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REPRESENTATIONS FOR THE DRAZIN INVERSE OF BLOCK CYCLIC MATRICES∗

M. CATRAL† AND P. VAN DEN DRIESSCHE‡

Abstract. A formula for the Drazin inverse of a block k-cyclic (k ≥ 2) matrix A with nonzeros only in blocks $A_{i,i+1}$, for $i = 1, \ldots, k \ (\text{mod } k)$ is presented in terms of the Drazin inverse of a smaller order product of the nonzero blocks of A, namely $B_i = A_{i,i+1} \cdots A_{i-1,i}$ for some $i$. Bounds on the index of A in terms of the minimum and maximum indices of these $B_i$ are derived. Illustrative examples and special cases are given.

Key words. Drazin inverse, Block cyclic matrix, Index.

AMS subject classifications. 15A09.

1. Introduction. We consider $k$-cyclic ($k \geq 2$) block real or complex matrices of the form

$$A = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $A_{12}, \ldots, A_{k1}$ are block submatrices and the diagonal zero blocks are square. It is easily verified that for any matrix $A$ of the form (1.1), the Moore-Penrose inverse $A^\dagger$ of $A$ is given by

$$A^\dagger = \begin{bmatrix} 0 & 0 & \cdots & 0 & A^\dagger_{k1} \\ A^\dagger_{12} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A^\dagger_{k-1,k} & 0 \end{bmatrix}.$$
where $A_{ij}^\dagger$ denotes the Moore-Penrose inverse of the block submatrix $A_{ij}$. Note that if each of the blocks $A_{ij}$ is square and invertible, then (1.2) gives the formula for the usual inverse $A^{-1}$ of $A$. We present a block formula for another type of generalized inverse, the Drazin inverse, of matrices of the form (1.1). Unlike the Moore-Penrose inverse, the Drazin inverse is defined only for square matrices.

Let $A$ be a real or complex square matrix. The Drazin inverse of $A$ is the unique matrix $A^D$ satisfying

\begin{align}
AA^D &= A^D A \\
A^D AA^D &= A^D \\
A^{q+1}A^D &= A^q,
\end{align}

where $q = \text{index } A$, the smallest nonnegative integer $q$ such that rank $A^{q+1} = \text{rank } A^q$. If index $A = 0$, then $A$ is nonsingular and $A^D = A^{-1}$. If index $A = 1$, then $A^D = A^#$, the group inverse of $A$. See [1], [2], [6] and references therein for applications of the Drazin inverse.

**Theorem 1.1.** [2, Theorem 7.2.3] Let $A$ be a square matrix with index $A = q$. If $p$ is a nonnegative integer and $X$ is a matrix satisfying $XAX = X$, $AX =XA$, and $A^{q+1}X = A^p$, then $p \geq q$ and $X = A^D$.

The problem of finding explicit representations for the Drazin inverse of a general $2 \times 2$ block matrix of the form

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \]

in terms of its blocks was posed by Campbell and Meyer in [2] and special cases of this problem were the focus of several recent papers, including [3]–[10], [13], [14] and [15]. In [4] and [14], representations for $2 \times 2$ block matrices matrices of the form (1.6) with $A_{11}$ and $A_{22}$ being square zero diagonal blocks were presented. Such block matrices were called bipartite (or 2-cyclic), and in this article, we extend the results given in [4] to general block $k$-cyclic matrices as defined in (1.1).

**2. Drazin inverse formula for block cyclic matrices.** Let $A$ be a block $k$-cyclic matrix of the form given in (1.1). For our Drazin inverse formula we introduce some notation that is also used in writing powers of $A$. For $i = 2, \ldots, k-1$, let $B_i$ be the square matrix defined by

\[ B_i = A_{i,i+1} \cdots A_{k-1,k} A_{k1} A_{12} \cdots A_{i-1,i}, \]

with $B_1 = A_{12} A_{23} \cdots A_{k-1,k} A_{k1}$ and $B_k = A_{k1} A_{12} \cdots A_{k-1,k}$, i.e., subscripts are taken mod $k$. For ease of notation, we define the matrix product

