Electronic Journal of Linear Algebra

Volume 23 *ELA Volume 23 (2012)*

Article 1

2012

Sign patterns that allow strong eventual nonnegativity

M. Catral

C. Erickson

L. Hogben
hogben@aimath.org

D. D. Olesky

P. van den Driessche

Recommended Citation

Catral, M.; Erickson, C.; Hogben, L.; Olesky, D. D.; and van den Driessche, P.. (2012), "Sign patterns that allow strong eventual nonnegativity", *Electronic Journal of Linear Algebra*, Volume 23.
DOI: https://doi.org/10.13001/1081-3810.1502

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 1-10, January 2012

ELA
http://math.technion.ac.il/iic/ela

SIGN PATTERNS THAT ALLOW STRONG EVENTUAL NONNEGATIVITY*

M. CATRAL†, C. ERICKSON‡, L. HOGBEN§, D.D. OLESKY¶, AND P. VAN DEN DRIESSCHE‖

Abstract. A new class of sign patterns contained in the class of sign patterns that allow eventual nonnegativity is introduced and studied. A sign pattern is *potentially strongly eventually nonnegative* (PSEN) if there is a matrix with this sign pattern that is eventually nonnegative and has some power that is both nonnegative and irreducible. Using Perron-Frobenius theory and a matrix perturbation result, it is proved that a PSEN sign pattern is either potentially eventually positive or r-cyclic. The minimum number of positive entries in an $n \times n$ PSEN sign pattern is shown to be n, and PSEN sign patterns of orders 2 and 3 are characterized.

Key words. Potentially eventually nonnegative, Potentially strongly eventually nonnegative, Sign pattern.

AMS subject classifications. 15B35, 15B48, 15A18.

1. Introduction.

A *sign pattern (matrix)* is a matrix having entries in $\{+, -, 0\}$. For a real matrix A, $\text{sgn}(A)$ is the sign pattern having entries that are the signs of the corresponding entries in A. If $\mathcal{A}$ is an $n \times n$ sign pattern, then the *sign pattern class* of $\mathcal{A}$, denoted $\mathcal{Q}(\mathcal{A})$, is the set of all $A \in \mathbb{R}^{n\times n}$ such that $\text{sgn}(A) = \mathcal{A}$. If $\mathscr{P}$ is a property of a real matrix, then a sign pattern $\mathcal{A}$ *requires* $\mathscr{P}$ if every real matrix $A \in \mathcal{Q}(\mathcal{A})$ has property $\mathscr{P}$, and $\mathcal{A}$ *allows* $\mathscr{P}$ or is *potentially* $\mathscr{P}$ if there is some $A \in \mathcal{Q}(\mathcal{A})$ that has property $\mathscr{P}$. Numerous properties have been investigated from the point of view of characterizing sign patterns that require or allow a particular property (see, e.g., [5, 9] and the references therein). Here we focus on the property of eventual nonnegativity, which we now define.

*Received by the editors on September 13, 2011. Accepted for publication on December 26, 2011. Handling Editor: Michael Tsatsomeros.

†Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH 45207, USA (catralm@xavier.edu).
‡Department of Mathematics, Iowa State University, Ames, IA 50011, USA (craig@iastate.edu).
§Department of Mathematics, Iowa State University, Ames, IA 50011, USA (lhogben@iastate.edu), and American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306, USA (hogben@aimath.org).
¶Department of Computer Science, University of Victoria, Victoria BC V8W 3P6, Canada (dolesky@cs.uvic.ca).
‖Department of Mathematics and Statistics, University of Victoria, Victoria BC V8W 3R4, Canada (pvdd@math.uvic.ca).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 1-10, January 2012

ELA
http://math.technion.ac.il/iic/ela

A matrix $A \in \mathbb{R}^{n\times n}$ is *eventually nonnegative* (resp., *eventually positive*) if there exists a positive integer k_0, the *power index* of A, such that for all $k \geq k_0$, $A^k \geq 0$ (resp., $A^k > 0$). Here inequalities are entrywise and all matrices are real and square unless otherwise stated. The *spectral radius* of a matrix A is denoted by $\rho(A)$, and an eigenvalue $\lambda \in \operatorname{spec}(A)$ is a *dominant eigenvalue* if $|\lambda| = \rho(A)$. A matrix A has the *strong Perron-Frobenius property* if A has a unique dominant eigenvalue that is positive, simple, and has a positive eigenvector. It is well known [10] that the set of matrices for which both A and A^T have the strong Perron-Frobenius property coincides with the set of eventually positive matrices. Eventually nonnegative matrices, which were introduced by Friedland [7], and eventually positive matrices, which have nice Perron-Frobenius structure, have applications to positive control theory (see, e.g., [12]).

