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A NEW EIGENVALUE BOUND FOR THE HADAMARD PRODUCT OF AN M-MATRIX AND AN INVERSE M-MATRIX

FUBIN CHEN†, YAOTANG LI‡, AND DEFENG WANG§

Abstract. If $A$ and $B$ are $n \times n$ nonsingular $M$-matrices, a new lower bound for the minimum eigenvalue $\tau(A \circ B^{-1})$ for the Hadamard product of $A$ and $B^{-1}$ is derived. This bound improves the result of [R. Huang. Some inequalities for the Hadamard product and the Fan product of matrices. Linear Algebra Appl., 428:1551–1559, 2008].

Key words. $M$-matrix, Hadamard product, Spectral radius, Lower bound.

AMS subject classifications. 15A06, 15A15, 15A48.

1. Introduction. For a positive integer $n$, $N$ denotes the set $\{1, 2, \ldots, n\}$. The set of all $n \times n$ complex matrices is denoted by $\mathbb{C}^{n \times n}$ and $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$ real matrices.

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and $B = (b_{ij}) \in \mathbb{R}^{n \times n}$. We write $A \geq B$ ($> B$) if $a_{ij} \geq b_{ij}$ ($> b_{ij}$) for all $i, j \in \{1, 2, \ldots, n\}$. If $0$ is the null matrix and $A \geq 0$ ($> 0$), we say that $A$ is a nonnegative (positive) matrix. The spectral radius of $A$ is denoted by $\rho(A)$. If $A$ is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A)$ is an eigenvalue of $A$.

We let $Z_n$ denote the class of all $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. An $n \times n$ matrix $A$ is called an $M$-matrix if there exists an $n \times n$ nonnegative matrix $B$ and a nonnegative real number $\lambda$ such that $A = \lambda I - B$ and $\lambda \geq \rho(B)$, $I$ is the identity matrix; if $\lambda > \rho(B)$, we call $A$ a nonsingular $M$-matrix; if $\lambda = \rho(B)$, we call $A$ a singular $M$-matrix. Denote by $M_n$ the set of nonsingular $M$-matrices.
Let \( A \in \mathbb{Z}^n \) and let \( \tau(A) = \min \{ \text{Re}(\lambda) : \lambda \in \sigma(A) \} \). Basic for our purpose are the following simple facts (see Problems 16, 19 and 28 in Section 2.5 of [4]):

1. \( \tau(A) \in \sigma(A) \); \( \tau(A) \) is called the minimum eigenvalue of \( A \).
2. If \( A, B \in M_n \), and \( A \geq B \), then \( \tau(A) \geq \tau(B) \).
3. If \( A \in M_n \), then \( \rho(A^{-1}) \) is the Perron eigenvalue of the nonnegative matrix \( A^{-1} \), and \( \tau(A) = \frac{1}{\rho(A^{-1})} \) is a positive real eigenvalue of \( A \).

Let \( A \) be an irreducible nonsingular \( M \)-matrix. It is known that there exist positive vectors \( u \) and \( v \) such that \( Au = \tau(A)u \) and \( v^T A = \tau(A)v^T \), \( u \) and \( v \) being called right and left Perron eigenvectors of \( A \), respectively.

For two real matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) of the same size, the Hadamard product of \( A \) and \( B \) is \( A \circ B = (a_{ij}b_{ij}) \). If \( A \) and \( B \) are two nonsingular \( M \)-matrices, then it is proved in [2] that \( A \circ B^{-1} \) is a nonsingular \( M \)-matrix.

If \( A = (a_{ij}) \) is a nonsingular \( M \)-matrix, we write \( N = D - A \), where \( D = \text{diag}(a_{ii}) \). Note that \( a_{ii} > 0 \) for all \( i \) if \( A \in M_n \). Thus, we define \( J_A = D^{-1}N; J_A \) is nonnegative.

Let \( A, B \in M_n \) and \( B^{-1} = (\beta_{ij}) \), in [4, Theorem 5.7.31] the following classical result is given:

\[
\tau(A \circ B^{-1}) \geq \tau(A) \min_{1 \leq i \leq n} \beta_{ii}.
\]

Recently, Huang [5, Theorem 9] improved this result and gave a new lower bound for \( \tau(A \circ B^{-1}) \), that is

\[
\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}}.
\]

In this paper, for two nonsingular \( M \)-matrices \( A \) and \( B \), we give a new lower bound for \( \tau(A \circ B^{-1}) \); some examples are given to illustrate our result.

