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ON 3-COLORED DIGRAPHS WITH EXACTLY ONE NONSINGULAR CYCLE*

DEBAJIT KALITA†

Abstract. The class of connected 3-colored digraphs containing exactly one nonsingular cycle is considered in this article. The main objective is to study the smallest Laplacian eigenvalue and the corresponding eigenvectors of such graphs. It is shown that the smallest Laplacian eigenvalue of such a graph can be realized as the algebraic connectivity (second smallest Laplacian eigenvalue) of a suitable undirected graph. The nonsingular unicyclic 3-colored digraph on n vertices, which minimize the smallest Laplacian eigenvalue over all such graphs is determined in this article.

Key words. Laplacian matrix, Mixed graph, Weighted directed graph, 3-Colored digraph, First eigenvector.

AMS subject classifications. 05C50, 05C05, 15A18.

1. Introduction. All our graphs are simple. All our directed graphs have simple underlying undirected graphs (except in Definitions 2.12 and 2.16). At times we use $V(G)$ (resp., $E(G)$) to denote the set of vertices (resp., edges) of a graph G (directed or undirected). In the absence of any specification $V(G)$ is assumed to be $\{1, 2, \ldots, n\}$. Let G be a directed graph. We write $(i, j) \in E(G)$ to mean the existence of the directed edge from the vertex i to the vertex j. With each edge (i, j), we associate a complex number w_{ij} of absolute value 1 with nonnegative imaginary part, that is, weights are chosen from the upper half part of the unit circle on the complex plane. We call it the *weight* of that edge. Henceforth, we shall understand that weights are complex numbers of unit modulus with nonnegative imaginary part, unless otherwise specified. We call the directed graph G with such a weight function w a *weighted directed graph*. The *adjacency matrix* $A(G)$ of G is the matrix with (i, j)-th entry

$$a_{ij} = \begin{cases} \mathsf{w}_{ij} & \text{if } (i,j) \in E(G), \\ \overline{\mathsf{w}}_{ji} & \text{if } (j,i) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this article, $\mathrm{i} = \sqrt{-1}$. Note that choosing the weights only from the "upper half part of the unit circle" is not really a restriction for the study of adjacency

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matrices. For example, if G has an edge (i,j) of weight $x+y\mathrm{i}$, then we may replace (i,j) by an edge (j,i) of weight $x-y\mathrm{i}$, while the adjacency matrix remains unchanged.

Let G be a weighted directed graph. In defining subgraph, walk, path, connectedness and degree of a vertex in G we focus only on the underlying unweighted undirected graph of G. Thus, the *degree* d_i of a vertex i in a weighted directed graph G is the number of edges incident with i. It may be viewed as the sum of the absolute values of the weights of the edges incident with the vertex i. The *Laplacian matrix* $L(G)$ of G is defined as the matrix $D(G)-A(G)$, where $D(G)$ is the diagonal matrix with d_i as the i-th diagonal entry.

DEFINITION 1.1. [2] Let G be a directed graph with edges having colors red, blue, or green. We assign a weight 1 to each red edge, a weight -1 to each blue edge, and a weight i to each green edge in G. We call this weighted directed graph a 3-*colored digraph.*

A *mixed graph* [1] is a graph with some directed and some undirected edges. Let G be a 3-colored digraph. Notice that if weight of each edge in G is 1, that is, if each edge in G has the color red, then $L(G)$ coincides with the usual Laplacian matrix of the underlying unweighted undirected uncolored graph of G. If the weights of the edges in G are ± 1, that is, if G does not have a green edge, then (*viewing the edges of color red as directed and the edges of color blue as undirected*) $L(G)$ coincides with the Laplacian matrix of a mixed graph introduced by Bapat et al. [1]. Thus, a mixed graph may be viewed as a 3-colored digraph without a green edge.

Note that with this set-up, the Laplacian matrix of a 3-colored digraph is positive semidefinite, see [2]. It was proved in [2] that unlike the Laplacian matrix of unweighted undirected graphs, the Laplacian matrix of a 3-colored digraph is sometimes nonsingular. A 3-colored digraph is said to be *singular (resp., nonsingular)* if its Laplacian matrix is singular (resp., nonsingular).

REMARK 1.2. Let G be a 3-colored digraph. If an edge (i,j) of G has a color red or color blue, then $a_{ij}=a_{ji}=1$ or -1, respectively. Thus, the adjacency (resp., Laplacian) matrix of G is indifferent about the orientations of the red and blue edges. In view of this fact, we write $ij \in E(G)$ to mean the existence of a red or a blue edge between the vertices i and j in G. We write $(i,j)\in E(G)$ to mean the existence of the green edge directed from the vertex i to the vertex j in G.

Denote by $\lambda_i(B)$ the i-th smallest eigenvalue of a Hermitian matrix B. For an undirected graph G, $\lambda_2(L(G))$ is popularly known as the algebraic connectivity of G, denoted by $a(G)$.

The article is organized as follows. In Section 2, we describe the structure of 3-colored digraphs containing exactly one nonsingular cycle. We show that given a

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connected 3-colored digraph G containing exactly one nonsingular cycle, there is a graph H with $a(H) = \lambda_1(L(G))$. In Section 3, we establish a monotonicity property on the real and imaginary parts of the eigenvectors corresponding to $\lambda_1(L(G))$, which is analogous to Fiedler's monotonicity theorem [10]. We describe the sign structure of the real and imaginary parts of the eigenvectors corresponding to $\lambda_1(L(G))$. In Section 4, we prove that among all nonsingular unicyclic 3-colored digraphs on n vertices, the smallest Laplacian eigenvalue is minimized by the cycle of weight $\pm\mathrm{i}$ and length n.

2. Smallest Laplacian eigenvalue. Let G be a 3-colored digraph and D be a diagonal matrix with the diagonal entries $d_{ii} \in \{\pm 1, \pm\mathrm{i}\}$, for each i. Note that $D^*L(G)D$ is the Laplacian matrix of another 3-colored digraph which we denote by ${}^{D}G$.

Definition 2.1. Let H and G be two 3-colored digraphs on n vertices. We say *H is D-similar to G*, if there exists a diagonal matrix D (with $d_{ii} \in \{\pm 1, \pm\mathrm{i}\}$), for each i) such that $H = {}^{D}G$.

Below we state a result due to Bapat et al. [2, Theorem 19] which shall be used.

Lemma 2.2. [2] *Let G be a connected 3-colored digraph. Then G is singular if and only if it is D-similar to ${}^{D}G$ with all edges red.*

Definition 2.3. [2] A *i_1-i_k-walk* W in a weighted directed graph G is a finite sequence $i_1, \ldots, i_k$ of vertices such that, for $1 \le p \le k-1$, either $(i_p, i_{p+1}) \in E(G)$ or $(i_{p+1}, i_p) \in E(G)$. If $e = (i_p, i_{p+1}) \in E(G)$, then we say e is *directed along* the walk, otherwise we say e is *directed opposite* to the walk. We call $\mathsf{w}_W = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}$ the *weight* of the walk W, where a_{ij} are the entries of $A(G)$.

The following lemma is essentially contained in [2, Theorem 23].

Lemma 2.4. *Let G be a connected 3-colored digraph. Then G is nonsingular if and only if G contains a cycle of weight other than 1.*

In view of Lemma 2.4, we call a cycle C in a 3-colored digraph *singular* if its weight $\mathsf{w}_C = 1$. Otherwise, we call it a *nonsingular* cycle.

The following lemmas are crucial in describing the structure of 3-colored digraphs containing exactly one nonsingular cycle.

Lemma 2.5. *Let G be a connected 3-colored digraph containing exactly one nonsingular cycle C. Then G is D-similar to ${}^{D}G$ with all edges red except one edge on C which is either blue or green if $\mathsf{w}_C = -1$ or $\mathsf{w}_C = \pm\mathrm{i}$, respectively.*

Proof. Let e be an edge on the cycle C in G. Take $G' = G - e$. Since C is the

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only nonsingular cycle in G, we see that G' does not contain a cycle of weight other than 1. Thus, G' is singular, by Lemma 2.4. By Lemma 2.2, each edge in ${}^{D}G'$ has color red, for some D. Consider ${}^{D}G$ for this D. Note that all the edges in ${}^{D}G$ except the edge corresponding to e have color red. Since weight of a cycle does not change in D-similar copies, we see that weight of the edge corresponding to e is w_C or $\overline{\mathsf{w}}_C$ in ${}^{D}G$. Hence, the result holds. □

Lemma 2.6. *Let G be a 3-colored digraph containing exactly one nonsingular cycle $C = [1, \dots, m, 1]$. Then the subgraph induced by C is C itself.*

Proof. Suppose that C has a chord joining the vertices i and j with $1 \le i < j \le m$. Take the cycles $C_1 = [1, \dots, i, j, \dots, m, 1]$ and $C_2 = [i, i+1, \dots, j, i]$. Note that $\mathsf{w}_{C_1}\mathsf{w}_{C_2} = \mathsf{w}_C \neq 1$, which implies one of C_1 and C_2 has weight other than 1. Hence, G contains at least two nonsingular cycles, which is a contradiction. □

Lemma 2.7. *Let G be a connected 3-colored digraph containing exactly one nonsingular cycle C. Let u be a vertex of G not on C. Then there is a vertex v on the cycle C such that $G - v$ is disconnected with at least two components, one containing u and another containing the remaining vertices of C.*

Proof. In view of Lemma 2.5, we assume that all the edges of G have color red except an edge e on the cycle C. Since G is connected, let v be a vertex in the cycle for which the distance $d(v, u)$ is minimum. Let P_{uv} be a shortest u-v-path in G. Then the vertex v is on every u-w-path, for each vertex w in C. If not, suppose G contains a u-w-path, say P_{uw} which does not contain v, for some vertex w in C. Let $P_{vw}(e)$ be the v-w-path on the cycle C containing the edge e. Take the cycle $C' = P_{uv} + P_{vw}(e) + P_{uw}$. Note that $\mathsf{w}_{C'} \neq 1$. So the cycle C' is nonsingular, by Lemma 2.4, which is a contradiction. Hence, $G - v$ is disconnected with at least two components, one containing u and another containing the remaining vertices of C. □

The next corollary follows immediately, which generalizes Lemma 2.1 in [7].

Corollary 2.8. *Let G be a connected 3-colored digraph with exactly one nonsingular cycle. If G has no cut vertex, then G is exactly the nonsingular cycle.*

Definition 2.9. Let G be a connected 3-colored digraph containing exactly one nonsingular cycle C and let i be a vertex on C. In view of Lemma 2.7, let H be the component of $G - i$, which contains the remaining vertices of C. Notice that if $G - i$ is connected, then $H = G - i$. We define G_i to be the graph $G - V(H)$.

The next lemma characterizes the structure of connected 3-colored digraphs containing exactly one nonsingular cycle.

Lemma 2.10. *Let G be a connected 3-colored digraph. Then the following statements are equivalent.*

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(a) G has exactly one nonsingular cycle $C = [1, \ldots, m, 1]$.
(b) G is obtained from a nonsingular cycle $C = [1, \ldots, m, 1]$ *by appending a singular connected graph* G_i *to the vertex i of C while identifying a vertex of* G_i *with i, for each* $i = 1, \ldots, m$.