Sign patterns that require eventual positivity or eventual nonnegativity were characterized in [6]. *Potentially eventually positive* (PEP) sign patterns were studied in [1], where several necessary or sufficient conditions are given for a sign pattern to be PEP, and orders 2 and 3 PEP sign patterns are characterized.

Much less is known about whether a sign pattern is *potentially eventually nonnegative* (PEN) as compared with whether it is PEP. The study of PEP sign patterns relies on the Perron-Frobenius eigenstructure of a positive matrix, and irreducible nonnegative matrices retain significant Perron-Frobenius properties (for discussion of Perron-Frobenius theory of nonnegative matrices, see, e.g., [2]). Motivated by this, we define a strongly eventually nonnegative (SEN) matrix below, for the purpose of studying sign patterns that allow such matrices.

Definition 1.1. A matrix $A \in \mathbb{R}^{n\times n}$ is *strongly eventually nonnegative* (SEN) if A is eventually nonnegative and there is some power of A that is both nonnegative and irreducible.

We note that the class of SEN matrices has been discussed in the literature (see, e.g., [13]), but not named. A test for a matrix to be in this class was given recently in [11]. Clearly an eventually positive matrix is strongly eventually nonnegative, and a matrix that is strongly eventually nonnegative is irreducible and not nilpotent. The nilpotent matrix $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ is eventually nonnegative but not SEN. The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is nonnegative and irreducible, so it is SEN, but not eventually positive. Additional examples are given below (e.g., Examples 2.2 and 2.3). In [13], it is shown that a real matrix A is eventually positive if and only if A^k is both irreducible and nonnegative for all sufficiently large k.

Our primary focus is the study of sign patterns that allow strongly eventually

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 1-10, January 2012

ELA
http://math.technion.ac.il/iic/ela

nonnegative matrices (PSEN sign patterns). In Section 3 we use matrix perturbation theory to show that PSEN sign patterns are either PEP or r-cyclic. This result is then applied to characterize PSEN sign patterns for orders 2 and 3. Section 2 presents properties of strongly eventually nonnegative matrices that are used in Section 3; these properties are consequences of well known properties of irreducible nonnegative matrices.

2. Matrix background. If p and q are relatively prime, then every integer m such that $m \geq (p-1)(q-1)$ is a nonnegative linear combination of p and q (see, e.g., [3, Lemma 3.5.5]). Thus, if $A^p, A^q \geq 0$, where p and q are relatively prime, then A is eventually nonnegative. In particular, if $A^k, A^{k+1} \geq 0$ for some positive integer k, then A is eventually nonnegative (the analogous result for eventual positivity is well known).

Let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be a multiset of complex numbers. The *radius* of σ is $\rho(\sigma) = \max_{i=1}^n |\lambda_i|$, $\overline{\sigma} = \{\overline{\lambda_1}, \ldots, \overline{\lambda_n}\}$, and the set σ is *self-conjugate* if $\overline{\sigma} = \sigma$. The value $\lambda_i \in \sigma$ is *dominant* if $|\lambda_i| = \rho(\sigma)$. Let r be the number of dominant values in σ, and let $\omega = e^{2\pi i/r}$. The set σ is a *Frobenius multiset* [7] if $\rho(\sigma) > 0$, the set of dominant values of σ is $\{\rho(\sigma), \rho(\sigma)\omega, \ldots, \rho(\sigma)\omega^{r-1}\}$, and $\omega\sigma = \sigma$. If a matrix is eventually nonnegative and not nilpotent, then its spectrum is a union of self-conjugate Frobenius multisets [7, 13].