2. Some lemmas and the main result. In order to prove our result, we first give some lemmas.

**Lemma 2.1.** [4, Lemma 5.1.2] Let \( A, B \in \mathbb{C}^{n \times n} \) and suppose that \( D \in \mathbb{C}^{n \times n} \) and \( E \in \mathbb{C}^{n \times n} \) are diagonal matrices, then

\[
D(A \circ B)E = (DAE) \circ (DB) = (AE) \circ (DB) = A \circ (DBE).
\]
Lemma 2.2. [5] Lemma 8] Let $B = (b_{ij}) \in M_n$ be irreducible, and let $y = (y_i)$ be a positive vector such that $J_B y = \rho(J_B) y$. Then for $B^{-1} = (\beta_{ij})$, we have
\[ |\beta_{ji}| \leq \rho(J_B) \rho(J_B^{-1}) b_{ii} y_j y_i, \quad i \neq j, \]
and
\[ \beta_{ii} \geq \frac{1}{b_{ii}(1 + \rho^2(J_B))}. \]

Lemma 2.3. [5] Theorem 6.4.7] Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$. Then all the eigenvalues of $A$ lie in the region:
\[ \bigcup_{i,j=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}| z - a_{jj} | \leq \sum_{k \neq i} |a_{ki}| \sum_{k \neq j} |a_{kj}| \right\}. \]

By the definition of $J_A$, we have
\[ \rho(J_A^T) = \rho(D^{-1} N^T) = \rho(N D^{-1}) = \rho(D^{-1} (N D^{-1}) D) = \rho(D^{-1} N) = \rho(J_A). \]

Theorem 2.4. Let $A = (a_{ij}), B \in \mathbb{R}^{n \times n}$ be two nonsingular $M$-matrices and let $B^{-1} = (\beta_{ij})$. Then
\[ \tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4 a_{ij} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{1/2} \right\}. \]

(2.1)

Proof. It is evident that (2.1) is an equality for $n = 1$.

We next assume that $n \geq 2$.

If $A \circ B^{-1}$ is irreducible, then $A$ and $B$ are irreducible. Then $J_A$ and $J_B$ are also irreducible and nonnegative, so there exists a positive vector $u = (u_i)$ such that $J_A^T u = \rho(J_A^T) u$. Note that $\rho(J_A^T) = \rho(J_A)$, so we have
\[ \sum_{j \neq i} \frac{|a_{ji}| u_j}{u_i} = a_{ii} \rho(J_A). \]
let \( \hat{A} = (\hat{a}_{ij}) = \hat{U}A\hat{U}^{-1} \) and \( \hat{B}^{-1} = (\hat{\beta}_{ij}) = \hat{V}B^{-1}\hat{V}^{-1} \) in which \( \hat{U} \) and \( \hat{V} \) are the nonsingular diagonal matrices \( \hat{U} = \text{diag}(u_1, u_2, \ldots, u_n) \) and \( \hat{V} = \text{diag}(\frac{1}{v_1}, \frac{1}{v_2}, \ldots, \frac{1}{v_n}) \).

Then, we have

\[
\hat{A} = (\hat{a}_{ij}) = \hat{U}A\hat{U}^{-1}
\]

\[
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
    \frac{1}{u_1} \\
    \frac{1}{u_2} \\
    \vdots \\
    \frac{1}{u_n}
\end{bmatrix}
\]

and

\[
\hat{B}^{-1} = (\hat{\beta}_{ij}) = \hat{V}B^{-1}\hat{V}^{-1}
\]

\[
\begin{bmatrix}
    \frac{1}{v_1} \\
    \frac{1}{v_2} \\
    \vdots \\
    \frac{1}{v_n}
\end{bmatrix}
\begin{bmatrix}
    \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\
    \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn}
\end{bmatrix}
\begin{bmatrix}
    v_1 \\
    v_2 \\
    \vdots \\
    v_n
\end{bmatrix}
\]

Also let \( W = \hat{V}\hat{U} \). Then, \( W \) is nonsingular. From Lemma 2.1, we have

\[
(VU)(A \circ B^{-1})(VU)^{-1} = VU(A \circ B^{-1})U^{-1}V^{-1} = (UAU^{-1}) \circ (VB^{-1}V^{-1}) = \hat{A} \circ \hat{B}^{-1}.
\]