Proof. $(a) \Rightarrow (b)$. Assume that G has exactly one nonsingular cycle $C = [1, \ldots, m, 1]$. By Lemma 2.6, the subgraph induced by C is C itself. In view of Lemma 2.7, for each $i = 1, \ldots, m$, let $G_i = G - V(H_i)$, where H_i is the component of the graph containing the remaining vertices of C. As G_i does not contain a nonsingular cycle, G_i is singular, for each $i = 1, \ldots, m$. Hence, the result holds. $(b) \Rightarrow (a)$ is trivial. ☐

Remark 2.11. Let G be a connected 3-colored digraph with exactly one nonsingular cycle $C = [1, \ldots, m, 1]$. In view of Lemma 2.5 and Lemma 2.10, ${}^DG$ is as in Figure 2.1 or Figure 2.2, if $\mathsf{w}_C = \pm \mathrm{i}$ or $\mathsf{w}_C = -1$, respectively. We denote the graph in Figure 2.1, by G_g and the graph in Figure 2.2, by G_b. Note that G_b is a mixed graph.

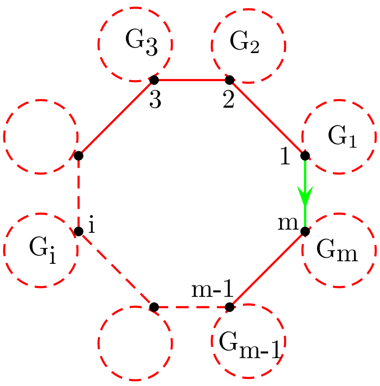

Fig. 2.1. G_g *has exactly one green edge, each* G_i *are connected.*

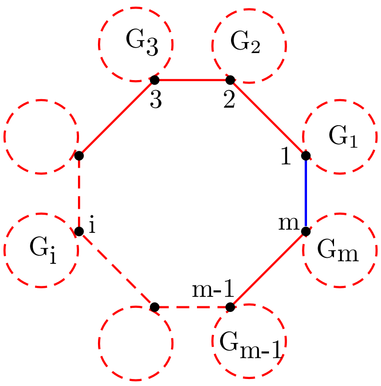

Fig. 2.2. G_b *has exactly one blue edge, each* G_i *are connected.*

Let G be a connected 3-colored digraph with exactly one nonsingular cycle. In view of Remark 2.11, in order to study the spectrum of G it is enough to study the spectrum of G_b or G_g. Note that G_b is nothing but a nonsingular mixed graph with exactly one nonsingular cycle. Fan has studied $\lambda_1(L(G_b))$ and its corresponding eigenvectors in [7]. He has proved that for the mixed graph G_b, there is an unweighted undirected graph H such that $a(H) = \lambda_1(L(G_b))$.

We now ask the following question: *Does there exist an unweighted undirected graph H whose algebraic connectivity* $a(H) = \lambda_1(L(G_g))$*?* In this section, we answer

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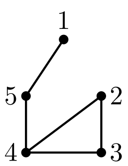
this question in the affirmative.

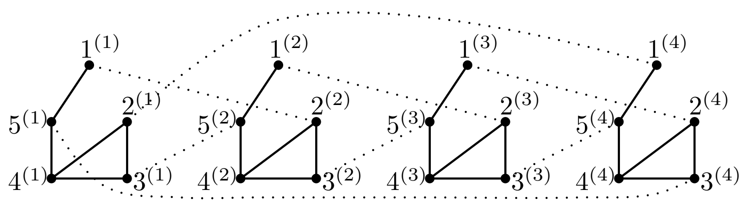
Definition 2.12. [13, Definition 37] Let G be an unweighted undirected graph on vertices $1, \ldots, n$ and F' be a set of pairs of nonadjacent vertices in G. Assign some orientations to the edges obtained by joining the nonadjacent pairs of vertices in F' and call it F. Let Γ denote the cycle $[1, \ldots, p, 1]$, where $(i, i+1) \in E(\Gamma)$ for each $i \in \mathbb{Z}_p = \{1, \ldots, p\}$ (with addition modulo p). Consider the disjoint union of p copies $G^{(1)}, \ldots, G^{(p)}$ of G. The label of each vertex u in $G^{(i)}$ is replaced by $u^{(i)}$. Add edges $u^{(i)}v^{(j)}$ whenever $(i, j) \in E(\Gamma)$ and $(u, v) \in F$. We denote the resulting graph by $G_p[F]$. We call an edge in $E(G_p[F]) \setminus E(G^{(1)} \cup \cdots \cup G^{(p)})$ a *pivotal* edge.

Definition 2.13. [13, Definition 40] Let G and F be as defined in Definition 2.12. We may view G as a weighted directed graph where each edge has a weight 1. The orientations of these edges are immaterial. Assign a weight ω to each edge in F. By G_ω we denote the weighted directed graph $G + F$.

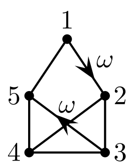
Example 2.14. Let G be the graph as shown in the following picture (a). Let $F = \{(1, 2), (3, 5)\}$. The graph $G_4[F]$ is supplied in picture (b). The dotted edges are the pivotal edges. A weighted directed graph G_ω is supplied in picture (c).



(a) G



(b) The graph $G_4[F]$



(c) G_ω

By $\mathsf{Re}\, x$ and $\mathsf{Im}\, x$ we mean the real part and imaginary part of x, respectively. We denote the vector $\begin{bmatrix} 1 \\ \omega \\ \vdots \\ \omega^{p-1} \end{bmatrix}$ by z_ω, where ω is a p-th root of unity.

The following proposition is essentially contained in [13, Lemma 45, Theorem 46].

Proposition 2.15. *Consider* $G_p[F]$ *as in* Definition 2.12. *Let* ω *be a* p*-th root*

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of unity and λ *be a Laplacian eigenvalue of* G_ω *with an eigenvector* x. *Then the following holds:*

(a) λ *is a Laplacian eigenvalue of* $G_p[F]$ *with an eigenvector* $\mathbf{x} = z_\omega \otimes x$. *Furthermore, if* $\omega \neq \pm 1$, *then* λ *is a Laplacian eigenvalue of* $G_p[F]$ *with linearly independent eigenvectors* $\operatorname{Re}\mathbf{x}$ *and* $\operatorname{Im}\mathbf{x}$.

(b) *If* $\omega^k \neq 1$, *for* $1 \leq k < p$, *then the Laplacian spectrum of* $G_p[F]$ *is the union of the Laplacian spectra of the weighted directed graphs* $G_{\omega^1}, \ldots, G_{\omega^p}$.

Definition 2.16. Consider the 3-colored digraph G_g as shown in Figure 2.1. Take the graphs $K_2[F]$, and $K_4[F]$, where $K = G_g - F$, $F = \{(1, m)\}$. We denote the graph $K_2[F]$ by $\widetilde{\mathsf{G}}$ (Figure 2.3) and the graph $K_4[F]$ by $\widehat{\mathsf{G}}$ (Figure 2.4). By G_r we denote the graph obtained from G_b changing the color of the blue edge in G_b to red. Note that G_r is singular and K_ω is nothing but G_g, G_b, or G_r if $\omega = \mathrm{i}$, $\omega = -1$, or $\omega = 1$, respectively.

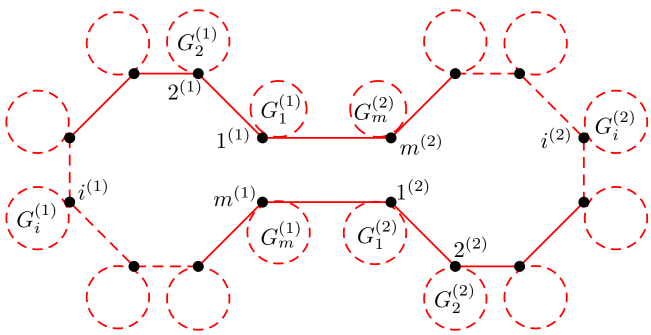

Fig. 2.3. *The graph* $\widetilde{\mathsf{G}}$.

The next result is one of our main results of this section which completely characterizes the Laplacian spectrum of the graph $\widehat{\mathsf{G}}$.

Theorem 2.17. *Consider* $\widehat{\mathsf{G}}$ *as in* Definition 2.16. *Then the following holds:*

(a) *If* λ *is an eigenvalue of* $L(G_g)$ *with an eigenvector* x, *then* λ *is an eigenvalue of* $L(\widehat{\mathsf{G}})$ *with linearly independent eigenvectors* $\begin{bmatrix} \operatorname{Re} x \\ -\operatorname{Im} x \\ -\operatorname{Re} x \\ \operatorname{Im} x \end{bmatrix}$ *and* $\begin{bmatrix} \operatorname{Im} x \\ \operatorname{Re} x \\ -\operatorname{Im} x \\ -\operatorname{Re} x \end{bmatrix}$.

(b) *If* μ *is an eigenvalue of* $L(G_b)$ *with an eigenvector* y, *then* μ *is an eigenvalue of* $L(\widehat{\mathsf{G}})$ *with an eigenvector* $z_\omega \otimes y$, *where* $\omega = -1$.

(c) *If* ν *is an eigenvalue of* $L(G_r)$ *with an eigenvector* z, *then* ν *is an eigenvalue of* $L(\widehat{\mathsf{G}})$ *with an eigenvector* $z_\omega \otimes z$, *where* $\omega = 1$.

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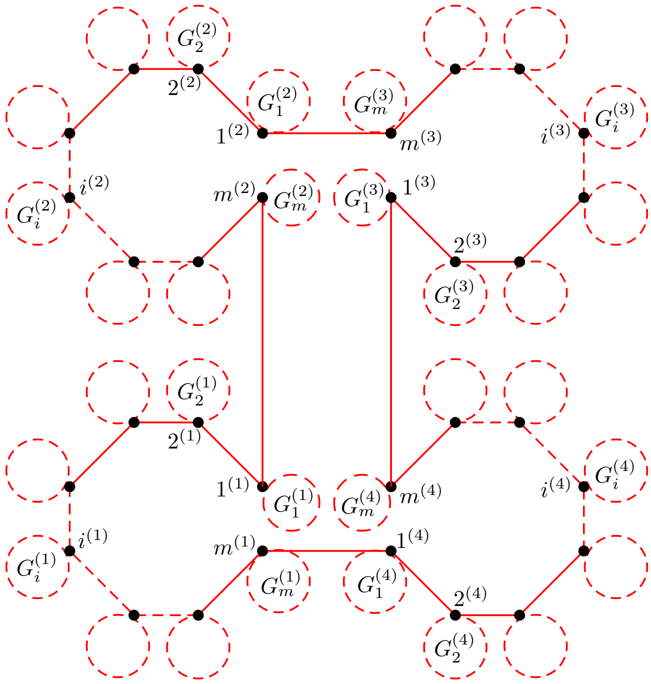

FIG. 2.4. *The graph* $\widehat{\mathsf{G}}$.