PROPOSITION 2.1. *Let A be a strongly eventually nonnegative matrix. Then $\rho(A)$ is a simple eigenvalue of A having positive left and right eigenvectors. With the notation $\rho = \rho(A)$, $r =$ the number of dominant eigenvalues of A, and $\omega = e^{2\pi i/r}$, the dominant eigenvalues of A are $\{\rho, \rho\omega, \ldots, \rho\omega^{r-1}\}$.*

Proof. Because $A^k \geq 0$ and irreducible for some $k > 0$, $\rho(A^k)$ is a simple eigenvalue of A^k having positive left and right eigenvectors. Thus, ρ is a simple eigenvalue of A, so in the expression for the spectrum of A as a union of Frobenius multisets [7, 13], the dominant eigenvalues must come from a single Frobenius multiset. ∎

For $r \geq 2$, a matrix $A \in \mathbb{R}^{n\times n}$ (or sign pattern) is called *r-cyclic* if there exists a permutation matrix P such that PAP^T has the block form

$$\begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{r-1,r} \\ A_{r1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{2.1}$$

where the diagonal 0 blocks are square (and the blocks A_{r1} and $A_{i,i+1}, i = 1, \ldots, r-1$ may be rectangular). If A is r-cyclic for some $r \geq 2$, then the *cyclic index* of A is the

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 1-10, January 2012

ELA
http://math.technion.ac.il/iic/ela

largest r for which A is r-cyclic; otherwise the cyclic index of A is 1. The definition of cyclic index of a sign pattern is analogous. Clearly an r-cyclic matrix is not eventually positive. Irreducible r-cyclic matrices provide a variety of interesting examples related to strong eventual nonnegativity.

Example 2.2. Let

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then ${A_1}^2$ and ${A_1}^3$ are nonnegative so A_1 is eventually nonnegative. Since ${A_1}^3$ is also irreducible, A_1 is strongly eventually nonnegative. Clearly A_1 is 2-cyclic, so not eventually positive.

Example 2.3. The matrix

$$A_2 = \begin{bmatrix} -2 & 1 & 0 & 1 & 1 & 1 \\ -4 & 2 & 0 & 2 & 2 & 2 \\ 4 & -2 & 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 \end{bmatrix}$$

is strongly eventually nonnegative because ${A_2}^k \geq 0$ for $k \geq 2$ and ${A_2}^3$ is irreducible. Since ${A_2}^2$ is reducible, A_2 is not eventually positive. Note that A_2 is not r-cyclic.

The matrix A_2 in Example 2.3, although not r-cyclic, has the property that the power $(A_2)^k$ behaves as the kth power of a 2-cyclic matrix for $k \geq 3$. The idea of powering up the matrix to eliminate a nilpotent part outside the cyclic structure is evident in [13, Example 6.1] and [4, Example 4.9]. Properties of nonnegative matrices show that what happens in Examples 2.2 and 2.3 is essentially the only way to have a strongly eventually nonnegative matrix that is not eventually positive. Let A be a strongly eventually nonnegative matrix with r dominant eigenvalues. If $r = 1$, then A is eventually positive. If $r \geq 2$, then for $k \geq n$ and $k \equiv 1 \mod r$, A^k is r-cyclic [11], i.e., A^k is permutationally similar to a matrix of the form (2.1).

Definition 2.4. An $n \times n$ strongly eventually nonnegative matrix having $r \geq 2$ dominant eigenvalues is a *canonical strongly eventually nonnegative matrix* if for $k \geq n$ and $k \equiv 1 \mod r$, the r-cyclic matrix A^k is of the form (2.1) (i.e., a permutation similarity is not needed).

Let $A, E \in \mathbb{C}^{n \times n}$. If λ is a simple eigenvalue of A, $A\mathbf{x} = \lambda\mathbf{x}$, and $\mathbf{y}^* A = \lambda \mathbf{y}^*$, then $(\mathbf{y}, \lambda, \mathbf{x})$ is called an *eigentriplet* of A. If $(\mathbf{y}, \lambda, \mathbf{x})$ is an eigentriplet of A (so λ is

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 1-10, January 2012

ELA
http://math.technion.ac.il/iic/ela

a simple eigenvalue of A) and we define $A(\varepsilon) = A + \varepsilon E$, then in a neighborhood of the origin there exist differentiable (and thus continuous) functions $\lambda(\varepsilon)$, $\mathbf{x}(\varepsilon)$, and $\mathbf{y}(\varepsilon)$ such that $(\mathbf{y}(\varepsilon), \lambda(\varepsilon), \mathbf{x}(\varepsilon))$ is an eigentriplet of $A(\varepsilon)$ and

$$\lambda(\varepsilon) \approx \lambda + \varepsilon \frac{\mathbf{y}^* E \mathbf{x}}{\mathbf{y}^* \mathbf{x}} \tag{2.2}$$

(see, e.g., [8, p. 323]). The following matrix perturbation result, which is used in the next section, is straightforward to prove.