\[ A_{i \rightarrow j} := A_{i,i+1} A_{i+1,i+2} \cdots A_{j-1,j}, \]
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for \( j \neq i \). Whenever it arises, we use the convention \( A_{i \rightarrow i} = I \), an identity matrix. For example, if \( k = 4 \) then \( A_{2 \rightarrow 3} = A_{23}, A_{3 \rightarrow 2} = A_{34}A_{41}A_{12}, A_{2 \rightarrow 1} = A_{23}A_{34}A_{41} \), and by \( B_3 = A_{34}A_{41}A_{12}A_{23} \). Observe that \( B_i = A_{i \rightarrow j} A_{j \rightarrow i} \), for any \( j \in \{ 1, \ldots, k \} \setminus \{ i \} \).

**Lemma 2.1.** For \( A \) given in (1.1) and with the notation above, for \( p \geq 0 \),

\[
A^{kp} = \begin{bmatrix}
B_1^p & 0 & \cdots & 0 \\
0 & B_2^p & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k^p
\end{bmatrix}
\]

\[
A^{kp+1} = \begin{bmatrix}
0 & B_1^p A_{1 \rightarrow 2} & 0 & \cdots & 0 \\
0 & 0 & B_2^p A_{2 \rightarrow 3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_{k-1}^p A_{k-1 \rightarrow k} \\
B_k^p A_{k \rightarrow 1} & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
A^{kp+2} = \begin{bmatrix}
0 & 0 & B_1^p A_{1 \rightarrow 3} & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & B_{k-2}^p A_{k-2 \rightarrow k} \\
B_{k-1}^p A_{k-1 \rightarrow 1} & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

and so on, until

\[
A^{kp+k-1} = \begin{bmatrix}
0 & 0 & \cdots & 0 & B_1^p A_{1 \rightarrow k} \\
B_2^p A_{2 \rightarrow 1} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & B_{k-1}^p A_{k-1 \rightarrow k-1} & 0 \\
0 & 0 & \cdots & B_k^p A_{k \rightarrow k-1} & 0
\end{bmatrix}
\]

**Lemma 2.2.** For all \( i \neq j, B_i^k A_{i \rightarrow j} = A_{i \rightarrow i} B_j^k \).

Proof. \( B_i^k A_{i \rightarrow j} = (A_{i \rightarrow j} A_{j \rightarrow i})^k A_{i \rightarrow j} = A_{i \rightarrow j} A_{j \rightarrow i} A_{i \rightarrow j} (A_{i \rightarrow j} A_{j \rightarrow i})^{k-1} A_{i \rightarrow j} \). \( \square \)

**Lemma 2.3.** For all \( i \neq j, B_i^D A_{i \rightarrow j} = A_{i \rightarrow j} B_j^D \). Hence, if \( \ell \neq i, j \) satisfies \( A_{i \rightarrow j} = A_{i \rightarrow \ell} A_{\ell \rightarrow j} \), then \( B_i^D A_{i \rightarrow j} = A_{i \rightarrow j} B_j^D = A_{i \rightarrow \ell} B_{\ell}^D A_{\ell \rightarrow j} \).
Proof. \( B_i^D A_{i\rightarrow j} = (A_{i\rightarrow j} A_{j\rightarrow i})^D A_{i\rightarrow j} = A_{i\rightarrow j} (A_{j\rightarrow i} A_{i\rightarrow j})^D = A_{i\rightarrow j} B_j^D \), where the second equality is due to [3, Lemma 2.4].

With the above notation, we now give a formula for the Drazin inverse of a \( k \)-cyclic matrix \( A \) given by (1.1).

**Theorem 2.4.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined as in (2.1) and \( A_{i\rightarrow j} \) defined in (2.2). Then, for all \( i = 1, \ldots, k \),

\[
A^D = \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1\rightarrow i} B_i^D A_{i\rightarrow k} \\
A_{2\rightarrow i} B_i^D A_{i\rightarrow 1} & 0 & \cdots & 0 & 0 \\
0 & A_{3\rightarrow i} B_i^D A_{i\rightarrow 2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k\rightarrow i} B_i^D A_{i\rightarrow k-1} & 0
\end{bmatrix}
\]

Moreover, if index \( B_i = s_i \), then index \( A \leq k s_i + k - 1 \).