(d) *The Laplacian spectrum of* $\widehat{\mathsf{G}}$ *is the union of the Laplacian spectra of* G_g, G_b *and* G_r.

Proof. Proofs of part (a) and part (d) of the theorem follows immediately from Proposition 2.15.

(b) Let $e = (1, m)$ be the green edge in G_g and $\omega = -1$. Observe that $L(K_\omega) = L(G_b)$, where $K = G_g - e$. Note that $\omega = -1$ is a 4-th root of unity and $\widehat{\mathsf{G}} = K_4[F]$, where $F = \{(1, m)\}$. Hence, μ is an eigenvalue of $L(\widehat{\mathsf{G}})$ with an eigenvector $z_\omega \otimes y$, by first part of Proposition 2.15(a). Proof of part (c) follows similarly. ◻

LEMMA 2.18. *Consider* G_b *as in* Figure 2.2. *Then* $\lambda_1(L(G_b)) \leq a(G_r)$.

Proof. Let $e = 1m$ be the blue edge in G_b. Note that the principal submatrices of $L(G)$ and $L(G_r)$ obtained by deleting the first row and column are the same. Let L_1 be the principal submatrix of $L(G)$ and $L(G_r)$ formed by deleting its first row and column. By interlacing theorem, we have $\lambda_1(L(G_b)) \leq \lambda_1(L_1) \leq a(G_r)$. Hence, $\lambda_1(L(G_b)) \leq a(G_r)$. ◻

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Definition 2.19. Let G be a 3-colored digraph. We call an eigenvector corresponding to the eigenvalue $\lambda_1(L(G))$ the *first eigenvector* of G.

The next lemma is essentially contained in [7, Lemma 2.4].

Lemma 2.20. *Consider the mixed graph G_b as in* Figure 2.2 *and let x be a first eigenvector of G_b. Then $a(\widetilde{\mathsf{G}}) = \lambda_1(L(G_b))$ and $\begin{bmatrix} x \\ -x \end{bmatrix}$ is a Fiedler vector of $\widetilde{\mathsf{G}}$.*

Proof. Let $e = 1m$ be the blue edge in G_b and $K = G_b - e$. Note that $\widetilde{\mathsf{G}} = K_2[F]$, where $F = \{(1, m)\}$. Let ω be a 2nd root of unity. Observe that $L(K_\omega) = L(G_b)$ for $\omega = -1$ and $L(K_\omega) = L(G_r)$ for $\omega = 1$, respectively. Thus, the Laplacian spectrum of $\widetilde{\mathsf{G}}$ is the union of the Laplacian spectrum of G_b and G_r, by Proposition 2.15. Since G_r is singular, we see that $a(\widetilde{\mathsf{G}}) = \min\{\lambda_1(L(G_b)), a(G_r)\} = \lambda_1(L(G_b))$, using Lemma 2.18. Hence, $\begin{bmatrix} x \\ -x \end{bmatrix}$ is a Fiedler vector of $\widetilde{\mathsf{G}}$, by first part of Proposition 2.15(a). ∎

Remark 2.21. Let G be a connected 3-colored digraph with exactly one nonsingular cycle C. Assume that C contains an even number of green edges. If x is a first eigenvector of G, then $\begin{bmatrix} \mathsf{Re}\, D^{-1}y \\ -\mathsf{Re}\, D^{-1}y \end{bmatrix}$, $\begin{bmatrix} \mathsf{Im}\, D^{-1}y \\ -\mathsf{Im}\, D^{-1}y \end{bmatrix}$ are Fiedler vectors of $\widetilde{\mathsf{G}}$, for some diagonal matrix D with diagonal entries $\pm 1, \pm \mathrm{i}$.

Let G be an unweighted undirected graph (3-colored digraph with all edges red) and let Y be a Fiedler vector of G. A vertex i of G is called a *characteristic vertex* of G if $Y(i) = 0$ and if there is a vertex j, adjacent to i, such that $Y(j) \neq 0$. An edge e with end vertices i, j is called a *characteristic edge* of G if $Y(i)Y(j) < 0$. By $\mathcal{C}(G, Y)$, we denote the *characteristic set* of G which is the collection of all characteristic vertices and characteristic edges of G.

Below we state a result for unweighted undirected graphs due to Bapat and Pati [3], which shall be used.

Lemma 2.22. [3, Theorem 12] *Let G be a connected graph and Y be a Fiedler vector. Let $S = \mathcal{C}(G, Y)$. Suppose S lies in the block B. Then $1 \leq |S| \leq \mathcal{N}_{\mathcal{B}} + 1$, where $\mathcal{N}_{\mathcal{B}} = |E(B)| - |V(B)| + 1$.*

The next lemma says the nonnegativity of first eigenvectors of G_b at the vertices of the nonsingular cycle.

Lemma 2.23. *Consider G_b as in* Figure 2.2. *Then G_b is D-similar to a mixed graph H_b such that a first eigenvector of H_b is nonnegative on the vertices of the nonsingular cycle in H_b, where H_b has all the edges red except one blue edge on the nonsingular cycle.*

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Proof. Proof is similar to that of [8, Theorem 2.9]. □

LEMMA 2.24. *Consider G_g and G_b as in* Figure 2.1 *and* Figure 2.2*, respectively. Then $\lambda_1(L(G_g)) \leq \lambda_1(L(G_b))$.*

Proof. In view of Lemma 2.23, let x be a first eigenvector of G_b such that x is nonnegative on the vertices of the cycle containing the blue edge $e = ij$ and $x^*x = 1$. Then

$$\begin{aligned}\lambda_1(L(G_b)) = x^*L(G_b)x &= \sum_{uv \in E(G_b - e)} |x_u - x_v|^2 + (x_i + x_j)^2 \\ &\geq \sum_{uv \in E(G_b - e)} |x_u - x_v|^2 + |x_i + \mathrm{i}x_j|^2 \geq \min_{y^*y=1} y^*L(G_g)y.\end{aligned}$$

Hence, the result holds. □

The following result says that the smallest Laplacian eigenvalue of G_g is nothing but the algebraic connectivity of the unweighted undirected graph $\widehat{\mathsf{G}}$.

THEOREM 2.25. *Consider G_g as in* Figure 2.1*. Let x be a first eigenvector of G_g. Then $a(\widehat{\mathsf{G}}) = \lambda_1(L(G_g))$ and $\hat{\mathbf{x}} = \begin{bmatrix} \mathsf{Re}\, x \\ -\mathsf{Im}\, x \\ -\mathsf{Re}\, x \\ \mathsf{Im}\, x \end{bmatrix}$, $\hat{\mathbf{y}} = \begin{bmatrix} \mathsf{Im}\, x \\ \mathsf{Re}\, x \\ -\mathsf{Im}\, x \\ -\mathsf{Re}\, x \end{bmatrix}$ are linearly independent Fiedler vectors.*

Proof. By Theorem 2.17, the Laplacian spectrum of $\widehat{\mathsf{G}}$ is the union of the Laplacian spectra of G_g, G_b and G_r. Note that $a(\widehat{\mathsf{G}})$ is the smallest nonzero eigenvalue of $L(\widehat{\mathsf{G}})$ and G_r is singular. Thus, $a(\widehat{\mathsf{G}}) = \min\{\lambda_1(L(G_g)), \lambda_1(L(G_b)), a(G_r)\}$. Using Lemma 2.24 and Lemma 2.18, we have $\lambda_1(L(G_g)) \leq \lambda_1(L(G_b)) \leq a(G_r)$. Hence, $a(\widehat{\mathsf{G}}) = \lambda_1(L(G_g))$. By Theorem 2.17, $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are linearly independent Fiedler vectors of $\widehat{\mathsf{G}}$. □

COROLLARY 2.26. *Let G be a 3-colored digraph containing exactly one nonsingular cycle C. Assume that C contains an odd number of green edges. If x is a first eigenvector of G, then $\begin{bmatrix} \mathsf{Re}\, D^{-1}x \\ -\mathsf{Im}\, D^{-1}x \\ -\mathsf{Re}\, D^{-1}x \\ \mathsf{Im}\, D^{-1}x \end{bmatrix}$ and $\begin{bmatrix} \mathsf{Im}\, D^{-1}x \\ \mathsf{Re}\, D^{-1}x \\ -\mathsf{Im}\, D^{-1}x \\ -\mathsf{Re}\, D^{-1}x \end{bmatrix}$ are Fiedler vectors of $\widehat{\mathsf{G}}$, for some diagonal matrix D with the diagonal entries $\pm 1, \pm\mathrm{i}$.*

Proof. Since C contains an odd number of green edges, we see that $\mathsf{w}_C = \pm\mathrm{i}$. Thus, G is D-similar to the 3-colored digraph G_g as shown in Figure 2.1, by Remark 2.11. Note that $D^{-1}L(G)D = L(G_g)$, which implies $D^{-1}x$ is a first eigenvector of G_g. Hence, the corollary holds, by Theorem 2.25. □

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3. First eigenvectors of G_g. Fan [7] has studied the structure of the first eigenvectors of G_b. In this section, we describe the structure of the real and imaginary parts of first eigenvectors of G_g. We prove a monotonicity property on the real and imaginary parts of the first eigenvectors of G_g, which is analogous to the Fiedler's monotonicity theorem in [10]. Finally, we prove that multiplicity of $\lambda_1(L(G_g))$ is one.

Below we state the Fiedler's monotonicity theorem for unweighted undirected graphs which shall be used.

PROPOSITION 3.1. [10, Fiedler's monotonicity theorem] *Let G be a connected graph and y be a Fiedler vector of G. Then exactly one of the following cases occurs:*

Case A: *There is exactly one block B in G which contains both positively and negatively valuated vertices (with respect to y). Every other block has either vertices with positive valuation only, or vertices with negative valuation only, or vertices with zero valuation only. Every pure path P which starts at a vertex i of B and contains no other vertex of B has the property that the valuations at points of articulation contained in P form either an increasing, or decreasing, or a zero sequence along this path according to whether $y(i) > 0$, $y(i) < 0$ or $y(i) = 0$; in the last case all vertices in P have valuation zero.*

Case B: *No block of G contains both positively and negatively valuated vertices (with respect to y). In this case, there exists a unique vertex k which has valuation zero and is adjacent to a vertex with nonzero valuation. This vertex k is a point of articulation. Each block with respect to y contains (with exception of k) either vertices with positive valuation only, vertices with negative valuation only, or vertices with zero valuation only. Every pure path P which starts at k has the property that the valuations at its points of articulation either increase, and then all valuations of vertices on P are (with exception of k) positive, or decrease, and then all valuations of vertices on P are (with exception of k) negative, or all valuations of vertices on P are zero. Every path containing both positively and negatively valued vertices passes through k.*

The next result is one of our main results of this section, which describes the structure of a first eigenvector of G_g.