Proposition 2.5. *Let A be a canonical strongly eventually nonnegative matrix with $\rho(A) = 1$, $r \geq 2$ dominant eigenvalues, and eigentriplet $(\mathbf{y} = [y_i], 1, \mathbf{x} = [x_j])$ with $\mathbf{x}, \mathbf{y} > 0$, and let $\omega = e^{2\pi i/r}$. Choose $k \equiv 1 \mod r$ such that $A^k \geq 0$, and thus, $B = A^k$ is in the block form* (2.1) *with the blocks denoted by B_{pq}.*

i. Partition $\mathbf{x}$ and $\mathbf{y}$ conformally with the r-cyclic structure (2.1) *of $B = A^k$ as $\mathbf{x}^T = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T & \cdots & \mathbf{x}_r^T \end{bmatrix}$ and $\mathbf{y}^T = \begin{bmatrix} \mathbf{y}_1^T & \mathbf{y}_2^T & \cdots & \mathbf{y}_r^T \end{bmatrix}$.
For $s = 0, \ldots, r-1$, define*

$$\mathbf{x}^{(s)} = \begin{bmatrix} \mathbf{x_1}^T & \omega^s \mathbf{x_2}^T & \omega^{2s} \mathbf{x_3}^T & \cdots & \omega^{(r-2)s} \mathbf{x}_{r-1}{}^T & \omega^{(r-1)s} \mathbf{x}_r{}^T \end{bmatrix}^T,$$
$$\mathbf{y}^{(s)} = \begin{bmatrix} \mathbf{y_1}^T & \omega^s \mathbf{y_2}^T & \omega^{2s} \mathbf{y_3}^T & \cdots & \omega^{(r-2)s} \mathbf{y}_{r-1}{}^T & \omega^{(r-1)s} \mathbf{y}_r{}^T \end{bmatrix}^T.$$

Then for $s = 0, \ldots, r-1$, $(\mathbf{y}^{(s)}, \omega^s, \mathbf{x}^{(s)})$ is an eigentriplet for A and $\mathbf{y}^{(s)*} \mathbf{x}^{(s)} = \mathbf{y}^T \mathbf{x}$.

ii. For $b \geq 1$, let $E = E_{i_1,j_1} + \cdots + E_{i_b,j_b}$, where E_{ij} denotes the matrix with the (i,j)-entry equal to one and zeros everywhere else, and $a_{i_u,j_u} \neq 0$ is in block A_{g_u,h_u} for $u = 1, \ldots, b$. Define $A(\varepsilon) = A + \varepsilon E$. For $s = 0, 1, \ldots, r-1$, let $\omega_s(\varepsilon)$ denote the eigenvalue of $A(\varepsilon)$ that for sufficiently small ε is closest to eigenvalue ω^s of A. Then

$$\omega_s(\varepsilon) \approx \omega^s + \sum_{u=1}^{b} \omega^{s(h_u - g_u)} \varepsilon \frac{y_{i_u} x_{j_u}}{\mathbf{y}^T \mathbf{x}}$$

for $s = 0, 1, \ldots, r-1$.

3. Sign patterns that allow strongly eventually nonnegative matrices. The solution to the “requires” problem for strongly eventually nonnegative matrices is immediate: If an irreducible pattern $\mathcal{A}$ requires eventual nonnegativity, then $\mathcal{A}$ is nonnegative [6, Corollary 2.2]. Thus, a sign pattern requires strong eventual nonnegativity if and only if it is an irreducible nonnegative pattern. Here, we consider the “allows” problem for the class of strongly eventually nonnegative matrices.

Definition 3.1. A sign pattern $\mathcal{A}$ is *potentially strongly eventually nonnegative* (PSEN) if there is a matrix $A \in \mathcal{Q}(\mathcal{A})$ that is strongly eventually nonnegative.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 1-10, January 2012

ELA
http://math.technion.ac.il/iic/ela

If $\mathcal{A}$ is PSEN, then $\mathcal{A}$ is irreducible, and the following is an immediate consequence of Proposition 2.1.

Observation 3.2. If $\mathcal{A}$ is PSEN, then every row of $\mathcal{A}$ has at least one $+$ and every column of $\mathcal{A}$ has at least one $+$.

Remark 3.3. The minimum number of $+$ entries in an $n \times n$ PSEN sign pattern is n, because n are necessary by Observation 3.2 and the sign pattern that has an n-cycle of $+$ entries and all other entries zero is PSEN.