**Proof.** Denote the matrix on the right hand side of (2.6) by \( X \). Performing block multiplication gives

\[
AX = \begin{bmatrix}
A_{12} A_{2\rightarrow i} B_i^D A_{i\rightarrow 1} & 0 & \cdots & 0 \\
0 & A_{23} A_{3\rightarrow i} B_i^D A_{i\rightarrow 2} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & A_{k1} A_{1\rightarrow i} B_i^D A_{i\rightarrow k}
\end{bmatrix}
\]

(by Lemma 2.3)

\[
= \begin{bmatrix}
B_1 B_1^D & 0 & \cdots & 0 \\
0 & B_2 B_2^D & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_k B_k^D
\end{bmatrix}
\]
and by using Lemma 2.3 again

\[
XA = \begin{bmatrix}
A_{1 \rightarrow i}, B_i^D A_{i \rightarrow k} A_{k1} & 0 & \cdots & 0 \\
0 & A_{2 \rightarrow i}, B_i^D A_{i \rightarrow 12} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & A_{k \rightarrow i}, B_i^D A_{i \rightarrow k-1} A_{k-1,k}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
B_i^D A_{1 \rightarrow i}, A_{i \rightarrow k} A_{k1} & 0 & \cdots & 0 \\
0 & B_i^D A_{2 \rightarrow i}, A_{i \rightarrow 12} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & B_i^D A_{k \rightarrow i}, A_{i \rightarrow k-1} A_{k-1,k}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
B_i^D B_1 & 0 & \cdots & 0 \\
0 & B_i^D B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_i^D B_k
\end{bmatrix}
\begin{bmatrix}
B_i^D B_1 & 0 & \cdots & 0 \\
0 & B_i^D B_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_i^D B_k
\end{bmatrix}
\]

\[
= AX,
\]

since \(B_i^D B_1 = B_i B_i^D\) by (1.3). Also, block-multiplying \(X\) with \(AX\) gives

\[
XAX = X(AX)
\]

\[
= \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1 \rightarrow k} B_i^D B_k B_i^D \\
0 & B_i^D B_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k \rightarrow k-1} B_i^D B_{k-1} B_{k-1}^D & 0
\end{bmatrix}
\]

\[
= X, \text{ by Lemma 2.3 and since } B_i^D B_i B_i^D = B_i^D \text{ by (1.4)}.
\]

Let \(i\) be any integer in \(\{1, \ldots, k\}\) and suppose that index \(B_i = s_i = s\). Then using \(2.3\) and Lemma 2.2

\[
A^{k+i} X = A^{k+i+1} X
\]

\[
= \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1 \rightarrow i} B_i^{i+1} B_i^D A_{i \rightarrow k} \\
0 & A_{2 \rightarrow i} B_i^{i+1} B_i^D A_{i \rightarrow 1} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k \rightarrow i} B_i^{i+1} B_i^D A_{i \rightarrow k-1} & 0
\end{bmatrix}
\]
Since index $B_i = s$, it follows by (1.5) that $B_i^{s+1}B_i^D = B_i^s$. Thus, using Lemma 2.2 and $A_{\ell\rightarrow i}, A_{i\rightarrow j} = A_{\ell\rightarrow j}$ for $\ell \neq j$,

$$A^{ks+k}X = \begin{bmatrix}
0 & 0 & \cdots & 0 & A_{1\rightarrow k}B_k^s \\
A_{2\rightarrow 1}B_1^s & 0 & \cdots & 0 & 0 \\
0 & A_{3\rightarrow 2}B_2^s & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{k\rightarrow k-1}B_{k-1}^s & 0
\end{bmatrix}$$

from (2.5) by using Lemma 2.2. By Theorem 1.1, index $A \leq ks+k-1$ and $X = A^D$.

Thus, the Drazin inverse of a $k$-cyclic matrix is reduced to calculating the Drazin inverse of the smallest order Drazin inverse of any of the matrix products $B_i$.

**Corollary 2.5.** If $A$ of the form in (1.1) is nonnegative and has at least one $B_i^D \geq 0$, then $A^D$ is nonnegative.