THEOREM 3.2. *Consider G_g as in* Figure 2.1. *Let $C = [1, \ldots, m, 1]$ be the nonsingular cycle with the green edge $e = (1, m)$. Let x be a first eigenvector of G_g. Then the following holds:*

- *(a) The cycle C contains a nonzero valuated vertex (with respect to x) and $\widehat{\mathsf{G}}$ satisfies Case A (with respect to $\hat{\mathbf{x}}$ as in* Theorem 2.25*) of* Proposition 3.1.
- *(b) Each block B of G_g other than C has either vertices of valuations with positive real part (resp., imaginary part) only, or vertices with negative real part (resp.,*

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imaginary part) only, or vertices with zero real part (resp., imaginary part) only.

(c) *Every path P which starts at a vertex v of C and contains no other vertex of C has the property that the real parts (resp., imaginary parts) of the valuations at the points of articulation contained in P form either an increasing, decreasing, or zero sequence along this path according to whether $\mathsf{Re}\, x(v) > 0$ (resp., $\mathsf{Im}\, x(v) > 0$), $\mathsf{Re}\, x(v) < 0$ (resp., $\mathsf{Im}\, x(v) < 0$), or $\mathsf{Re}\, x(v) = 0$ (resp., $\mathsf{Im}\, x(v) = 0$), respectively. In the last case, all vertices in P have real part (resp., imaginary part) zero.*

Proof. (a) Suppose that the valuations of x at the vertices of C are all zero. Let $P_{yz}^{(i)}$ denote the $y^{(i)}$-$z^{(i)}$-path in $\widehat{\mathsf{G}}$ corresponding to the y-z-path in $G_g - e$, for $i = 1, 2, 3, 4$. Take the cycle

$$\widehat{C} = P_{m1}^{(1)} + 1^{(1)}m^{(2)} + P_{m1}^{(2)} + 1^{(2)}m^{(3)} + P_{m1}^{(3)} + 1^{(3)}m^{(4)} + P_{m1}^{(4)} + 1^{(4)}m^{(1)}$$

in $\widehat{\mathsf{G}}$. Note that $\hat{\mathbf{x}} = \begin{bmatrix} \mathsf{Re}\, x \\ -\mathsf{Im}\, x \\ -\mathsf{Re}\, x \\ \mathsf{Im}\, x \end{bmatrix}$ is a Fiedler vector of $\widehat{\mathsf{G}}$, by Theorem 2.25. Since $\hat{\mathbf{x}}(u^{(1)}) = \mathsf{Re}\, x(u) = -\hat{\mathbf{x}}(u^{(3)})$, $\hat{\mathbf{x}}(u^{(2)}) = -\mathsf{Im}\, x(u) = -\hat{\mathbf{x}}(u^{(4)})$ for each vertex u of G_g, we see that the valuations of $\hat{\mathbf{x}}$ at the vertices of $\widehat{C}$ are all zero.

Case 1: If $\widehat{\mathsf{G}}$ satisfies Case A of Proposition 3.1 with respect to $\hat{\mathbf{x}}$, then there exists a single block B_0 in $\widehat{\mathsf{G}}$ containing both positively and negatively valuated vertices and each other block has either vertices with positive valuation, or negative valuation, or vertices zero valuations only. Clearly $\widehat{C}$ can not be the block B_0. So B_0 must be contained in some $G_k^{(i)}$ (where G_k is as in Definition 2.9) for some k with $1 \le k \le m$. Thus, $\widehat{\mathsf{G}}$ has at least four blocks containing both positively and negatively valuated vertices (with respect to $\hat{\mathbf{x}}$), which is a contradiction.

Case 2: If $\widehat{\mathsf{G}}$ satisfies Case B of Proposition 3.1 with respect to $\hat{\mathbf{x}}$, then there exists a unique vertex $u^{(i)}$ in $\widehat{\mathsf{G}}$ such that $\hat{\mathbf{x}}(u^{(i)}) = 0$ and $u^{(i)}$ is adjacent to a vertex $w^{(j)}$ with $\hat{\mathbf{x}}(w^{(j)}) \neq 0$ for some i, j with $1 \le i, j \le 4$. So $w^{(j)}$ can not be a vertex in $\widehat{C}$. Thus, $w^{(j)}$ must be contained in $G_k^{(j)}$ for some k with $1 \le k \le m$. From the structure of $\widehat{\mathsf{G}}$ and G_g it follows that $i = j$, that is, $u^{(i)}$ is a vertex in $G^{(j)}$, where $G^{(j)}$ is as in Definition 2.16. Thus, $\hat{\mathbf{x}}(u^{(i)}) = 0$, $\hat{\mathbf{x}}(w^{(i)}) \neq 0$, and $u^{(i)}$ is adjacent to $w^{(i)}$ in $\widehat{\mathsf{G}}$ for $i = 1, 2, 3, 4$, which is contradiction. Hence, the cycle C contains at least one nonzero valuated vertex (with respect to x).

To prove the second part of (a), let $x(u) \neq 0$ for some vertex u in C. Thus, either $\mathsf{Re}\, x(u) \neq 0$ or $\mathsf{Im}\, x(u) \neq 0$. Note that $\hat{\mathbf{x}}(u^{(1)}) = \mathsf{Re}\, x(u) = -\hat{\mathbf{x}}(u^{(3)})$ and $\hat{\mathbf{x}}(u^{(2)}) = -\mathsf{Im}\, x(u) = -\hat{\mathbf{x}}(u^{(4)})$. Thus, $\widehat{C}$ contains both positively and negatively valuated vertices (with respect to $\hat{\mathbf{x}}$). Hence, the graph $\widehat{\mathsf{G}}$ satisfies Case A of Proposition 3.1,

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with respect to $\hat{\mathbf{x}}$.

(b) Let $B \neq C$ be a block in G_g and let $u_1, \ldots, u_k$ be the vertices of B. Then B is contained in some G_i (where G_i is as in Definition 2.9). Observe that $u_1^{(1)}, \ldots, u_k^{(1)}$ are vertices of $\widehat{\mathsf{G}}$ in $B^{(1)}$, where $B^{(1)}$ is a block in $\widehat{\mathsf{G}}$ corresponding to B. By part(a), $\widehat{C}$ is the only block in $\widehat{\mathsf{G}}$ containing both positively and negatively valuated vertices (with respect to $\hat{\mathbf{x}}$). Since $B^{(1)} \neq \widehat{C}$, we see that $B^{(1)}$ contains the vertices either with positive valuation only, vertices with negative valuation only, or vertices with zero valuation only (with respect to $\hat{\mathbf{x}}$), that is, either $\hat{\mathbf{x}}(u_j^{(1)}) > 0$, $\hat{\mathbf{x}}(u_j^{(1)}) < 0$, or $\hat{\mathbf{x}}(u_j^{(1)}) = 0$ for each $j = 1, \ldots, k$. Note that $\mathsf{Re}\, x(u_j) = \hat{\mathbf{x}}(u_j^{(1)})$ for each $j = 1, \ldots, k$. Thus, the real parts of the valuations (with respect to x) at all the vertices in B are either positive, negative, or zero. With a similar argument, since $u_j^{(4)}$ are vertices of $B^{(4)}$ and $\mathsf{Im}\, x(u_j) = \hat{\mathbf{x}}(u_j^{(4)})$ for $j = 1, \ldots, k$, where $B^{(4)}$ is a block in $\widehat{\mathsf{G}}$ corresponding to B, it follows that the imaginary parts of the valuations (with respect to x) at all the vertices in B are either positive, negative, or zero.

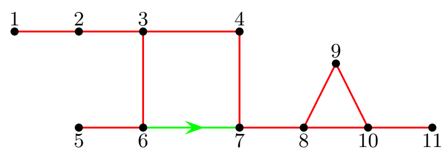
(c) Let $P^{(1)}$ and $P^{(4)}$ be the paths corresponding to P which starts at the vertices $v^{(1)}$ and $v^{(4)}$, respectively in $\widehat{\mathsf{G}}$. By part(a), $\widehat{\mathsf{G}}$ satisfies Case A of Proposition 3.1, with respect to $\hat{\mathbf{x}}$. Thus, the points of articulation contained in $P^{(1)}$ (resp., $P^{(4)}$) forms either an increasing, decreasing, or zero sequence along this path according to whether $\hat{\mathbf{x}}(v^{(1)}) > 0$ (resp., $\hat{\mathbf{x}}(v^{(4)}) > 0$), $\hat{\mathbf{x}}(v^{(1)}) < 0$, (resp., $\hat{\mathbf{x}}(v^{(4)}) < 0$), or $\hat{\mathbf{x}}(v^{(1)}) = 0$ (resp., $\hat{\mathbf{x}}(v^{(4)}) = 0$), by Proposition 3.1. Note that $\mathsf{Re}\, x(u) = \hat{\mathbf{x}}(u^{(1)})$, and $\mathsf{Im}\, x(u) = \hat{\mathbf{x}}(u^{(4)})$ for any vertex u of P. Thus, the real parts (resp., imaginary parts) of the valuations at the points of articulation contained in P form either an increasing, decreasing, or zero sequence along this path according to whether $\mathsf{Re}\, x(v) > 0$ (resp., $\mathsf{Im}\, x(v) > 0$), $\mathsf{Re}\, x(v) < 0$, (resp., $\mathsf{Im}\, x(v) < 0$), or $\mathsf{Re}\, x(v) = 0$ (resp., $\mathsf{Im}\, x(v) = 0$). □

The following example illustrates the Theorem 3.2.

EXAMPLE 3.3. Consider G as shown below. Real and imaginary parts of a first eigenvector x of G are supplied here.

$$\mathsf{Re}\, x = \begin{bmatrix} 1.160 & 1.106 & 1 & 0.906 & 0.986 & 0.940 & 0.770 & 0.964 & 1.030 & 1.047 & 1.099 \end{bmatrix}^t$$

$$\mathsf{Im}\, x = \begin{bmatrix} 0 & 0 & 0 & -0.404 & 0.424 & 0.404 & -0.790 & -0.989 & -1.057 & -1.074 & -1.127 \end{bmatrix}^t$$



REMARK 3.4. By Theorem 3.2, the characteristic set $\mathcal{C}(\widehat{\mathsf{G}}, \hat{\mathbf{x}})$ lies on the cycle $\widehat{C}$

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and by Lemma 2.22, $|\mathcal{C}(\widehat{\mathsf{G}}, \hat{\mathbf{x}})| \leq 2$.

The next result describes the sign structure of the real and imaginary parts of first eigenvectors of G_g.