The next theorem is our main result about the structure of a PSEN sign pattern.

Theorem 3.4. *If $\mathcal{A}$ is PSEN and the cyclic index of $\mathcal{A}$ is 1, then $\mathcal{A}$ is PEP.*

Proof. Suppose $\mathcal{A}$ is a PSEN sign pattern and the cyclic index of $\mathcal{A}$ is 1. Since $\mathcal{A}$ is PSEN, there exists $A \in \mathcal{Q}(\mathcal{A})$ such that A is SEN. Let r be the number of dominant eigenvalues of A. If $r = 1$ then A is eventually positive [10] and $\mathcal{A}$ is PEP, so assume $r \geq 2$. We show that we can find a perturbation $A(\varepsilon) \in \mathcal{Q}(\mathcal{A})$ of A that is eventually positive, so $\mathcal{A}$ is PEP. By multiplying A by a scalar, assume $\rho(A) = 1$. Choose $k > k_0$ (where k_0 is the power index of A) and $k \equiv 1 \mod r$. Then $A^k \geq 0$ and the dominant eigenvalues of A^k are distinct (in fact, they are the same as the dominant eigenvalues for A), so A^k is irreducible and r-cyclic. There exists a permutation matrix P such that PA^kP^T has the form (2.1). Let $A' = PAP^T$, so $(A')^k = PA^kP^T$, and $\mathcal{A}' = P\mathcal{A}P^T$. Partition $A' = [A'_{pq}]$ as a block matrix conformally with $(A')^k$.

Let $p_1, \ldots, p_b$ be the distinct primes that divide r. For each $u = 1, \ldots, b$, $\mathcal{A}'$ is not p_u-cyclic (because the cyclic index of $\mathcal{A}'$ is 1). Therefore there exists a nonzero block $\mathcal{A}'_{g_u,h_u}$ with $h_u - g_u \not\equiv 1 \mod p_u$. Let the row and column indices of a nonzero element of $\mathcal{A}'_{g_u,h_u}$ be denoted by i_u and j_u. Define $E = E_{i_1,j_1} + \cdots + E_{i_b,j_b}$. Adopt the notation of Proposition 2.5 applied to A', with $\omega = e^{2\pi i/r}$, $s = 0, \ldots, r-1$, and eigentriplets $(\mathbf{y}^{(s)}, \omega^s, \mathbf{x}^{(s)})$. Then for small $\varepsilon > 0$ the eigenvalues of $A'(\varepsilon) = A' + \varepsilon E$ are

$$\omega_s(\varepsilon) \approx \omega^s + \sum_{u=1}^{b} \omega^{s(h_u-g_u)} \varepsilon \frac{y_{i_u} x_{j_u}}{\mathbf{y}^T\mathbf{x}}.$$

For each $s = 1, \ldots, r-1$, there exists u such that $p_u | \frac{r}{d}$ where $d = \gcd(s, r)$. Since $p_u \not| \, (h_u - g_u - 1)$, it follows that $(\frac{r}{d}) \not| \, (h_u - g_u - 1)$. Therefore, $r \not| \, d(h_u - g_u - 1)$. Since $d = \gcd(s, r)$, $r \not| \, s(h_u - g_u - 1)$. Therefore, $\omega^{s(h_u-g_u)} \neq \omega^s$. By the triangle inequality applied to the scalars $\omega^s + \sum_{u=1}^{b} \omega^{s(h_u-g_u)} \varepsilon \frac{y_{i_u} x_{j_u}}{\mathbf{y}^T\mathbf{x}}$ viewed as vectors in the complex plane,

$$1 + \sum_{u=1}^{b} \varepsilon \frac{y_{i_u} x_{j_u}}{\mathbf{y}^T\mathbf{x}} > \left| \omega^s + \sum_{u=1}^{b} \omega^{s(h_u-g_u)} \varepsilon \frac{y_{i_u} x_{j_u}}{\mathbf{y}^T\mathbf{x}} \right|$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 1-10, January 2012