Thus, we have \( \tau(A \circ B^{-1}) = \tau(\hat{A} \circ \hat{B}^{-1}) \) and

\[
\hat{A} \circ \hat{B}^{-1} = (c_{ij}) =
\begin{bmatrix}
    a_{11}\beta_{11} & a_{12}\beta_{12} & \cdots & a_{1n}\beta_{1n} \\
    a_{21}\beta_{21} & a_{22}\beta_{22} & \cdots & a_{2n}\beta_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1}\beta_{n1} & a_{n2}\beta_{n2} & \cdots & a_{nn}\beta_{nn}
\end{bmatrix}
\]
We next consider the minimum eigenvalue of $\hat{A} \circ \hat{B}^{-1}$. Let $\tau(\hat{A} \circ \hat{B}^{-1}) = \lambda$, so that $0 < \lambda < a_i \beta_i$, $\forall i \in N$. Thus, by Lemma 2.3, there is a pair $(i, j)$ of positive integers with $i \neq j$ such that

$$|\lambda - a_{ii} \beta_{ii}| |\lambda - a_{jj} \beta_{jj}| \leq \sum_{k \neq i} |c_{ki}| \sum_{k \neq j} |c_{kj}|.$$  

Observe that

$$\sum_{k \neq i} |c_{ki}| \sum_{k \neq j} |c_{kj}| = \left( \sum_{k \neq i} \left| \frac{a_{kk} \beta_{k1} u_k v_i}{u_i v_k} \right| \right) \left( \sum_{k \neq j} \left| \frac{a_{kk} \beta_{k1} u_k v_j}{u_j v_k} \right| \right)$$

$$\leq \left( \sum_{k \neq i} \left| \frac{a_{kk} u_k}{u_i} \right| \rho(J_B) \beta_{ii} \right) \left( \sum_{k \neq j} \left| \frac{a_{kk} u_k}{u_j} \right| \rho(J_B) \beta_{jj} \right)$$

$$= a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B).$$

Thus, we have

$$|\lambda - a_{ii} \beta_{ii}| |\lambda - a_{jj} \beta_{jj}| \leq a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B).$$

Then, we have

$$\lambda \geq \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{1/2} \right\}.$$  

That is,

$$\tau(A \circ B^{-1}) \geq \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{1/2} \right\}$$

$$\geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii} a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{1/2} \right\}.$$  

Now, assume that $A \circ B^{-1}$ is reducible. It is known that a matrix in $Z_n$ is a nonsingular $M$-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by $D = (d_{ij})$ the $n \times n$ permutation matrix with $d_{11} = d_{23} = \cdots = d_{n-1,n} = d_{n1} = 1$, then both $A - tD$ and $B - tD$ are irreducible nonsingular $M$-matrices for any chosen positive real number $t$, sufficiently small such that all the leading principal minors of both $A - tD$ and $B - tD$ are positive. Now we substitute $A - tD$ and $B - tD$ for $A$ and $B$, respectively in the previous case, and then letting $t \to 0$, the result follows by continuity. \[\blacksquare\]
From (2.1) and (2.3), we have

\[ (2.3) \]

\[ (2.2) \]

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**Theorem 2.5.** Let \( A = (a_{ij}), B \in \mathbb{R}^{n \times n} \) be two nonsingular \( M \)-matrices and let \( B^{-1} = (\beta_{ij}) \). Then

\[
\min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\} \\
\geq \frac{1 - \rho(J_A) \rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}}.
\]

**Proof.** Without loss of generality, for \( i \neq j \), assume that

\[(2.2) \quad a_{ii} \beta_{ii} - a_{ii} \beta_{ii} \rho(J_A) \rho(J_B) \leq a_{jj} \beta_{jj} - a_{jj} \beta_{jj} \rho(J_A) \rho(J_B).\]

Thus, (2.2) is equivalent to

\[(2.3) \quad a_{jj} \beta_{jj} \rho(J_A) \rho(J_B) \leq a_{ii} \beta_{ii} \rho(J_A) \rho(J_B) + a_{jj} \beta_{jj} - a_{ii} \beta_{ii}\]