THEOREM 3.5. *Consider G_g as in* Figure 2.1. *Let $C = [1, \ldots, m, 1]$ be the cycle with the green edge $e = (1, m)$. Let x be a first eigenvector of G_g. Then x satisfies the following properties:*

(a) On the vertices of the cycle C, $\mathsf{Re}\, x \neq 0$ and $\mathsf{Im}\, x \neq 0$.
(b) No entry of x at the vertices of the cycle C is zero.
(c) $\mathsf{Re}\, x$, (resp., $\mathsf{Im}\, x$) can be zero on at most one vertex of C.
(d) If v is a vertex on C such that $\mathsf{Re}\, x(v) = 0$ (resp., $\mathsf{Im}\, x(v) = 0$), then $\mathsf{Im}\, x$ (resp., $\mathsf{Re}\, x$) is either positive or negative on the vertices of C.
(e) If there exist two vertices u and v on C such that $\mathsf{Re}\, x(u) \,\mathsf{Re}\, x(v) < 0$ (resp., $\mathsf{Im}\, x(u) \,\mathsf{Im}\, x(v) < 0$), then $\mathsf{Im}\, x$ (resp., $\mathsf{Re}\, x$) is either positive or negative on the vertices of C.
(f) Let $\mathsf{Re}\, x(1) \,\mathsf{Re}\, x(m) < 0$ (resp., $\mathsf{Im}\, x(1) \,\mathsf{Im}\, x(m) < 0$). Then $\mathsf{Im}\, x$ (resp., $\mathsf{Re}\, x$) is either positive or negative, according as $\mathsf{Re}\, x(1)$ is negative or positive on the vertices of C..
(g) Let $\mathsf{Re}\, x(1) \,\mathsf{Re}\, x(m) > 0$ (resp., $\mathsf{Im}\, x(1) \,\mathsf{Im}\, x(m) > 0$), then $\mathsf{Re}\, x$ (resp., $\mathsf{Im}\, x$) is either positive or negative on the vertices of C, according as $\mathsf{Re}\, x(1)$ (resp., $\mathsf{Im}\, x(1)$) is positive or negative, respectively.

Proof. Consider the graph $\widehat{\mathsf{G}}$ corresponding to G_g as in Definition 2.16. Note that $\hat{\mathbf{x}}$ (as in Theorem 2.25) is a Fiedler vector of $\widehat{\mathsf{G}}$. For vertices y, z on C, let $P_{yz}^{(i)}$ denote the $y^{(i)}$-$z^{(i)}$-path in $\widehat{\mathsf{G}}$ corresponding to the y-z-path in $G_g - e$, for $i = 1, \ldots, 4$.

(a) Assume that $\mathsf{Re}\, x = 0$ on the vertices of C. By Theorem 3.2, x is nonzero on C. Let $\mathsf{Im}\, x(u) \neq 0$ for some vertex u on C. Observe that $\hat{\mathbf{x}}(u^{(2)}) = -\mathsf{Im}\, x(u) \neq 0$ and $\hat{\mathbf{x}}(u^{(4)}) = \mathsf{Im}\, x(u) \neq 0$. Also, $\hat{\mathbf{x}}(j^{(1)}) = \mathsf{Re}\, x(j) = 0$, and $\hat{\mathbf{x}}(j^{(3)}) = -\mathsf{Re}\, x(j) = 0$ for each vertex j of C. Thus, the paths $1^{(1)}m^{(2)} + P_{mu}^{(2)}$, $P_{u1}^{(2)} + 1^{(2)}m^{(3)}$ and $1^{(3)}m^{(4)} + P_{mu}^{(4)}$ in $\widehat{\mathsf{G}}$ contain at least one characteristic vertex each of $\widehat{\mathsf{G}}$. Thus, $|\mathcal{C}(\widehat{\mathsf{G}}, \hat{\mathbf{x}})| \geq 3$, a contradiction. Similarly, we can prove that $\mathsf{Im}\, x \neq 0$ on the vertices of C.

(b) Assume that u is a vertex on the cycle C such that $x(u) = 0$. Then $\mathsf{Re}\, x(u) = 0, \mathsf{Im}\, x(u) = 0$. By part(a), $\mathsf{Re}\, x \neq 0$ on the vertices of C. Let w be a vertex on C such that $\mathsf{Re}\, x(w) \neq 0$. Since $\hat{\mathbf{x}}(u^{(1)}) = \mathsf{Re}\, x(u) = 0$ and $\hat{\mathbf{x}}(w^{(1)}) = \mathsf{Re}\, x(w) \neq 0$, we see that $P_{uw}^{(1)}$ contains at least one characteristic vertex each of $\widehat{\mathsf{G}}$. Similarly, $P_{uw}^{(3)}$ contains at least one characteristic vertex each of $\widehat{\mathsf{G}}$. Note that $\hat{\mathbf{x}}(u^{(2)}) = -\mathsf{Im}\, x(u) = 0$. Thus, if w lies on the u-1-path in $G_g - e$, then the path $P_{w1}^{(1)} + 1^{(1)}m^{(2)} + P_{mu}^{(2)}$ contains at least one characteristic vertex of $\widehat{\mathsf{G}}$. Thus, $|\mathcal{C}(\widehat{\mathsf{G}}, \hat{\mathbf{x}})| \geq 3$, a contradiction. Similarly, if w lies on the m-u-path in $G_g - e$, we get a contradiction.

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(c) Assume that u and v are two vertices on C such that $\mathsf{Re}\,x(u) = 0 = \mathsf{Re}\,x(v)$, where $u < v$. By part(b), $\mathsf{Im}\,x(u) \neq 0$ and $\mathsf{Im}\,x(v) \neq 0$. Note that u is on the v-1-path in $G_g - e$. Since $\hat{\mathbf{x}}(u^{(1)}) = \mathsf{Re}\,x(u) = 0$ and $\hat{\mathbf{x}}(v^{(2)}) \neq 0$, we see that the path $P_{u1}^{(1)} + 1^{(1)}m^{(2)} + P_{mv}^{(2)}$ in $\widehat{\mathsf{G}}$ contains at least one characteristic vertex. Similarly the path $P_{u1}^{(3)} + 1^{(3)}m^{(4)} + P_{mv}^{(4)}$ contains at least one characteristic vertex of $\widehat{\mathsf{G}}$. Note that $\hat{\mathbf{x}}(u^{(2)}) = -\,\mathsf{Im}\,x(v) \neq 0$ and $\hat{\mathbf{x}}(v^{(3)}) = -\,\mathsf{Re}\,x(v) = 0$. Thus, the path $P_{u1}^{(2)} + 1^{(2)}m^{(3)} + P_{mv}^{(3)}$ contains at least one characteristic vertex of $\widehat{\mathsf{G}}$. Hence, $|\mathcal{C}(\widehat{\mathsf{G}}, \hat{\mathbf{x}})| \geq 3$, a contradiction. Similarly, we can prove for $\mathsf{Im}\,x$.

(d) Let v be a vertex on C such that $\mathsf{Re}\,x(v) = 0$. Assume that $\mathsf{Im}\,x(w) = 0$ for some vertex w on C. By part(b), $x(v) \neq 0$, $x(w) \neq 0$ which implies $\mathsf{Im}\,x(v) \neq 0$, $\mathsf{Re}\,x(w) \neq 0$. Note that $\hat{\mathbf{x}}(v^{(1)}) = \mathsf{Re}\,x(v) = 0$ and $\hat{\mathbf{x}}(w^{(1)}) = \mathsf{Re}\,x(w) \neq 0$. Thus, the path $P_{vw}^{(1)}$ contains at least one characteristic vertex of $\widehat{\mathsf{G}}$. Since $\hat{\mathbf{x}}(v^{(2)}) = -\,\mathsf{Im}\,x(v) \neq 0$ and $\hat{\mathbf{x}}(w^{(2)}) = -\,\mathsf{Im}\,x(w) = 0$, we see that $P_{vw}^{(2)}$ contains at least one characteristic vertex of $\widehat{\mathsf{G}}$. Similarly, we see that $P_{vw}^{(4)}$ contains at least one characteristic vertex of $\widehat{\mathsf{G}}$. Thus, $|\mathcal{C}(\widehat{\mathsf{G}}, \hat{\mathbf{x}})| > 2$, a contradiction. Similarly, if $\mathsf{Im}\,x(i)\,\mathsf{Im}\,x(j) < 0$ for some vertices i, j on C, we see that $|\mathcal{C}(\widehat{\mathsf{G}}, \hat{\mathbf{x}})| > 2$, a contradiction. Similarly, we can prove for $\mathsf{Im}\,x$.

(e) Let $\mathsf{Re}\,x(u)\,\mathsf{Re}\,x(v) < 0$, for some vertices u, v on the cycle C. Since $\hat{\mathbf{x}}(w^{(1)}) = \mathsf{Re}\,x(w)$ and $\hat{\mathbf{x}}(w^{(3)}) = -\,\mathsf{Re}\,x(w)$ for any vertex w in G, we see that $\hat{\mathbf{x}}(u^{(i)})\hat{\mathbf{x}}(v^{(i)}) < 0$ for $i = 1, 3$. Thus, the paths $P_{uv}^{(i)}$, for $i = 1, 3$ contain at least two characteristic elements of $\widehat{\mathsf{G}}$. If $\mathsf{Im}\,x(i) = 0$ for some vertex i on C, then by part (d), $\mathsf{Re}\,x$ is positive or negative on the vertices of C, a contradiction to our hypothesis. If $\mathsf{Im}\,x(i)\,\mathsf{Im}\,x(j) < 0$ for some vertices i, j on C, then $P_{ij}^{(2)}$ contains at least one characteristic element of $\widehat{\mathsf{G}}$. Thus, $|\mathcal{C}(\widehat{\mathsf{G}}, \hat{\mathbf{x}})| \geq 3$, a contradiction. Hence, the result holds. Similarly, we can prove for $\mathsf{Im}\,x$.

(f) Proof follows with similar argument.

(g) Suppose that $\mathsf{Re}\,x(1)\,\mathsf{Re}\,x(m) > 0$ and $\mathsf{Re}\,x(1) > 0$. Then $\mathsf{Re}\,x(m) > 0$. Since $\hat{\mathbf{x}}(1^{(1)}) = \mathsf{Re}\,x(1) > 0$ and $\hat{\mathbf{x}}(m^{(3)}) = -\,\mathsf{Re}\,x(m) < 0$, we see that the path $1^{(1)}m^{(2)} + P_{m1}^{(2)} + 1^{(2)}m^{(3)}$ contains at least one characteristic element of $\widehat{\mathsf{G}}$. Further, $\hat{\mathbf{x}}(1^{(3)}) = -\,\mathsf{Re}\,x(1) < 0$ and $\hat{\mathbf{x}}(m^{(1)}) = \mathsf{Re}\,x(m) > 0$ implies that the path $1^{(3)}m^{(4)} + P_{m1}^{(4)} + 1^{(4)}m^{(1)}$ contains at least one characteristic elements of $\widehat{\mathsf{G}}$. If $\mathsf{Re}\,x(u) = 0$ for some vertex u on C, then $\hat{\mathbf{x}}(u^{(3)}) = -\,\mathsf{Re}\,x(u) = 0$. In that case, the path $P_{um}^{(3)}$ contains a characteristic vertex of $\widehat{\mathsf{G}}$, which implies $|\mathcal{C}(\widehat{\mathsf{G}}, \hat{\mathbf{x}})| \geq 3$, a contradiction. Similarly, if $\mathsf{Re}\,x(i)\,\mathsf{Re}\,x(j) < 0$ for some vertices i, j on C we get a contradiction. Similarly, we can prove for $\mathsf{Im}\,x$. ∎

In our next result, we prove the nonnegativity of the real and imaginary parts on

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the nonsingular cycle of the first eigenvectors of G_g.