The following example illustrates Theorem 2.4 and Corollary 2.5.

**Example 2.6.** Let

$$A = \begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & A_{12} & 0 \\
0 & 0 & A_{23} \\
A_{31} & 0 & 0
\end{bmatrix}.$$

Then $B_1 = A_{12}A_{23}A_{31} = 1, B_2 = A_{23}A_{31}A_{12} = \frac{1}{2}J_2$ (where $J_2$ is a $2 \times 2$ all ones matrix) and $B_3 = A_{31}A_{12}A_{23} = 1$. Note that index $B_1 = 0$ and $B_1^D = B_1^{-1} = 1$.

Using Theorem 2.2,

$$A^D = \begin{bmatrix}
0 & B_1^DA_{12}A_{23} \\
A_{24}A_{31}B_1^D & 0 & 0 \\
0 & A_{31}B_1^DA_{12} & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0
\end{bmatrix} = A^2.$$

In fact, rank $A = \text{rank } A^2$, hence $A^D = A^\# = A^2$ agreeing with Theorem 2.2 in [11].

**3. Index of $A$ in relation to the indices of the block products.** With $A$ as in (1.1), for $j \geq 0$, by (2.5) and (2.4),

\begin{align*}
(3.1) \quad & \text{rank } A^{kj} = \text{rank } B_1^j + \text{rank } B_2^j + \cdots + \text{rank } B_k^j \\
(3.2) \quad & \text{rank } A^{kj+1} = \text{rank } B_1^j A_{12} + \text{rank } B_2^j A_{23} + \cdots + \text{rank } B_k^j A_{k1}.
\end{align*}
The following rank inequality is used throughout the proof of Lemma 3.2 and can be found in standard linear algebra texts (see, e.g., [12, page 13]).

**Lemma 3.1. (Frobenius Inequality)** If \( U \) is \( m \times n \), \( V \) is \( n \times p \) and \( W \) is \( p \times q \), then
\[
\text{rank}(UV) + \text{rank}(WV) \leq \text{rank}(V) + \text{rank}(UVW).
\]

**Lemma 3.2.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1), and let \( s = \text{index} B_i \geq 1 \) for some \( i \in \{1, \ldots, k\} \). Then \( \text{rank}(A^{k_s-k+1}) < \text{rank}(A^{k_s-k}) \).

**Proof.** Let \( s = \text{index} B_i \) for some \( i \in \{1, \ldots, k\} \). From (3.2),
\[
\text{rank}(A^{k_s-k+1}) = \text{rank}(A^{k(s-1)+1}) = \text{rank}(B_1^{s-1}A_{12} + \text{rank}(B_2^{s-1}A_{23} + \text{rank}(B_3^{s-1}A_{34} + \cdots + \text{rank}(B_k^{s-1}A_{ki})),
\]
where the terms can be reordered as
\[
\text{rank}(B_1^{s-1}A_{i1}) + \text{rank}(B_2^{s-1}A_{i1,i+2}) + \cdots + \text{rank}(B_k^{s-1}A_{i1,k}) + \text{rank}(B_1^{s-1}A_{12} + \cdots + \text{rank}(B_k^{s-1}A_{i1,k})),
\]
using Lemma 3.2 the first two terms in the expression in (3.3) can be written as
\[
\text{rank}(A_{i1})B_{i1} + \text{rank}(A_{i1,i+2}B_{i1} + \cdots + \text{rank}(A_{i1,k}B_{i1} + \text{rank}(A_{12} + \cdots + \text{rank}(A_{i1,k})),
\]
and using the Frobenius inequality (Lemma 3.1),
\[
\text{rank}(B_1^{s-1}A_{i1}) + \text{rank}(B_2^{s-1}A_{i1,i+2}) \leq \text{rank}(B_1^{s-1} + \text{rank}(A_{i1}B_1^{s-1}A_{12} + \cdots + \text{rank}(B_k^{s-1}A_{i1,k})),
\]
where the equality is again due to Lemma 3.2. Thus,
\[
\text{rank}(A^{k_s-k+1}) \leq \text{rank}(B_1^{s-1} + \text{rank}(B_2^{s-1}A_{i1,i+2} + \text{rank}(B_3^{s-1}A_{i1,i+3} + \cdots + \text{rank}(B_k^{s-1}A_{i1,k})),
\]
Applying Lemma 3.2 and the Frobenius inequality again to the second and third terms on the righthand side of the inequality in (3.4) gives
\[
\text{rank}(B_1^{s-1}A_{i1,i+2} + \text{rank}(B_2^{s-1}A_{i1,i+3} + \cdots + \text{rank}(B_k^{s-1}A_{i1,k} + \text{rank}(A_{i1,i+3}B_{i1}^{s-1})),
\]
Continuing in this manner gives
\[
\text{rank}(A^{k_s-k+1}) \leq \text{rank}(B_1^{s-1} + \text{rank}(B_2^{s-1} + \cdots + \text{rank}(B_k^{s-1} + \text{rank}(A_{i1,i}, B_{i1}^{s-1}A_{i1,i})),
\]
(3.5)
Using Lemma 2.2, the last term on the righthand side of the inequality in (3.5) becomes
\[ \text{rank} B_{i+1}^{s+1} A_{i+1} - \text{rank} B_i^{s+1} B_i = \text{rank} B_i^s < \text{rank} B_i^{s-1}, \]
since index \( B_i = s \). Thus,
\[ \text{rank} A_{k+1}^{s-k+1} < \text{rank} B_i^{s+1} + \text{rank} B_i^{s-1} + \ldots + \text{rank} B_i^{s-1} + \text{rank} B_i^s, \]
where the equality follows from (3.1).