ELA
http://math.technion.ac.il/iic/ela

for $s = 1, \dots, r-1$. Therefore, ε_0 can be chosen so that $\omega_0(\varepsilon_0)$ is strictly greater than the absolute value of each eigenvalue $\omega_s(\varepsilon_0)$ for $s \geq 1$, $\mathbf{x}^{(0)}(\varepsilon_0) > 0$, $\mathbf{y}^{(0)}(\varepsilon_0) > 0$, and $A'(\varepsilon_0) \in \mathcal{Q}(\mathcal{A}')$. Thus, $A'(\varepsilon_0) = A' + \varepsilon_0 E_{ij}$ has a unique dominant eigenvalue that is positive and simple, and has positive left and right eigenvectors, so $A'(\varepsilon_0)$ is eventually positive. Thus, $\mathcal{A}'$ is PEP, and since $\mathcal{A}$ is permutation similar to $\mathcal{A}'$, $\mathcal{A}$ is PEP. □

Corollary 3.5. *If $\mathcal{A}$ is PSEN, then $\mathcal{A}$ is either PEP or r-cyclic for some $r \geq 2$.*

The behavior of matrices differs from that of sign patterns in this regard: The matrix A_2 in Example 2.3 is an SEN matrix that is neither eventually positive nor r-cyclic. The next example exhibits an eventually positive matrix $\tilde{A}_2 \in \mathcal{Q}(\mathrm{sgn}(A_2))$, as required by Corollary 3.5.

Example 3.6. The matrix

$$\tilde{A}_2 = \begin{bmatrix} -1.9 & 1 & 0 & 1 & 1 & 1 \\ -4 & 2 & 0 & 2 & 2 & 2 \\ 4 & -2 & 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 \end{bmatrix},$$

obtained from A_2 by perturbing the (1,1)-entry, is eventually positive, because $(\tilde{A}_2)^5 > 0$ and $(\tilde{A}_2)^6 > 0$.

One significant difference between PSEN and PEP sign patterns is that any superpattern of a PEP sign pattern is PEP [1, Theorem 3.1], whereas this need not be true for PSEN sign patterns, as the next example shows. (The sign pattern $\hat{\mathcal{A}} = [\hat{\alpha}_{ij}]$ is a *superpattern* of $\mathcal{A} = [\alpha_{ij}]$ if $\alpha_{ij} \neq 0$ implies $\hat{\alpha}_{ij} = \alpha_{ij}$.)

Example 3.7. The pattern $\mathcal{A}_3 = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ is PSEN, but the superpattern $\mathcal{A}_4 = \begin{bmatrix} - & + \\ + & 0 \end{bmatrix}$ is not PSEN because the dominant eigenvalue of any matrix with this pattern is negative.

We now apply Corollary 3.5 to characterize 2×2 and 3×3 PSEN sign patterns. Two $n \times n$ sign patterns $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if $\mathcal{B} = P^T \mathcal{A} P$, or $\mathcal{B} = P^T \mathcal{A}^T P$ (where P is a permutation matrix); if $\mathcal{B}$ is equivalent to $\mathcal{A}$, then $\mathcal{B}$ is PSEN if and only if $\mathcal{A}$ is PSEN. We use the following notation: For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, the *positive part* of $\mathcal{A}$ is $\mathcal{A}^+ = [\alpha_{ij}^+]$ where α_{ij}^+ is $+$ if $\alpha_{ij} = +$, and 0 if $\alpha_{ij} = 0$ or $\alpha_{ij} = -$; the *negative part* of $\mathcal{A}$ is defined analogously (see [1]). Here ? denotes one of $0, +, -$; $\oplus$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 1-10, January 2012

ELA
http://math.technion.ac.il/iic/ela

denotes one of $0,+$; $\ominus$ denotes one of $0,+$.

THEOREM 3.8. *For an $n \times n$ sign pattern $\mathcal{A}$ with $n \leq 3$, $\mathcal{A}$ is PSEN if and only if one of the following is true:*

i. $\mathcal{A}^+$ *is primitive.*

ii. $n = 3$ *and* $\mathcal{A}$ *is equivalent to a sign pattern of the form* $\mathcal{B} = \begin{bmatrix} + & - & \ominus \\ + & ? & - \\ - & + & + \end{bmatrix}$.

