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THE M-COMPETITION INDICES OF SYMMETRIC PRIMITIVE DIGRAPHS WITHOUT LOOPS∗

YANLING SHAO†, YUBIN GAO†, AND ZHONGSHAN LI‡

Abstract. For positive integers $m$ and $n$ with $1 \leq m \leq n$, the $m$-competition index (generalized competition index) of a primitive digraph $D$ of order $n$ is the smallest positive integer $k$ such that for every pair of vertices $x$ and $y$ in $D$, there exist $m$ distinct vertices $v_1, v_2, \ldots, v_m$ such that there exist walks of length $k$ from $x$ to $v_i$ and from $y$ to $v_i$ for each $i = 1, \ldots, m$. In this paper, we study the generalized competition indices of symmetric primitive digraphs without loops. We determine the generalized competition index set and characterize the digraphs in this class with largest generalized competition index.

Key words. Competition index, $m$-Competition index, Scrambling index, Generalized competition index.

AMS subject classifications. 05C50, 15A48, 05C20.

1. Introduction. For terminology and notation used here, we follow those in [3, 5]. Let $D = (V, E)$ denote a digraph on $n$ vertices with the vertex set $V = V(D)$ and the arc set $E = E(D)$. Loops are permitted but multiple arcs are not. A walk in $D$ is a sequence $w = v_0v_1 \cdots v_k$ such that for $1 \leq i \leq k$, there exists an arc from $v_{i-1}$ to $v_i$. A digraph $D$ is called primitive if there exists a positive integer $k$ such that for any pair of vertices $u$ and $v$, there is a walk of length $k$ from vertex $u$ to vertex $v$. The smallest such $k$ is called the exponent of $D$, and it is denoted by $\exp(D)$. It is well known that $D$ is primitive if and only if $D$ is strongly connected and the greatest common divisor of the lengths of all the cycles in $D$ is 1.

The length of a walk $w$ is denoted by $l(w)$. The distance from vertex $u$ to vertex $v$ in $D$ is the length of a shortest walk from $u$ to $v$, and is denoted by $d_D(u, v)$ (or simply $d(u, v)$). For $X \subseteq V(D)$, set $d(u, X) = \min_{v \in X} d(u, v)$. The notation $u \xrightarrow{k} v$ indicates that there is a walk of length $k$ from $u$ to $v$. For distinct $r$ vertices $v_1, v_2, \ldots, v_r$, the notation $C_r = v_1v_2 \cdots v_rv_1$ means that $C_r$ is the $r$-cycle consisting of the arcs $(v_r, v_1)$ and $(v_i, v_{i+1}), 1 \leq i \leq r - 1$. Let $D$ be a primitive digraph of

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order $n$. For positive integers $m$ and $n$ with $1 \leq m \leq n$, we define the $m$-competition index (generalized competition index) of the primitive digraph $D$, denoted by $k_m(D)$, as the smallest positive integer $k$ such that for every pair of vertices $x$ and $y$, there exist $m$ distinct vertices $v_1, v_2, \ldots, v_m$ such that $x \xrightarrow{k} v_i$ and $y \xrightarrow{k} v_i$ in $D$ for each $i = 1, \ldots, m$.

Akelbek and Kirkland \cite{1, 2} introduced the scrambling index of a primitive digraph $D$, denoted by $k(D)$. Kim \cite{5} introduced the $m$-competition index as a generalization of the competition index. In the case of primitive digraphs, the definitions of the scrambling index and 1-competition index are identical, i.e., $k(D) = k_1(D)$.

For a positive integer $k$ and a primitive digraph $D$, we define the $k$-step outneighborhood of a vertex $x$ as

$$N^+(D^k : x) = \{v \in V(D) \mid x \xrightarrow{k} v\}.$$ 

The $k$-step common outneighborhood of vertices $x$ and $y$ is defined as

$$N^+(D^k : x, y) = N^+(D^k : x) \cap N^+(D^k : y).$$

The local $m$-competition index of vertices $x$ and $y$ is defined as

$$k_m(D : x, y) = \min\{k : |N^+(D^t : x, y)| \geq m, \text{ for all } t \geq k\},$$