**Theorem 3.3.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1). Then, the following statements hold.

(i) If index \( B_i = 0 \) for all \( i = 1, \ldots, k \), then \( A \) is nonsingular and index \( A = 0 \).

(ii) If index \( B_i = s_i \geq 1 \) for some \( i \in \{1, \ldots, k\} \), then index \( A \geq ks_i - k + 1 \).

**Proof.** The first statement follows immediately from (2.3) and (3.1). For the second statement, let index \( B_i = s_i \geq 1 \) for some \( i \in \{1, \ldots, k\} \). Then \( \text{rank} A^{ks_i-k+1} < \text{rank} A^{ks_i-k} \), by Lemma 3.2. From the strict inequality, index \( A \geq ks_i - k + 1 \).

The next result follows immediately from Theorem 3.3(ii).

**Corollary 3.4.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1). If index \( A \leq 1 \), then index \( B_i \leq 1 \) for all \( i = 1, \ldots, k \). That is, if the group inverse \( A^\# \) exists, then the group inverses \( B_i^\# \) exist for all \( i = 1, \ldots, k \).

Note however that the converse to Corollary 3.4 is false (see, e.g., [4, Example 4.3]).

**Remark 3.5.** If \( A \) of the form (1.1) is nonnegative and all matrices with the same +, 0 sign pattern as \( A \) have index 1 have at least one \( B_i^\# \) nonnegative, then these group inverses are nonnegative (Corollary 2.5) and \( A \) is conditionally \( S^2 \) GI in the notation of Zhou et al. [15].

**Corollary 3.6.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1), and let \( s = \min_{1 \leq i \leq k} \text{index} \ B_i \) and \( s' = \max_{1 \leq i \leq k} \text{index} \ B_i > 0 \). Then \( ks' - k + 1 \leq \text{index} \ A \leq ks + k - 1 \). If \( s' = 0 \), then index \( A = 0 \).

Corollary 3.6 leads to a result about the indices of \( B_i \) that is of independent interest.