iii. $\mathcal{A} \geq 0$ *and* $\mathcal{A}$ *is irreducible.*

Proof. All the listed sign patterns are PSEN (by [1, Theorems 6.1 and 6.4] for (i) and (ii); (iii) is clear), so assume $\mathcal{A}$ is PSEN. By Corollary 3.5, $\mathcal{A}$ is PEP or r-cyclic. If $\mathcal{A}$ is PEP and $n \leq 3$, then either $\mathcal{A}^+$ is primitive, or $n = 3$ and $\mathcal{A}$ is equivalent to a sign pattern of the form $\mathcal{B}$ [1, Theorems 6.1 and 6.4]. So assume $\mathcal{A}$ is r-cyclic, where $r = 2$ or $r = 3$ (the latter requires $n = 3$). Then since $\mathcal{A}$ must have a $+$ in each row and column, $\mathcal{A} \geq 0$. Since $\mathcal{A}$ is PSEN, $\mathcal{A}$ is irreducible. □

In the proof of Theorem 3.8, it was shown that for $n \leq 3$, if $\mathcal{A}^+$ is r-cyclic then $\mathcal{A} \geq 0$. There are other conditions on the positive part $\mathcal{A}^+$ of a PSEN sign pattern $\mathcal{A}$ that require $\mathcal{A} \geq 0$.

THEOREM 3.9. *Let $\mathcal{A}$ be an $n \times n$ PSEN sign pattern having exactly n positive entries. Then $\mathcal{A} \geq 0$, i.e., $\mathcal{A}$ is the sign pattern of a permutation matrix.*

Proof. Since $\mathcal{A}$ is PSEN, there exists a matrix $A \in \mathcal{Q}(\mathcal{A})$ that is SEN. Let $B = \frac{1}{\rho(A)}A$. Then $B \in \mathcal{Q}(\mathcal{A})$, B is SEN, and $\rho(B) = 1$ with $B\mathbf{v} = \mathbf{v}$, $\mathbf{w}^T B = \mathbf{w}^T$, and $\mathbf{v}, \mathbf{w} > 0$. Now let $C = D^{-1}BD$ for $D = \operatorname{diag}(v_1, \dots, v_n)$. Then $C \in Q(\mathcal{A})$ is SEN and $\rho(C) = 1$ and $C\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ is the $n \times 1$ all ones vector. Since C is SEN, by Observation 3.2 C has a plus in each row and column, and since C has only n positive entries, no two plusses can be in the same row or in the same column. Thus, there exists a permutation matrix P chosen so that all the diagonal entries $\widehat{c}_{ii}$ of $\widehat{C} := PC$ are positive (and all off-diagonal entries $\widehat{c}_{ij}, i \neq j$, are nonpositive). Then $\widehat{c}_{ii} = 1 - \sum_{j \neq i} \widehat{c}_{ij} \geq 1$. By Gershgorin's theorem, $\operatorname{spec}(\widehat{C})$ lies on the union of circles centered at $\widehat{c}_{ii}$ that all pass through the point $(1, 0)$. Thus, either all of the eigenvalues of $\widehat{C}$ are equal to 1, or $|\det \widehat{C}| > 1$. The latter is not possible as $|\det \widehat{C}| = |\det C| \leq 1$. Thus, all of the eigenvalues of $\widehat{C}$ are equal to 1. Since the trace is the sum of the eigenvalues, $\widehat{c}_{ii} = 1$ for all i and $\widehat{C}$ is the identity matrix. It follows that $C = P^T \in \mathcal{Q}(\mathcal{A})$ and $C \geq 0$, so $\mathcal{A} \geq 0$. □

Let $\mathcal{A} = [\mathcal{A}_{ij}]$ be a block sign pattern. If $\mathcal{A}_{ij} = {\mathcal{A}_{ij}}^+$, we write ${\mathcal{A}_{ij}}^+$ in the partition, and similarly for $\mathcal{A}_{ij} = {\mathcal{A}_{ij}}^-$. A sign pattern $\mathcal{A}$ of the form $\begin{bmatrix} {\mathcal{A}_{11}}^+ & {\mathcal{A}_{12}}^- \\ {\mathcal{A}_{21}}^- & {\mathcal{A}_{22}}^+ \end{bmatrix}$ with both diagonal blocks square (note that ${\mathcal{A}_{21}}^-$ and ${\mathcal{A}_{12}}^-$ may be rectangular) is

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 1-10, January 2012

ELA
http://math.technion.ac.il/iic/ela

called *positive block checkerboard*, and one of the form $-\mathcal{A}$ is called *negative block checkerboard*. A block of the form ${\mathcal{A}_{ij}}^-$ is a *negative block*, and one of the form ${\mathcal{A}_{ij}}^+$ is a *positive block*.