and the local $m$-competition index of $x$ is defined as

$$k_m(D : x) = \max_{y \in V(D)} \{k_m(D : x, y)\}.$$

Then we have

$$k_m(D) = \max_{x \in V(D)} k_m(D : x) = \max_{x, y \in V(D)} k_m(D : x, y).$$

The $m$-competition index is a generalization of the scrambling index and the exponent of a primitive digraph. It was known that for $1 \leq m \leq n$ (for example see \cite{5}),

$$k(D) = k_1(D) \leq k_2(D) \leq \cdots \leq k_n(D) = \exp(D).$$

A symmetric digraph is a digraph such that for any vertices $u$ and $v$, $(u, v)$ is an arc if and only if $(v, u)$ is an arc. An undirected graph (possibly with loops) can be regarded as a symmetric digraph.

There has been interest recently in generalized competition index \cite{5, 6, 7, 9}. Let $S^0_n$ denote the set of all symmetric primitive digraphs of order $n$ without loops. In this paper, we study the $m$-competition indices of digraphs in $S^0_n$ with $n \geq 6$. We determine the $m$-competition index set for $S^0_n$, and characterize the digraphs in $S^0_n$ with the largest $m$-competition index.
2. The generalized competition indices for special graphs. In this section, we study the generalized competition indices for some special graphs. Let \( S_n(r) \) denote the set of all symmetric primitive digraphs of order \( n \) having a cycle of odd length \( r \) but no cycle of any odd length less than \( r \).

**Lemma 2.1.** Let \( G \in S_n(r) \) and \( C_r \) be an \( r \)-cycle in \( G \). For any positive integer \( k \) and any vertex \( u \in V(C_r) \),

\[
|N^+(G^k : u) \cap V(C_r)| = \min\{k + 1, r\}.
\]

**Proof.** Set \( C_r = v_1v_2 \cdots v_r v_1 \). Note that for any positive integer \( k \) and any vertex \( v_i \in V(C_r) \),

\[
N^+(G^k : v_i) \cap V(C_r) = \begin{cases} 
\{v_i, v_{i+2}, \ldots, v_{i+k}, v_{i-2}, v_{i-4}, \ldots, v_{i-k}\}, & \text{if } k \text{ is even}, \\
\{v_{i+1}, v_{i+3}, \ldots, v_{i+k}, v_{i-1}, v_{i-3}, \ldots, v_{i-k}\}, & \text{if } k \text{ is odd},
\end{cases}
\]

where the vertex subscripts are taken modulo \( r \). Then

\[
|N^+(G^k : v_i) \cap V(C_r)| = \begin{cases} 
k + 1, & \text{if } 1 \leq k \leq r - 1, \\
r, & \text{if } k \geq r,
\end{cases}
\]

that is, \( |N^+(G^k : v_i) \cap V(C_r)| = \min\{k + 1, r\} \). \( \square \)

Now, we study the graphs in Figure 1, where \( r \) is odd with \( 3 \leq r \leq n - 1 \), and \( 1 \leq l \leq n - r \).

![Figure 1. Graphs \( G_r,l \).](image)

**Theorem 2.2.** Let \( G_{r,l} \) be primitive graphs as shown in Figure 1. For \( 2 \leq m \leq n - 1 \),

\[
k_m(G_{r,l}) = \begin{cases} 
l + \left\lfloor \frac{r-m-2}{2} \right\rfloor, & \text{if } 2 \leq m \leq r - 1, \\
l + m - 1, & \text{if } r \leq m \leq r + l - 1, \\
2l + r - 1, & \text{if } m \geq r + l.
\end{cases}
\]

**Proof.** We prove the statement case by case.

*Case 1. \( 2 \leq m \leq r - 1 \).*
Let $C_r$ be the only $r$-cycle in $G_{r,l}$. For any $v_i, v_j \in V(G_{r,l})$, note that
\[ d(v_i, C_r) \leq l, \quad d(v_j, C_r) \leq l, \quad \text{and} \quad \frac{r - 1}{2} \leq \left\lfloor \frac{r + m - 2}{2} \right\rfloor \leq r - 2. \]