**Theorem 3.7.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1), and let \( s_\ell = \text{index} \ B_\ell \) for \( \ell \in \{1, \ldots, k\} \). Then \( |s_i - s_j| \leq 1 \) for all \( i, j \in \{1, \ldots, k\} \).
Proof. Let \( s = \min_{1 \leq i \leq k} \text{index } B_i \) and \( s' = \max_{1 \leq i \leq k} \text{index } B_i \), and suppose that \( s' = s + t \) where \( t \geq 0 \). By Corollary 3.6

\[
 k(s + t) - k + 1 \leq \text{index } A \leq ks + k - 1.
\]

It follows that

\[
 k(s + t) - k + 1 \leq ks + k - 1,
\]

or equivalently,

\[
 k(t - 2) + 2 \leq 0.
\]

As \( k \geq 2 \), the inequality above is possible only if \( t \leq 1 \). Thus, \( s' - s = t \leq 1 \) and \( |\text{index } B_i - \text{index } B_j| \leq 1 \) for all \( i, j \). \( \Box \)

The next result gives tight bounds on \( \text{index } A \) in terms of the minimum index of the block products \( B_i \). The proof is immediate from Corollary 3.6 and Theorem 3.7.

**Theorem 3.8.** Let \( A \) be as in (1.1) with associated matrices \( B_i \) defined in (2.1), and let \( s = \min_{1 \leq i \leq k} \text{index } B_i \). Then, exactly one of the following holds:

(i) \( \text{index } B_i = s \) for all \( i = 1, \ldots, k \), or
(ii) \( \text{index } B_i = s + 1 \) for some \( i = 1, \ldots, k \).

If (i) holds, then \( ks - k + 1 \leq \text{index } A \leq ks + k - 1 \). If (ii) holds, then \( ks + 1 \leq \text{index } A \leq ks + k - 1 \).

The above result generalizes bounds found in [4, Section 3] and shows that if \( k = 2 \) and (ii) holds, then \( \text{index } A = 2s + 1 \).

We now give examples that illustrate Theorem 3.8.

**Example 3.9.** Let \( A \) be the matrix in Example 2.6. Using the notation in Theorem 3.8, \( s = 0 = \text{index } B_1 = \text{index } B_3 \) and \( \text{index } B_2 = 1 = s + 1 \). Applying the result with \( k = 3 \) gives the bounds \( 1 \leq \text{index } A \leq 2 \). Since \( \text{rank } A = \text{rank } A^2 \), \( \text{index } A = 1 = ks + 1 \), which is the lower bound of Theorem 3.8 case (ii).

**Example 3.10.** Let

\[
 A = \begin{bmatrix}
 0 & 1 & -1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 2 \\
 0 & 0 & 0 & 1 & -1 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
 A = \begin{bmatrix}
 0 & A_{12} & 0 \\
 0 & 0 & A_{23} \\
 A_{31} & 0 & 0
\end{bmatrix}.
\]
Then $B_1 = 3, B_2 = \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$ and $B_3 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Note that index $B_1 = 0$ and $B_2 = \frac{1}{3}$. Using Theorem 2.4

$$A_D = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 \end{bmatrix}.$$ 

Using the notation in Theorem 3.3, $s = 0 = \text{index } B_1$ and index $B_2 = \text{index } B_3 = 1 = s + 1$. Applying the theorem with $k = 3$ gives the bounds $1 \leq \text{index } A \leq 2$. It can be computed that index $A = 2 = ks + k - 1$, which is the upper bound of Theorem 3.3 case (ii).

**Example 3.11.** Let

$$A = \begin{bmatrix} 0 & B & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ I & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $B$ is a square matrix and $I$ is an identity matrix of the same order as $B$. Note that $B_i = B$ for all $i$. Suppose that index $B = s$. Then index $A = ks$, the midpoint of the interval $[ks - k + 1, ks + k - 1]$ in Theorem 3.3 case (i), and from Theorem 2.4

$$A_D = \begin{bmatrix} 0 & 0 & \cdots & 0 & B^D B \\ B^D & 0 & \cdots & 0 & 0 \\ 0 & B^D B & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B^D B & 0 \end{bmatrix}. $$
Example 3.12. Let

\[ A = \begin{bmatrix} 0 & F & 0 & \cdots & 0 \\ 0 & 0 & F & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & F \\ F & 0 & 0 & \cdots & 0 \end{bmatrix} \]

where \( F \) is a square matrix. Then index \( A = \text{index } F \) and \( B_i = F^k \) for \( i = 1, \ldots, k \). Setting index \( A = \ell \) and index \( B_i = s \) gives \( s = \left\lceil \frac{\ell}{k} \right\rceil \). Thus, index \( A \) can take any value in the interval \([ks - k + 1, ks]\), which is half the range given in Theorem 3.8 case (i).