THEOREM 3.6. *Consider* G_g *as in* Figure 2.1. *Then* G_g *is* D*-similar to a* 3*-colored digraph* H*, such that a first eigenvector of* H *has nonnegative real and imaginary parts on the vertices of the nonsingular cycle in* H.

Proof. Let x be a first eigenvector of G_g.

If $\mathsf{Re}\, x \geq 0$, $\mathsf{Im}\, x \leq 0$ on the vertices of C, then we take $D = -\mathrm{i}I_n$. If $\mathsf{Re}\, x \leq 0$, $\mathsf{Im}\, x \geq 0$ on the vertices of C, then we take $D = \mathrm{i}I_n$. In these cases D^*x is nonnegative on the nonsingular cycle of ${}^DG_g$.

First we assume that $\mathsf{Im}\, x$ is not nonnegative on C. Let u, v be two vertices on C, such that $\mathsf{Im}\, x(u) > 0$ and $\mathsf{Im}\, x(v) < 0$ with $1 \leq u < v \leq m$. By Lemma 3.5(e), $\mathsf{Re}\, x$ is either positive or negative on C. Without loss of generality suppose that $\mathsf{Re}\, x > 0$ on C.

Case 1: If $\mathsf{Im}\, x(i) \neq 0$ for $i = 1, \ldots, m$, then the u-v-path in $G_g - e$ has an edge f joining vertices i_0, i_0+1 with $\mathsf{Im}\, x(i_0) > 0$ and $\mathsf{Im}\, x(i_0+1) < 0$. Since $|\mathcal{C}(\widehat{\mathsf{G}}, x)| \leq 2$, we see that $\mathsf{Im}\, x(j) > 0$ for each $j = 1, \ldots, i_0$ and $\mathsf{Im}\, x(j) < 0$ for each $j = i_0 + 1, \ldots, m$. Let G_1 and G_2 be the two components of $G_b - f - e$, such that G_1 contains the vertex 1 and G_2 contains the vertex m. Take $D = -\mathrm{i}I_{|V(G_2)|} \oplus I_{|V(G_1)|}$. Then $\mathsf{Re}\, D^*x$ and $\mathsf{Im}\, D^*x$ are nonnegative on the vertices of C, and D^*x is a first eigenvector of ${}^DG_g$ with the green edge f.

Case 2: If $\mathsf{Im}\, x(i) = 0$, for some vertex i on C. Let f be the edge on C, joining the vertices $i_0, i_0 + 1$. Since $|\mathcal{C}(\widehat{\mathsf{G}}, x)| \leq 2$, we see that $\mathsf{Im}\, x(j) \neq 0$, for each vertex $j \neq i_0$ on C. Moreover, $\mathsf{Im}\, x(j) > 0$ for each $j = 1, \ldots, i_0 - 1$ and $\mathsf{Im}\, x(j) < 0$ for each $j = i_0 + 1, \ldots, m$. Let G_1 and G_2 be the two components of $G_b - f - e$, such that G_1 contains the vertex 1 and G_2 contains the vertex m. Take $D = -\mathrm{i}I_{|V(G_2)|} \oplus I_{|V(G_1)|}$. Then $\mathsf{Re}\, D^*x$ and $\mathsf{Im}\, D^*x$ are nonnegative on the vertices of C, and D^*x is a first eigenvector of ${}^DG_g$ with the green edge f. The rest of the proof follows with similar argument. ◻

In the next theorem, we prove that $\lambda_1(L(G_g))$ has multiplicity one.

THEOREM 3.7. *Consider* G_g *as in* Figure 2.1. *Let* $C = [1, \ldots, m, 1]$ *be the cycle in* G_g *with the green edge* $e = (1, m)$. *Then multiplicity of* $\lambda_1(L(G_g))$ *is one.*

Proof. Let Z_1, Z_2 be two linearly independent first eigenvectors of G_g. By Theorem 3.5(a), $Z_1(1) \neq 0$, $Z_2(1) \neq 0$. Let $Z = \alpha Z_1 + \beta Z_2$, where $\alpha = Z_2(1)$, $\beta = -Z_1(1)$. Then Z is a first eigenvector of G_g such that $Z(1) = 0$, a contradiction to Theorem 3.5(a). ◻

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Next, we discuss the multiplicity of $\lambda_1(L(G_b))$.

REMARK 3.8. Consider G_b as in Figure 2.2 and take $\widetilde{\mathsf{G}}$. Let x be a first eigenvector of G_b. By Lemma 2.20, $Y = \begin{bmatrix} x \\ -x \end{bmatrix}$ is a Fiedler vector of $\widetilde{\mathsf{G}}$. For vertices y, z on C, let $P_{yz}^{(i)}$ denote the $y^{(i)}$-$z^{(i)}$-path in $\widetilde{\mathsf{G}}$ corresponding to the y-z-path in $G_b - e$, for $i = 1, 2$. Consider the cycle $\widetilde{C} = P_{1m}^{(1)} + m^{(1)}1^{(2)} + P_{1m}^{(2)} + m^{(2)}1^{(1)}$ in $\widetilde{\mathsf{G}}$. Note that $Y(j^{(1)}) = x(j)$ and $Y(j^{(2)}) = -x(j)$ for each vertex j of G_b. If valuations of x at the vertices of C are all zero, then the valuations of Y at the vertices of $\widetilde{C}$ are all zero. In that case, $\widetilde{\mathsf{G}}$ neither satisfies Case A nor Case B of Proposition 3.1, with respect to Y, a contradiction. Thus, there exists a vertex u on the nonsingular cycle in G_b such that $x(u) \neq 0$. Hence, the characteristic set of $\widetilde{\mathsf{G}}$ lies on $\widetilde{C}$. By Lemma 2.22, $|\mathcal{C}(\widetilde{\mathsf{G}}, Y)| \leq 2$.

The following result is a generalization of [8, Corollary 2.8].

LEMMA 3.9. *Consider G_b as in* Figure 2.2. *Let x be a first eigenvector of G_b. Then there exists at most one vertex i on the nonsingular cycle C such that $x(i) = 0$.*

Proof. Let $C = [1, \ldots, m, 1]$ and $e = 1m$ be the blue edge. Assume that there exist two vertices u and v with $u < v$ on C such that $x(u) = x(v) = 0$. By Remark 3.8, $x(w) \neq 0$, for some vertex w on C. By Lemma 2.20, $Y = \begin{bmatrix} x \\ -x \end{bmatrix}$ is a Fiedler vector of $\widetilde{\mathsf{G}}$. Thus, $Y(j^{(1)}) = x(j)$ and $Y(j^{(2)}) = -x(j)$ for each vertex j of G_b. If w is on the u-v-path in $G_b - e$ path, then the paths $P_{uw}^{(1)}$, $P_{uw}^{(2)}$, $P_{wv}^{(1)}$ and $P_{wv}^{(2)}$ in $\widetilde{\mathsf{G}}$ contain at least one characteristic vertex each. Thus, $|\mathcal{C}(\widetilde{\mathsf{G}}, Y)| \geq 3$, a contradiction. If w is not on the u-v-path in $G_b - e$, then the paths $P_{u1}^{(1)} + 1^{(1)}m^{(2)} + P_{mv}^{(2)}$ and $P_{vm}^{(1)} + m^{(1)}1^{(2)} + P_{1u}^{(2)}$ in $\widetilde{\mathsf{G}}$ contain at least two characteristic vertex each, which implies $|\mathcal{C}(\widetilde{\mathsf{G}}, Y)| > 2$, a contradiction. Hence, the result holds. ☐

In [8], the authors proved that multiplicity of $\lambda_1(L(G_b)$ is at most two when G_b is unicyclic. Here G_b is not necessarily unicyclic. Hence, the following result is a generalization of [8, Theorem 2.14].

LEMMA 3.10. *Consider G_b as in* Figure 2.2. *Then $\lambda_1(L(G_b))$ has multiplicity at most two.*

Proof. Proof is similar to that of [8, Theorem 2.14] using Lemma 3.9. ☐

4. Minimizing the smallest Laplacian eigenvalue. In this section, we consider the nonsingular unicyclic 3-colored digraphs. In view of Lemma 2.5, throughout this section, all our nonsingular unicyclic 3-colored digraphs have all edges red except one blue or green edge on the cycle. Fan [6] has obtained the nonsingular unicyclic mixed graphs on n vertices with a fixed girth, which minimizes the smallest Laplacian

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eigenvalue over all such graphs. The unique mixed graph which minimizes the smallest Laplacian eigenvalue over all nonsingular unicyclic mixed graphs on n vertices was determined in [8].

We ask the following question: *Does there exist a unique nonsingular unicyclic 3-colored digraph on n vertices, which minimizes the smallest Laplacian eigenvalue over all such graphs?* In this section, we answer this question in the affirmative.

The next lemma is an immediate application of the interlacing theorem, see [12].

LEMMA 4.1. *Let G be a 3-colored digraph and let G' be the graph obtained from G by adding a pendent vertex. Then $\lambda_1(L(G')) \leq \lambda_1(L(G))$.*

LEMMA 4.2. [5] *Consider the cycle C on n vertices with $\mathsf{w}_C = -1$. Then the Laplacian spectrum of C is $\{2[1 - \cos\frac{(2i-1)\pi}{n}] : i = 1, \dots, n\}$.*

DEFINITION 4.3. [13] Let G be a 3-colored digraph on vertices $1, \dots, n$ and let F_g be the set of green edges in G. Let G' be a copy of G, in which we replace the label of the vertex i by i', for each $i = 1, \dots, n$. Let F_g' denote the green edges in G' corresponding to F_g. Construct the mixed graph on $2n$ vertices obtained from $(G - F_g) \cup (G' - F_g')$ by inserting a red edge uv' and a blue edge $u'v$, for each green edge $(u, v) \in F_g$. We denote this graph by $\mathsf{G}[\mathsf{g}]$.

Below we sate a result due to Kalita and Pati contained in [13] which shall be used.

LEMMA 4.4. *Let G be a 3-colored digraph. Then the Laplacian spectrum of G, without considering multiplicities, is the same as that of $\mathsf{G}[\mathsf{g}]$. Further, if x is an eigenvector of G corresponding to an eigenvalue λ, then λ is an eigenvalue of $\mathsf{G}[\mathsf{g}]$ with linearly independent eigenvectors* $\begin{bmatrix} \mathsf{Re}\, x \\ -\mathsf{Im}\, x \end{bmatrix}, \begin{bmatrix} \mathsf{Im}\, x \\ \mathsf{Re}\, x \end{bmatrix}$.

LEMMA 4.5. *Consider the cycle C on n vertices with $\mathsf{w}_C = \pm\mathrm{i}$. Then the Laplacian spectrum of C is* $\left\{2[1 - \cos\frac{(2i-1)\pi}{2n}] : i = 1, \dots, n\right\}$.