From (2.1) and (2.3), we have

\[
\frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\} \\
\geq \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho(J_A) \rho(J_B) \left( a_{ii} \beta_{ii} \rho(J_A) \rho(J_B) + a_{jj} \beta_{jj} - a_{ii} \beta_{ii} \right) \right]^{\frac{1}{2}} \right\} \\
= \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho(J_A) \rho(J_B) \right]^{\frac{1}{2}} \right\} \\
= \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{jj} \beta_{jj} - a_{ii} \beta_{ii} + 2a_{ii} \beta_{ii} \rho(J_A) \rho(J_B))^2 \right]^{\frac{1}{2}} \right\} \\
= \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - (a_{jj} \beta_{jj} - a_{ii} \beta_{ii} + 2a_{ii} \beta_{ii} \rho(J_A) \rho(J_B)) \right\} \\
= a_{ii} \beta_{ii} - a_{ii} \beta_{ii} \rho(J_A) \rho(J_B) \\
= a_{ii} \beta_{ii} (1 - \rho(J_A) \rho(J_B)) \\
\geq \frac{1 - \rho(J_A) \rho(J_B) a_{ii}}{1 + \rho^2(J_B)} \frac{a_{ii}}{b_{ii}}.
\]

Thus, we have

\[
\tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{\frac{1}{2}} \right\} \\
\geq \frac{1 - \rho(J_A) \rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}}. \quad \blacksquare
\]
Remark 2.6. Theorem 2.5 shows that the result of Theorem 2.4 is better than the result of Theorem 9 in [5].

3. Examples.

Example 3.1. Let
\[
A = \begin{bmatrix}
1 & -0.5 & 0 & 0 \\
-0.5 & 1 & -0.5 & 0 \\
0 & -0.5 & 1 & -0.5 \\
0 & 0 & -0.5 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
4 & -1 & -1 & -1 \\
-2 & 5 & -1 & -1 \\
0 & -2 & 4 & -1 \\
-1 & -1 & -1 & 4
\end{bmatrix}.
\]

Then
\[
A \circ B^{-1} = \begin{bmatrix}
0.4 & -0.1 & 0 & 0 \\
-0.1167 & 0.3667 & -0.1 & 0 \\
0 & -0.1167 & 0.4 & -0.1 \\
0 & 0 & -0.1 & 0.4
\end{bmatrix}.
\]

By calculating with Matlab 7.0, we have \(\rho(J_A) = 0.809, \rho(J_B) = 0.7652\), and \(\tau(A \circ B^{-1}) = 0.2148\). By Theorem 9 in [5], we have
\[
\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} \frac{a_{ii}}{b_{ii}} = 0.048.
\]

By our Theorem 2.4, we have
\[
\tau(A \circ B^{-1}) \geq \min_{i \neq j} \frac{1}{2} \left\{ a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A) \rho^2(J_B) \right]^{1/2} \right\} = 0.1524.
\]

which approaches the real value 0.2148. This numerical example shows that the result in Theorem 2.4 is better than that in Theorem 9 in [5] in some cases.

Example 3.2. Let
\[
A = \begin{bmatrix}
2 & -2 \\
-1 & 2
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & -0.5 \\
-0.5 & 1
\end{bmatrix}.
\]

Then
\[
A \circ B^{-1} = \begin{bmatrix}
1.7142 & -0.5714 \\
-0.2857 & 2.2858
\end{bmatrix}.
\]
By calculating with Matlab 7.0, we have $\rho(J_A) = 0.7071$, $\rho(J_B) = 0.3536$, and $\tau(A \circ B^{-1}) = 1.0144$. By Theorem 9 in [5], we have

$$\tau(A \circ B^{-1}) \geq \frac{1 - \rho(J_A)\rho(J_B)}{1 + \rho^2(J_B)} \min_{1 \leq i \leq n} a_{ii} = 0.6666.$$ 

By our Theorem 2.4, we have

$$\tau(A \circ B^{-1}) \geq \min_{i \neq j} \left\{ \frac{1}{2} \left( a_{ii} \beta_{ii} + a_{jj} \beta_{jj} - \left[ (a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4a_{ii}a_{jj} \beta_{ii} \beta_{jj} \rho^2(J_A)\rho^2(J_B) \right]^{\frac{1}{2}} \right) \right\} = 1.0144.$$ 

It is a surprise to see that our bound is the minimum eigenvalue of $A \circ B^{-1}$. This numerical example shows that the bound of Theorem 2.4 is sharp.

REFERENCES