Examples 3.11 and 3.12 have \( B_i \), and thus index \( B_i \), the same for all \( i \). The following result determines index \( A \) in this case, and the necessary and sufficient conditions reduce to the result of [3], Theorem 3.5] for \( k = 2 \).

Theorem 3.13. Let \( A \) be a block \( k \)-cyclic matrix of the form in (1.7) with associated matrices \( B_i \) defined in (2.7), and suppose that \( s = \min_{1 \leq i \leq k} \text{index } B_i \geq 1 \). Then index \( A = ks \) if and only if

(i) index \( B_i = s \) for all \( i = 1, \ldots, k \), and

(ii) rank \( B_j^s < \text{rank } B_j^{s-1}A_{j-1,j-1} \) for some \( j \in \{1, \ldots, k\} \).

If (i) holds, then rank \( B_i^s = \text{rank } B_j^s \) for all \( i, j = 1, \ldots, k \). If (i) holds but (ii) does not hold, then index \( A < ks \).

Proof. Suppose that index \( A = ks \). Then rank \( A^{ks} < \text{rank } A^{ks-1} \). It follows, using (2.3), (2.5) and (5.1), that \( \sum_{i=1}^{k} \text{rank } B_i^s < \sum_{i=1}^{k} \text{rank } B_i^{s-1}A_{i,i-1} \). Thus, rank \( B_j^s < \text{rank } B_j^{s-1}A_{j,j-1} \) for some \( j \in \{1, \ldots, k\} \), hence (ii) holds. Suppose on the contrary that (i) does not hold. Then, for some \( j \in \{1, \ldots, k\} \), index \( B_j = s + 1 \) (by Theorem 3.8). Thus, rank \( B_j^s > \text{rank } B_j^{s+1} \), hence by (2.3) rank \( A^{ks} = \sum_{i=1}^{k} \text{rank } B_i^s > \sum_{i=1}^{k} \text{rank } B_i^{s+1} = \text{rank } A^{k(s+1)} \). This implies that rank \( A^{ks} > \text{rank } A^{ks+k} \), so index \( A > ks \), a contradiction. Hence, (i) and (ii) must hold.

For the reverse implication, suppose that (i) and (ii) hold. Then rank \( A^{ks} = \sum_{i=1}^{k} \text{rank } B_i^s < \sum_{i=1}^{k} \text{rank } B_i^{s-1}A_{i,i-1} = \text{rank } A^{k(s-1)+(k-1)} = \text{rank } A^{ks-1} \), where the strict inequality is due to (ii). Thus, index \( A \geq ks \). Note that since rank \( B_i^s \geq \text{rank } B_i^s A_{i,j} \geq \text{rank } B_i^{s+1} \) and rank \( B_i^s A_{i,j} = \text{rank } A_{i,j}B_j^s \) (by Lemma 2.2), it follows using (i) that rank \( B_i^{s+1} \) = rank \( B_i^s A_{i,j} \) = rank \( A_{i,j}B_j^s \) = rank \( B_j^{s+1} \) for all \( i, j \). Thus, rank \( A^{ks} = \sum_{i=1}^{k} \text{rank } B_i^s = \sum_{i=1}^{k} \text{rank } B_i^{s}A_{i,i-1} = \text{rank } A^{ks+1} \), using (3.1) and (3.2) . Hence, rank \( A^{ks} = \text{rank } A^{ks+1} \), and so index \( A \leq ks \). This
proves that index $A = ks$. The last two statements of the theorem follow from the proof above.

The result of Theorem 3.13 is illustrated by Example 3.11 since rank $\mathbf{B}_s < \text{rank} \mathbf{B}_{s+1} = \text{rank} \mathbf{B}_{s+1}$; it follows that rank $A = ks$. Example 3.12 also illustrates Theorem 3.13 since rank $A^{ks} = \text{rank} A^{ks+1}$ and rank $F^{ks} < \text{rank} F^{k(s-1)} F^{k-1} = \text{rank} F^{ks-1}$ if and only if index $F = \text{index} A = ks$; otherwise index $A < ks$.

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