Proof. By Lemma 4.4, the set of Laplacian eigenvalues of C is the same as that of $C[\mathsf{g}]$. Note that $C[\mathsf{g}]$ is a cycle of length $2n$ and weight -1. Hence, the result holds, by Lemma 4.2. ☐

DEFINITION 4.6. Consider the 3-colored digraph obtained by appending a cycle $C = [1, \dots, m, 1]$ to a pendent vertex of the undirected path on $n - m$ vertices. Call this 3-colored digraph a *lollipop graph*, *b-lollipop graph*, or *g-lollipop graph* if $\mathsf{w}_C = 1$, $\mathsf{w}_C = -1$, or $\mathsf{w}_C = \pm\mathrm{i}$, respectively. By $\mathsf{C_{n,m}}$, $^{\mathsf{b}}\mathsf{C_{n,m}}$, and $^{\mathsf{g}}\mathsf{C_{n,m}}$ we mean a lollipop graph, b-lollipop graph, and g-lollipop graph, respectively.

EXAMPLE 4.7. The following picture is an example of a g-lollipop graph $^{\mathsf{g}}\mathsf{C_{n,m}}$

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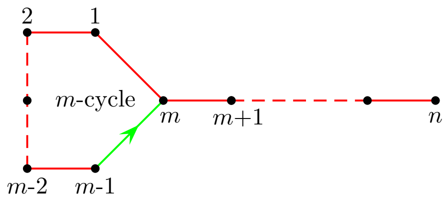

Note that $\mathsf{C_{n,m}}$ is singular and henceforth, we assume that all the edges in $\mathsf{C_{n,m}}$ are red, in view of Theorem 2.2. Fallat, Kirkland and Pati [4] have proved that for $n \geq \frac{3m-1}{2}$ and $m \geq 4$, $a(\mathsf{C_{n,m}}) > a(\mathsf{C_{n,m-1}})$. Recently Guo, Shiu and Li [11] have proved the following result.

Lemma 4.8. [11, Theorem 2.11] *Let $\mathsf{C_{n,m}}$ be the lollipop graph on n vertices with girth m. Then $a(\mathsf{C_{n,m-1}}) < a(\mathsf{C_{n,m}})$, for $m \geq 4$.*

Lemma 4.9. *Let G be a unicyclic 3-colored digraph of girth m with the cycle C of weight $\pm$i. Then $\lambda_1(L(G)) \leq 2(1 - \cos\frac{\pi}{2m}) < \frac{1}{2}$. Equality holds if and only if G is a cycle of weight $\pm$i on m vertices.*

Proof. Note that G can be obtained from C by adding pendent vertices repeatedly. By Lemma 4.1 and Lemma 4.5, $\lambda_1(L(G)) \leq 2(1 - \cos\frac{\pi}{2m}) < \frac{1}{2}$.

Assume that G is not a cycle. By Theorem 2.25, $\lambda_1(L(G)) = a(\widehat{\mathsf{G}})$, where $\widehat{\mathsf{G}}$ is as in Definition 2.16, constructed from G_g (as in Figure 2.1) corresponding to G. Since G has at least one pendent vertex, we see that $\widehat{\mathsf{G}}$ can be obtained from the lollipop graph $C_{4m+1,4m}$ by adding pendents sequentially. Thus, by Lemma 4.1 and 4.8,

$$a(\widehat{\mathsf{G}}) \leq a(C_{4m+1,4m}) < 2(1 - \cos\frac{2\pi}{4m+1}) < 2(1 - \cos\frac{\pi}{2m}).$$

Hence, $\lambda_1(L(G)) < 2(1 - \cos\frac{\pi}{2m})$. So the equality holds if and only if G has no pendent vertices. ◻

Lemma 4.10. [6, Theorem 2.6] *Let G be a nonsingular unicyclic mixed graph with a fixed girth m. Then $\lambda_1(L(G)) \leq 2(1 - \cos\frac{\pi}{m})$, where equality holds if and only if G is a nonsingular cycle on m vertices.*

Remark 4.11. Let G be a unicyclic 3-colored digraph with fixed girth m and the cycle contains an even number of green edges. Then G is D-similar to a nonsingular unicyclic mixed graph. Hence, by Lemma 4.10, $\lambda_1(L(G)) \leq 2(1 - \cos\frac{\pi}{3})$.

Lemma 4.12. *Let G be a unicyclic 3-colored digraph on n vertices with a fixed girth m. Let T be the tree on k vertices attached to a vertex j on the cycle in G. If G' is the unicyclic 3-colored digraph obtained from G by replacing T with the path P_k, then $\lambda_1(L(G)) \geq \lambda_1(L(G'))$.*

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Proof. Let $i_1, \ldots, i_k$ be the vertices T with $j = i_1$. Let x be a normalized eigenvector of $L(G)$ corresponding to $\lambda_1(L(G))$ with $\mathsf{Im}\, x(j) = 0$. By Theorem 3.2, $\mathsf{Im}\, x(i_r) = 0$, for $1 \leq r \leq k$. By Theorem 3.5(d), we have $\mathsf{Re}\, x > 0$. In view of Theorem 3.2, we can arrange the vertices of T with $0 < \mathsf{Re}\, x(i_1) \leq \cdots \leq \mathsf{Re}\, x(i_k)$. Thus,

$$\begin{aligned}
\lambda_1(L(G)) &= x^* L(G - E(T))x + \sum_{ij \in E(T)} |x(i) - x(j)|^2 \\
&\geq x^* L(G - E(T))x + \sum_{j=1}^{k-1} |x(i_j) - x(i_{j+1})|^2 \\
&= x^* L(G - E(T) + P_k)x = x^* L(G')x \geq \lambda_1(L(G')). \quad \square
\end{aligned}$$

The next Lemma follows from Lemma 4.12 which generalizes [6, Theorem 3.2].

Lemma 4.13. *Among all nonsingular unicyclic* 3*-colored digraphs on n vertices with a fixed girth m, the smallest Laplacian eigenvalue is minimized by a nonsingular unicyclic* 3*-colored digraph with girth m having the following property: there are at most two connected components at every vertex on the cycle, and the components not including the vertices on the cycle (if exists) is a path.*

The next lemma is a generalization of [8, Lemma 3.2].

Lemma 4.14. *Let G be a nonsingular unicyclic* 3*-colored digraph on n vertices with a fixed girth m obtained from a cycle C by attaching at most one path to each vertex i of C. Let $P_i = i, i_1, i_2, \ldots, i_r$ $(r \geq 1)$ and $P_j = j, j_1, \ldots, j_s$, $(s \geq 1)$ be the paths attached to the vertex i and j on C, respectively. Let x be a first eigenvector of G with $|x(j)| \geq |x(i)|$. If $G_1 = G - ii_1 + j_s i_1$, then $\lambda_1(L(G)) \geq \lambda_1(L(G_1))$.*

Proof. Without loss of generality, we assume that $\mathsf{Im}\, x(i) = 0$. Then, by Theorem 3.5(d), $\mathsf{Re}\, x > 0$ and by Theorem 3.2, $x(i_k)$ are real for $k = 1, \ldots, r$ and

$$x(i) \leq x(i_1) \leq \cdots \leq x(i_r).$$

Let $y \in \mathbb{C}^n$ be defined on the vertices of G_1 such that

$$\begin{cases} y(i_k) = x(i_k) + x(j_s) - x(i) \text{ for } k = 1, \ldots, r, \\ y(v) = x(v) \text{ for } v \neq i_k, k = 1, \ldots, r. \end{cases}$$

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Observe that $x^*L(G)x = y^*L(G_1)y$ and

$$\begin{aligned} y^*y &= \sum_{v \in V(G)\setminus\{i_1,i_2,\ldots,i_r\}} |x(v)|^2 + \sum_{k=1}^{r} |x(i_k) + x(j_s) - x(i)|^2 \\ &= x^*x + r\,|x(j_s) - x(i)|^2 + 2\,\mathsf{Re}\left(\Big(x(j_s) - x(i)\Big)\sum_{k=1}^{r}\overline{x(i_k)}\right) \\ &\geq x^*x + r\Big(|x(j_s)|^2 + |x(i)|^2 - 2x(i)\,\mathsf{Re}\,x(j_s)\Big) + 2rx(i)\Big(\mathsf{Re}\,x(j_s) - x(i)\Big) \\ &\geq x^*x + r\Big(|x(j_s)|^2 - |x(i)|^2\Big). \end{aligned}$$

By Theorem 3.2, $|x(i)| \leq |x(j)| \leq |x(j_1)| \leq \cdots \leq |x(j_s)|$. So we have $y^*y \geq x^*x$. Hence,

$$\lambda_1(L(G)) = \frac{x^*L(G)x}{x^*x} \geq \frac{y^*L(G_1)y}{y^*y} \geq \lambda_1(L(G_1)). \quad \square$$

By $^{\mathsf{g}}\mathsf{C_n}$ and $^{\mathsf{b}}\mathsf{C_n}$ we mean a cycle of weight $\pm\mathrm{i}$ and a cycle of weight -1, on n vertices, respectively.

In the next result, we prove that among all nonsingular unicyclic 3-colored digraphs on n vertices with a fixed girth m, $^{\mathsf{g}}\mathsf{C_{n,m}}$ minimizes the smallest eigenvalue.

Theorem 4.15. *Let G be a nonsingular unicyclic 3-colored digraph on n vertices with a fixed girth m. Then $\lambda_1(L(G)) \geq \lambda_1(L(^{\mathsf{g}}\mathsf{C_{n,m}}))$.*

Proof. Let $C = [1,\ldots,m,1]$ be the cycle in G and let $H := C + \sum_{i=1}^{k}(w_iv_i + P_{n_i})$, where P_{n_i} is a path on n_i vertices with pendent vertices u_i and v_i, w_i is a vertex on C, for $i = 1,\ldots,k$ and $n = m + \sum_{i=1}^{k} n_i$. Let x be a first eigenvector of H such that $\max\limits_{1\leq i\leq k} |x(w_i)| = |x(w_0)|$. Thus, by Theorem 4.14, $\lambda_1(L(H)) \geq \lambda_1(L(H_1))$, where $H_1 = (H - w_0v_0) + w_1v_0 + P_{n_0}$. Note that the graph H_1 has exactly $k-1$ paths attached to $k-1$ vertices of the cycle. By Lemma 2.24, $\lambda_1(L(^{\mathsf{b}}\mathsf{C_{n,m}})) \geq \lambda_1(L(^{\mathsf{g}}\mathsf{C_{n,m}}))$. Thus, with similar argument, after a finite number of steps we have

$$\lambda_1(L(H)) \geq \lambda_1(L(H_1)) \geq \cdots \geq \lambda_1(L(^{\mathsf{b}}\mathsf{C_{n,m}})) \geq \lambda_1(L(^{\mathsf{g}}\mathsf{C_{n,m}})).$$

Note that $\lambda_1(L(G)) \geq \lambda_1(L(H))$, by Lemma 4.14. Hence, the result holds. $\square$

Definition 4.16. Consider the path $P_n, n \geq 3$ on vertices $1,\ldots,n$. Add the blue colored edge joining the vertices k and $n-k+1$ of P_n, for $1 \leq k < \frac{n}{2}$ to obtain a mixed graph. We denote this graph by $G(n,k)$.

Example 4.17. Mixed graphs $G(7,3)$ and $G(8,4)$ are shown in the following picture.