REMARK 3.10. A positive block checkerboard sign pattern $\mathcal{A}$ is not PSEN because $\mathcal{A}^k$ is positive block checkerboard with the same block structure for all positive integers k, so if $\mathcal{A}^k \geq 0$, then A^k is reducible.

THEOREM 3.11. *Let $\mathcal{A}$ be a negative block checkerboard sign pattern. Then $\mathcal{A}$ is PSEN if and only if $\mathcal{A} \geq 0$ and $\mathcal{A}$ is irreducible.*

Proof. If $\mathcal{A} = \mathcal{A}^+$ and $\mathcal{A}$ is irreducible, then clearly $\mathcal{A}$ is PSEN. For the converse, assume $\mathcal{A}$ is PSEN, and let $A \in \mathcal{Q}(\mathcal{A})$ be SEN. Then A^ℓ is negative block checkerboard for odd ℓ and A^ℓ is positive block checkerboard for even ℓ, both with the same block structure as A. Furthermore, if the (i, j)-entry in A^ℓ is in a negative block (resp., positive block) of A^ℓ, then this entry is computed as the sum of products, with each product being zero or negative (resp., zero or positive), i.e., no cancellation occurs.

For all $\ell \geq k_0$, $A^\ell \geq 0$ (where k_0 is the power index of A), and there exists a (necessarily odd) $k \geq k_0$ such that $A^k \geq 0$ is irreducible. Then the digraph of A^k is strongly connected, so there is a walk from any vertex to any other. In particular, there is a walk from vertex j to itself, i.e., there exists an $m \geq 0$ such that the (j, j)-entry of A^{km} is positive (since it is nonzero and $k \geq k_0$). So if the (i, j)-entry of A were negative, then the (i, j)-entry of A^{km+1} would also be negative. Therefore, $\mathcal{A} = \mathcal{A}^+$. ☐

Acknowledgement. Much of this research was carried out at the Banff International Research Station and the authors thank BIRS for their support. The research of MC was supported in part by Xavier University Faculty Development Fund. The research of CE and LH was supported in part by NSF DMS 0502354. The research of DDO and PvdD was supported in part by NSERC Discovery grants.

REFERENCES

[1] A. Berman, et al. Sign patterns that allow eventual positivity. *Electronic Journal of Linear Algebra*, 19:108–120, 2010.
[2] A. Berman and B. Plemmons. *Nonnegative Matrices in the Mathematical Sciences.* SIAM, Philadelphia, 1994.
[3] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory.* Cambridge University Press, New York, 1991.
[4] S. Carnochan Naqvi and J.J. McDonald. The combinatorial structure of eventually nonnegative matrices. *Electronic Journal of Linear Algebra*, 9:255–269, 2002.
[5] M. Catral, D.D. Olesky, and P. van den Driessche. Allow problems concerning spectral proper-

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 1-10, January 2012

ELA
http://math.technion.ac.il/iic/ela

ties of sign pattern matrices: A survey. *Linear Algebra and its Applications*, 430:3080–3094, 2009.

[6] E.M. Ellison, L. Hogben, and M.J. Tsatsomeros. Sign patterns that require eventual positivity or require eventual nonnegativity. *Electronic Journal of Linear Algebra*, 19:98–107, 2010.

[7] S. Friedland. On an inverse problem for nonnegative and eventually nonnegative matrices. *Israel Journal of Mathematics*, 29:43–60, 1978.

[8] G.H. Golub and C.F. Van Loan. *Matrix Computations*, 3rd edition. Johns Hopkins University Press, Baltimore, 1996.

[9] F.J. Hall and Z. Li. Sign pattern matrices. In *Handbook of Linear Algebra*, L. Hogben (editor). Chapman & Hall/CRC Press, Boca Raton, 2007.

[10] D. Handelman. Positive matrices and dimension groups affiliated to C^*-algebras and topological Markov chains. *Journal of Operator Theory*, 6:55–74, 1981.

[11] L. Hogben. Eventually cyclic matrices and a test for strong eventual nonnegativity. *Electronic Journal of Linear Algebra*, 19:129–140, 2010.

[12] B.G. Zaslavsky and J.J. McDonald. A characterization of Jordan canonical forms which are similar to eventually nonnegative matrices with the properties of nonnegative matrices. *Linear Algebra and its Applications*, 372:253–285, 2003.

[13] B.G. Zaslavsky and B.-S. Tam. On the Jordan form of an irreducible matrix with eventually non-negative powers. *Linear Algebra and its Applications*, 302/303:303–330, 1999.