If $i \geq r + l$ and $j \geq r + l$, then by Lemma 2.1,
\[ |N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_i, v_j)| \geq |N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_i, v_j) \cap V(C_r)| \]
\[ = |N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_i) \cap V(C_r)| \]
\[ = |N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_r+l) \cap V(C_r)| \]
\[ = |N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_r) \cap V(C_r)| \]
\[ = \min\left\{ \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 1, r \right\} \]
\[ = \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 1 \]
\[ \geq \left\lfloor \frac{m + 1 + m - 2}{2} \right\rfloor + 1 = m. \]

If $i < r + l$, then $l - d(v_i, C_r) \geq 1$. By Lemma 2.1,
\[ |N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_i) \cap V(C_r)| \]
\[ = \left\{ \begin{array}{cl}
|N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_i) \cap V(C_r)|, & \text{if } r \leq i < r + l \\
|N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_i) \cap V(C_r)|, & \text{if } i < r - 1
\end{array} \right. \]
\[ = \min\{ l - d(v_i, C_r) + \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 1, r \} \]
\[ \geq \min\{ \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 2, r \} \]
\[ = \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 2. \]

So, if $i < r + l$ or $j < r + l$, then
\[ |N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_i, v_j)| \]
\[ \geq |N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_i, v_j) \cap V(C_r)| \]
\[ = \left| N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_i) \cap V(C_r) \right| \cap \left( N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_j) \cap V(C_r) \right) \]
\[ \geq \left| N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_i) \cap V(C_r) \right| + \left| N^+(G_{r,l}^{+\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_j) \cap V(C_r) \right| - r \]
\[ \geq 2 \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 3 - r \geq m. \]
Therefore, \( k_m(G_{r,l}) \leq l + \left\lfloor \frac{r+m-2}{2} \right\rfloor \).

We now show that \( k_m(G_{r,l}) > l + \left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 \). If \( \left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 \) is even, then

\[
N^+(G_{r,l}^\left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 : v_r) \cap V(C_r) = \{v_r, v_{r+2}, \ldots, v_{\left\lfloor \frac{r+m-2}{2} \right\rfloor + 1} : v_r, v_{r-2}, v_{r-4}, \ldots, v_{r-\left\lfloor \frac{r+m-2}{2} \right\rfloor + 1} \}
\]

\[
N^+(G_{r,l}^\left\lfloor \frac{r+m-2}{2} \right\rfloor : v_r) \cap V(C_r) = \{v_1, v_3, \ldots, v_{\left\lfloor \frac{r+m-2}{2} \right\rfloor}, v_{r-1}, v_{r-3}, \ldots, v_{r-\left\lfloor \frac{r+m-2}{2} \right\rfloor} \}
\]

and

\[
\left( N^+(G_{r,l}^\left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 : v_r) \cap V(C_r) \right) \cap \left( N^+(G_{r,l}^\left\lfloor \frac{r+m-2}{2} \right\rfloor : v_r) \cap V(C_r) \right) = \begin{cases} \{v_r - \left\lfloor \frac{r+m-2}{2} \right\rfloor, \ldots, v_{\left\lfloor \frac{r+m-2}{2} \right\rfloor} \}, & \text{if } m \geq 3, \\
\phi, & \text{if } m = 2. \end{cases}
\]

If \( \left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 \) is odd, then the result follows in a similar manner. Thus,

\[
|N^+(G_{r,l}^\left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 : v_{r+l}, v_{r+l-1})| = |N^+(G_{r,l}^\left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 : v_r) \cap V(C_r)| \\
= \left| \left( N^+(G_{r,l}^\left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 : v_{r+l}) \cap V(C_r) \right) \cap \left( N^+(G_{r,l}^\left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 : v_{r+l-1}) \cap V(C_r) \right) \right| \\
\leq m - 1.
\]

So \( k_m(G_{r,l}) > l + \left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 \). Then \( k_m(G_{r,l}) = l + \left\lfloor \frac{r+m-2}{2} \right\rfloor \).

Case 2. \( r \leq m \leq r + l - 1 \).