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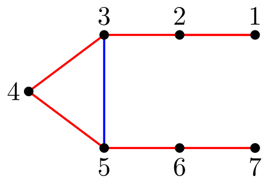
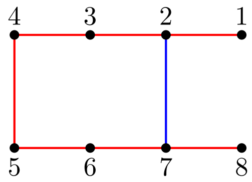


$G(7,3)$



$G(8,2)$

The following lemma is essentially contained in [8, Theorem 3.4].

LEMMA 4.18. *Consider the b-lollipop graph* ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m}}$. *Then the following holds:*

(i) Multiplicity of the smallest Laplacian eigenvalue of ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m}}$ *is one, for* $m < n$.
(ii) If x *is a first eigenvector of* ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m}}$, *then* $x(i) = -x(m-i)$, *for* $i = 1, \ldots, m-1$.
(iii) $\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m-1}})) < \lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n,m}}))$, *for* $m > 3$.

DEFINITION 4.19. Let G be a 3-colored digraph. By F_r, F_b, and F_g we mean the set of red edges, blue edges and green edges in G, respectively. Let x be an eigenvector of $L(G)$ corresponding to an eigenvalue λ. By *eigen-condition* at a vertex i, we mean the following equation.

$$(d_i - \lambda)x(i) = \sum_{ij \in F_r} x(j) - \sum_{ij \in F_b} x(j) + \mathrm{i} \sum_{(i,j) \in F_g} x(j) - \mathrm{i} \sum_{(j,i) \in F_g} x(j).$$

REMARK 4.20. Observe that if Y is an eigenvector of $L(P_n)$ corresponding to an eigenvalue λ, then we see that $Y' = \begin{bmatrix} Y(n) & Y(n-1) & \cdots & Y(1) \end{bmatrix}^t$ is an eigenvector of $L(P_n)$ corresponding to the same eigenvalue λ. Note that $L(P_n)$ has distinct eigenvalues $2(1 - \cos\frac{\pi j}{n})$, for $j = 1, \ldots, n$. Thus, $Y = \alpha Y'$, for some $\alpha \in \mathbb{R}$, which implies $\alpha = \pm 1$. Hence, $Y(i) = \pm Y(n-i+1)$ for $i = 1, \ldots, n$. Moreover, if T is a tree on n vertices then $a(P_n) = 2(1 - \cos\frac{\pi}{n}) \le a(T)$, see [9].

In the next lemma, we prove that the smallest Laplacian eigenvalue of $G(n,k)$ for $k \neq 1, n$ is simple and if n is even, then $\lambda_1(L(G(n,k)))$ is equal to the algebraic connectivity of P_n for each k.

LEMMA 4.21. *Consider the mixed graph* $G(n,k)$ *as in* Definition 4.16

(i) $\lambda_1(L(G(n,k)))$ *has multiplicity one, for each* $k \neq 1, n$.
(ii) If n *is even, then* $\lambda_1(L(G(n,k)) = a(P_n)$.

Proof. (i) Let Y be a first eigenvector of $G(n,k)$. Assume that $Y(n) = 0$. Using Definition 4.19, the eigen-condition at the vertex n of $G(n,k)$, we have

$$[1 - \lambda_1(L(n,k))]Y(n) = Y(n-1),$$

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which implies $Y(n-1)=0$. Similarly, using the eigen-conditions at the vertices $n-1,\dots,n-k+2$ of $G(n,k)$ we have $Y(n-2)=0,\ \dots,\ Y(n-k+1)=0$, respectively. Thus, $Y'=\begin{bmatrix}Y(1) & Y(2) & \cdots & Y(n-k+1)\end{bmatrix}^t$ is an eigenvector of ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}}$ corresponding to the eigenvalue $\lambda_1(L(G(n,k))$. So

$$\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}})) \leq \lambda_1(L(G(n,k))).$$

Note that $G(n,k)$ can be obtained from ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}}$ by adding pendent vertices sequentially. Thus, by Lemma 4.1,

$$\lambda_1(L(G(n,k))) \leq \lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}})).$$

Hence, $\lambda_1(L(G(n,k))) = \lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}}))$ and Y' is an eigenvector corresponding to $\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n-k+1,n-2k+2}}))$. Observe that $Y'(k+1)=-Y'(n-k+1)$, by Lemma 4.18(ii). So $Y(k+1)=-Y(n-k+1)=0$, a contradiction to Lemma 3.9. Thus, $Y(n)\neq 0$ for any first eigenvector Y of $G(n,k)$. Assume that there exist two linearly independent first eigenvectors Z_1,Z_2 of $G(n,k)$. Let $Z=\alpha Z_1+\beta Z_2$, where $\alpha=Z_2(n)\neq 0$, $\beta=-Z_1(n)\neq 0$. Then Z is a first eigenvector of $G(n,k)$ with $Z(n)=0$, a contradiction. Hence, $\lambda_1(L(G(n,k)))$ has multiplicity one, for $k\neq 1,n$.

(ii) Let n be even. If $k=1,n$, then $G(n,k)={}^{\mathsf{b}}\mathsf{C}_{\mathsf{n}}$. By Lemma 4.2, $\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n}}))=2(1-\cos\frac{\pi}{n})=a(P_n)$. Assume that $k\neq 1,n$. Let y be a Fiedler vector of P_n. Then $y(k)=-y(n-k+1)$. Hence, y is an eigenvector of $L(G(n,k)))$ corresponding to $a(P_n)$. Thus, $\lambda_1(L(G(n,k)))\leq a(P_n)$. Assume that $\lambda_1(L(G(n,k)))<a(P_n)$. Let z be an eigenvector of $L(G(n,k)))$ corresponding to the eigenvalue $\lambda_1(L(G(n,k)))$. Observe that $z'=\begin{bmatrix}z(n) & z(n-1) & \cdots & z(1)\end{bmatrix}^t$ is an eigenvector of $L(G(n,k)))$ corresponding to the eigenvalue $\lambda_1(L(G(n,k)))$. By part (i), multiplicity of $\lambda_1(L(G(n,k)))$ is one. Thus, $z=\alpha z'$ for some $\alpha\in\mathbb{R}$, which implies $\alpha=\pm 1$. If $\alpha=1$, then $z(i)=z(n-i+1)$ for each i, $1\leq i\leq\frac{n}{2}$. In that case, z is an eigenvalue of $L(G')$, where G' is the graph obtained from $G(n,k)$ by removing the red edge joining the vertices $\frac{n}{2}$ and $\frac{n}{2}+1$. Note that the underlying graph of G' is a tree. By Lemma 2.4, G' is singular. So $\lambda_2(L(G'))\leq\lambda_1(L(G(n,k)))$. By Lemma 2.2, ${}^{D}G'$ has all edges red for some D. Using Remark 4.20, $a(P_n)\leq a({}^{D}G')=\lambda_2(L(G'))\leq\lambda_1(L(G(n,k)))$, a contradiction. If $\alpha=-1$, then $z(k)=-z(n-k+1)$. In that case, z is an eigenvector of $L(P_n)$ corresponding to the eigenvalue $\lambda_1(L(G(n,k)))$. Thus, $a(P_n)\leq\lambda_1(L(G(n,k)))$, a contradiction. Hence, the result holds. ☐

The next lemma is crucial in proving our main result of this section.

LEMMA 4.22. $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}))\geq\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}}))$.

Proof. Observe that ${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}={}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}}$ for $n=m$. Hence, the lemma holds for $m=n$. Let $m<n$ and let $x\in\mathbb{C}^n$ be a first eigenvector of ${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}$ such that $\mathsf{Im}\,x(m)=0$. By Theorem 3.5, $\mathsf{Re}\,x(m)\neq 0$ and $\mathsf{Im}\,x(m)=\mathsf{Im}\,x(m+1)=\cdots=\mathsf{Im}\,x(n)=0$. By

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Lemma 4.4, $\lambda_1(L(\mathsf{G[g]})) = \lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}))$, where $G={}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}$. Note that $Y = \begin{bmatrix} \mathsf{Re}\, x \\ -\mathsf{Im}\, x \end{bmatrix}$ is an eigenvector of $\mathsf{G[g]}$ corresponding to the eigenvalue $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}))$, by Lemma 4.4. Thus, Y is a first eigenvector of $\mathsf{G[g]}$. Notice that $Y(m') = Y(m'+1) = \cdots = Y(n') = 0$, where $m', \ldots, n'$ are the vertices in $\mathsf{G[g]}$ corresponding to the vertices $m, \ldots, n$ (see Definition 4.3). So $\lambda_1(L(\mathsf{G[g]}))$ is an eigenvalue of $L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n+m,2m}})$ with an eigenvector $Z = \begin{bmatrix} Y(1) & \cdots & Y(n), & Y(1') & \cdots & Y(m') \end{bmatrix}^t$, which implies $\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n+m,2m}})) \leq \lambda_1(L(\mathsf{G[g]}))$. Note that ${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}}$ can be obtained from ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n+m,2m}}$ by adding pendent vertices sequentially. Thus, by Lemma 4.1, $\lambda_1(L(\mathsf{G[g]})) \leq \lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n+m,2m}}))$. Hence,

$$\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}})) = \lambda_1(L(\mathsf{G[g]})) = \lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n+m,2m}})).$$

Observe that $G(2n, n-m+1)$ can be obtained from ${}^{\mathsf{b}}\mathsf{C}_{\mathsf{n+m,2m}}$ by adding pendent vertices sequentially. Thus, $\lambda_1(L({}^{\mathsf{b}}\mathsf{C}_{\mathsf{n+m,2m}})) \geq \lambda_1(L(G(2n, n-m+1))) = a(P_{2n})$, using Lemma 4.1 and Lemma 4.21(ii). Hence, $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}})) \geq a(P_{2n}) = 2(1-\cos\frac{\pi}{2n})$. Since ${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}}$ is a cycle of weight $\pm\mathrm{i}$, we see that $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}})) = 2(1-\cos\frac{\pi}{2n})$, by Lemma 4.5. Hence, the lemma holds. ☐

The next lemma provides a lower bound for the smallest Laplacian eigenvalue of unicyclic 3-colored digraphs.

Lemma 4.23. *Let G be a nonsingular unicyclic 3-colored digraph on n vertices. Then*

$$\lambda_1(L(G)) \geq 2(1-\cos\frac{\pi}{2n}).$$

Equality holds if and only if G is a cycle ${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}}$.

Proof. Using Theorem 4.15 and Lemma 4.22 we have,

$$\lambda_1(L(G)) \geq \lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n,m}})) \geq \lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}})).$$

Note that $\lambda_1(L({}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}})) = 2(1-\cos\frac{\pi}{2n})$, by Lemma 4.5. Hence, the result holds. ☐

The next theorem is our main result of this section which provides the unique graph minimizing the smallest Laplacian eigenvalue over all nonsingular unicyclic 3-colored digraphs on n vertices.

Theorem 4.24. *Among all nonsingular unicyclic 3-colored digraphs on n vertices, the smallest Laplacian eigenvalue is uniquely minimized by the cycle ${}^{\mathsf{g}}\mathsf{C}_{\mathsf{n}}$.*

Proof. Proof follows from Lemma 4.23. ☐

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