 2nd Edition, Springer-Verlag, New York, 2002.
[3] D.S. Djordjević. Further results on the reverse order law for generalized inverses. *SIAM J. Matrix Anal. Appl.*, 29(4):1242–1246, 2007.
[4] T.N.E. Greville. Note on the generalized inverse of a matrix product. *SIAM Rev.*, 8:518–521, 1966.
[5] R.E. Hartwig. The reverse order law revisited. *Linear Algebra Appl.*, 76:241–246, 1986.
[6] Q. Liu and M. Wei. Reverse order law for least squares g-inverses of multiple matrix products. *Linear Multilinear Algebra*, 56(5):491-506, 2008.
[7] G. Marsaglia and G.P.H. Styan. Equalities and inequalities for ranks of matrices. *Linear Multilinear Algebra*, 2:269–292, 1974.
[8] W. Sun and Y. Wei. Inverse order rule for weighted generalized inverse. *SIAM J. Matrix Anal. Appl.*, 19:772–775, 1998.
[9] W. Sun and Y. Wei. Triple reverse-order law for weighted generalized inverses. *Appl. Math. Comput.*, 125:221–229, 2002.
[10] Y. Tian. Reverse order laws for the generalized inverses of multiple matrix products. *Linear Algebra Appl.*, 211:85–100, 1994.

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[11] Y. Tian. Upper and lower bounds for ranks of matrix expressions using generalized inverses. *Linear Algebra Appl.*, 355:187–214, 2002.

[12] Y. Tian. More on maximal and minimal ranks of Schur complements with applications. *Appl. Math. Comput.*, 152:675–692, 2004.

[13] Y. Tian. On mixed-type reverse order laws for the Moore-Penrose inverse of a matrix product. *Int. J. Math. Math. Sci.*, 58:3103–3116, 2004.

[14] Y. Tian and Y. Liu. On a group of mixed-type reverse order laws for generalized inverses of a triple matrix product with applications. *Electron. J. Linear Algebra*, 16:73–89, 2007.

[15] G. Wang, Y. Wei, and S. Qiao. *Generalized Inverses: Theory and Computations*. Science Press, Beijing, 2004.

[16] G. Wang and B. Zheng. The reverse order law for the generalized inverse $A^{(2)}_{T,S}$. *Appl. Math. Comput.*, 157:295–305, 2004.

[17] M. Wei. Reverse order laws for generalized inverse of multiple matrix products. *Linear Algebra Appl.*, 293:273–288, 1999.

[18] Z. Xiong and C. Jiang. The mixed-type reverse order laws for weighted generalized inverses of a triple matrix product. *J. Appl. Math. Comput.*, 29:53–66, 2009.

[19] Z. Xiong and B. Zheng. The reverse order laws for $\{1,2,3\}$- and $\{1,2,4\}$-inverses of two matrix product. *Appl. Math. Lett.*, 21:649–655, 2008.

[20] B. Zheng and Z. Xiong. A new equivalent condition of the reverse order law for G-inverses of multiple matrix products. *Electron. J. Linear Algebra*, 17:1–8, 2008.

[21] B. Zheng and Z. Xiong. The reverse order laws for $\{1,2,3\}$- and $\{1,2,4\}$-inverses of multiple matrix products. *Linear Multilinear Algebra*, 58(6):765–782, 2010.