For any \( v_i \in V(G_{r,l}) \), we will show that \( \{v_1, v_2, \ldots, v_m\} \subseteq N^+(G_{r,l}^{l+m-1} : v_i) \). First, consider a vertex \( u \in \{v_{r+l}, v_{r+l+1}, \ldots, v_n\} \). Note that \( w = uv_{r+l-1} \cdots v_m \) is the walk from \( u \) to \( v_m \) of length \( r + l - m \). Since \( r + l - m \) and \( l + m - 1 \) have the same parity, and \( r + l - m \leq l + m - 1 \), we have \( v_m, v_{r+l} \in N^+(G_{r,l}^{l+m-1} : u) \). Note that there exist two walks \( w_1 \) and \( w_2 \) from \( u \) to \( v_i \) (\( 1 \leq i \leq m - 1 \)) such that \( l(w_1) \leq l + m - 1 \), \( l(w_2) \leq l + m - 1 \), and \( l(w_1) \) and \( l(w_2) \) have different parity. So \( v_i \in N^+(G_{r,l}^{l+m-1} : u) \).

Next, for any vertex \( v_i, 1 \leq i \leq r + l - 1 \), we can show \( \{v_1, v_2, \ldots, v_m\} \subseteq N^+(G_{r,l}^{l+m-1} : v_i) \) similarly. Then \( |N^+(G_{r,l}^{l+m-1} : v_i, v_j)| \geq m \) for any \( v_i, v_j \in V(G_{r,l}) \), and \( k_m(G_{r,l}) \leq l + m - 1 \).

We now show that \( k_m(G_{r,l}) > l + m - 2 \). If \( l + m - 2 \) is even, then

\[
N^+(G_{r,l}^{l+m-2} : v_{r+l})
\]
If \( k \) so

Case 3. \( m \)

On the one hand, it is easy to see that for each vertex \( v_i \), \( N^+(G_{r,l}^{2l+r-1} : v_i) = V(G_{r,l}) \), so \( k_m(G_{r,l}) \leq 2l + r - 1 \). On the other hand, since

\[ |N^+(G_{r,l}^{2l+r-2} : v_{r+l})| = |\{v_1, v_2, \ldots, v_{r+l-1}\}| = r + l - 1 < m, \]

we have \( k_m(G_{r,l}) > 2l + r - 2 \), and \( k_m(G_{r,l}) = 2l + r - 1 \). This completes the proof.

**Corollary 2.3.** For \( 2 \leq m \leq n - 1 \),

\[ k_m(G_{r,n-r}) = \begin{cases} n - r + \left\lfloor \frac{r + m - 2}{2} \right\rfloor, & \text{if } 2 \leq m \leq r - 1, \\ n - r + m - 1, & \text{if } r \leq m \leq n - 1. \end{cases} \]

Next, we study the graphs in Figure 2, where \( r \) is odd with \( 3 \leq r \leq n - 3 \), and \( 1 \leq l \leq n - r - 2 \).

**Theorem 2.4.** Let \( \overline{G}_{r,l} \) be primitive graphs as shown in Figure 2. For \( 2 \leq m \leq n - 1 \),

\[ k_m(\overline{G}_{r,l}) = \begin{cases} l + \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 1, & \text{if } 2 \leq m \leq r - 1, \\ l + m, & \text{if } r \leq m \leq r + l - 1, \\ 2l + r, & \text{if } r + l \leq m \leq n - 1. \end{cases} \]
Proof. We prove the statement case by case.

Case 1. \(2 \leq m \leq r - 1\).

Let \(C_r\) be the only \(r\)-cycle in \(\overrightarrow{G}_{r,l}\). Note that \(d(v_n, C_r) = l + 1\), \(d(v, C_r) \leq l\) for any \(v_i \neq v_n\), and

\[
\frac{r-1}{2} \leq \left\lfloor \frac{r+m-2}{2} \right\rfloor \leq r - 2.
\]

For any \(v_i \neq v_n\), by Lemma 2.1

\[
|N^+(\overrightarrow{G}_{r,l} : (r \cdot m - 2) + 1 : v_i, v_n)| 
\geq |N^+(\overrightarrow{G}_{r,l} : (r \cdot m - 2) + 1 : v_i, v_n) \cap V(C_r)| 
\geq |N^+(\overrightarrow{G}_{r,l} : (r \cdot m - 2) + 1 : v_i) \cap V(C_r)| + |N^+(\overrightarrow{G}_{r,l} : (r \cdot m - 2) + 1 : v_n) \cap V(C_r)| - r 
= \min\{l + \left\lfloor \frac{r+m-2}{2} \right\rfloor + 1 - d(v_i, C_r) + 1, r\} + \min\left\lfloor \frac{r+m-2}{2} \right\rfloor + 1, r\} - r 
\geq \min\left\lfloor \frac{r+m-2}{2} \right\rfloor + 2, r\} + \left\lfloor \frac{r+m-2}{2} \right\rfloor + 1 \geq r \geq m.
\]

For any \(v_i \neq v_n\), and \(v_j \neq v_n\),

\[
|N^+(\overrightarrow{G}_{r,l} : (r \cdot m - 2) + 1 : v_i, v_j)| 
\geq |N^+(\overrightarrow{G}_{r,l} : (r \cdot m - 2) + 1 : v_i, v_j) \cap V(C_r)| 
\geq |N^+(\overrightarrow{G}_{r,l} : (r \cdot m - 2) + 1 : v_i) \cap V(C_r)| + |N^+(\overrightarrow{G}_{r,l} : (r \cdot m - 2) + 1 : v_j) \cap V(C_r)| - r 
\geq 2 \min\left\lfloor \frac{r+m-2}{2} \right\rfloor + 2, r\} - r 
\geq 2\left(\left\lfloor \frac{r+m-2}{2} \right\rfloor + 2\right) - r \geq m.
\]

Therefore, \(k_m(\overrightarrow{G}_{r,l}) \leq l + \left\lfloor \frac{r+m-2}{2} \right\rfloor + 1\).

Now, we will show that \(k_m(\overrightarrow{G}_{r,l}) > l + \left\lfloor \frac{r+m-2}{2} \right\rfloor\). If \(\left\lfloor \frac{r+m-2}{2} \right\rfloor - 1\) is even, then

\[
N^+(\overrightarrow{G}_{r,l} : \left\lfloor \frac{r+m-2}{2} \right\rfloor - 1 : v_r) \cap V(C_r) 
= \{v_r, v_2, v_4, \ldots, v_{r-2}, v_{r-4}, \ldots, v_{r-\left\lfloor \frac{r+m-2}{2} \right\rfloor + 1}\},
\]

\[
N^+(\overrightarrow{G}_{r,l} : \left\lfloor \frac{r+m-2}{2} \right\rfloor : v_r) \cap V(C_r) 
= \{v_1, v_3, \ldots, v_{r-1}, v_{r-3}, \ldots, v_{r-\left\lfloor \frac{r+m-2}{2} \right\rfloor}\},
\]

\[
N^+(\overrightarrow{G}_{r,l} : \left\lfloor \frac{r+m-2}{2} \right\rfloor + 1 : v_r) \cap V(C_r) 
= \{v_1, v_3, \ldots, v_{r-1}, v_{r-3}, \ldots, v_{r-\left\lfloor \frac{r+m-2}{2} \right\rfloor + 1}\},
\]
On the one hand, it is easy to see that for each vertex $v$, if $m$ is odd, then the result follows in a similar manner. Thus,

$$\left| N^+\left(\overline{G}_{r,l}\right)^{\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_r \right| \cap V(C_r) \right| = \left| \left( N^+\left(\overline{G}_{r,l}\right)^{\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_r \right) \cap V(C_r) \right|$$

If $\left\lfloor \frac{r+m-2}{2} \right\rfloor - 1$ is odd, then the result follows in a similar manner. Thus,

$$\left| N^+\left(\overline{G}_{r,l}\right)^{\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_n, v_{n-1} \right| = \left| N^+\left(\overline{G}_{r,l}\right)^{\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_n, v_{n-1} \right| \cap V(C_r) \right| = \left( N^+\left(\overline{G}_{r,l}\right)^{\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_r \right) \cap V(C_r) \right| \cap \left( N^+\left(\overline{G}_{r,l}\right)^{\left\lfloor \frac{r+m-2}{2} \right\rfloor} : v_r \right) \cap V(C_r) \right|$$

If $\left\lfloor \frac{r+m-2}{2} \right\rfloor = 0$, then $m = 2$.

Now, we will show that $k_m(\overline{G}_{r,l}) > l + \left\lfloor \frac{r+m-2}{2} \right\rfloor$. Then $k_m(\overline{G}_{r,l}) = l + \left\lfloor \frac{r+m-2}{2} \right\rfloor + 1$.

Case 2. $r \leq m \leq r + l - 1$.

For any $v_i \in V(\overline{G}_{r,l})$, by similar arguments as in Case 2 of the proof of Theorem 2.2, we have

$$\{v_1, v_2, \ldots, v_m\} \subseteq N^+\left(\overline{G}_{r,l}^m : v_i\right)$$

So $k_m(\overline{G}_{r,l}) \leq l + m$.

Now, we will show that $k_m(\overline{G}_{r,l}) > l + m - 1$. If $l + m - 1$ is even, then

$$N^+\left(\overline{G}_{r,l}^{l+m-1} : v_n\right)$$

$$= \{v_1, v_2, \ldots, v_{r+l-1}, v_n\} \setminus \{v_k \mid m \leq k \leq l + r - 2, \text{ and } k \equiv (l + r - 2)(\mod 2)\}$$

$$N^+\left(\overline{G}_{r,l}^{l+m-1} : v_{n-1}\right)$$

$$= \{v_1, v_2, \ldots, v_{n-1}\} \setminus \{v_k \mid m + 1 \leq k \leq r + l - 1, \text{ and } k \equiv (l + r - 1)(\mod 2)\}$$

Therefore,

$$N^+\left(\overline{G}_{r,l}^{l+m-1} : v_n\right) \cap N^+\left(\overline{G}_{r,l}^{l+m-1} : v_{n-1}\right) = \{v_1, v_2, \ldots, v_{m-1}\}$$

If $l + m - 1$ is odd, then the result follows in a similar manner. So $k_m(\overline{G}_{r,l}) > l + m - 1$, and $k_m(\overline{G}_{r,l}) = l + m$.

Case 3. $r + l \leq m \leq n - 1$.

On the one hand, it is easy to see that for each vertex $v_i$, $V(\overline{G}_{r,l}) \setminus \{v_n\} \subseteq N^+\left(\overline{G}_{r,l}^{2l+r} : v_i\right)$, so $k_m(\overline{G}_{r,l}) \leq 2l + r$. On the other hand, since

$$N^+\left(\overline{G}_{r,l}^{2l+r-1} : v_n\right) = \{v_1, v_2, \ldots, v_{r+l-1}\} \cup \{v_n\}$$

$$N^+\left(\overline{G}_{r,l}^{2l+r-1} : v_{n-1}\right) = \{v_1, v_2, \ldots, v_{n-1}\}$$

...
we have
\[ |N^+(G_{r,l}^{2l+r-1} : v_n) \cap N^+(G_{r,l}^{2l+r-1} : v_{n-1})| = |\{v_1, v_2, \ldots, v_{r+l-1}\}| = r + l - 1 < m. \]

So, \( k_m(G_{r,l}) > 2l + r - 1 \), and \( k_m(G_{r,l}) = 2l + r \). This completes the proof. \( \Box \)

Now, we consider the graph \( G_r \) as shown in Figure 3, where \( r \) is odd with \( 3 \leq r \leq n \).

![Figure 3. Graph \( G_r \).](image)

**Theorem 2.5.** For \( 2 \leq m \leq n - 1 \),

\[ k_m(G_r) = \begin{cases} \left\lfloor \frac{r + m - 1}{2} \right\rfloor, & \text{if } 2 \leq m \leq r - 1, \\ r - 1, & \text{if } r \leq m \leq n - 1. \end{cases} \]

**Proof.** We prove the statement case by case.

**Case 1.** \( 2 \leq m \leq r - 1 \).

Write \( C_r = v_1v_2\cdots v_r v_1 \). Note that for any integer \( 1 \leq l \leq r - 1 \) and any vertex \( v_i \), if \( r \leq i \leq n \), by Lemma 2.1,

\[ |N^+(G_r^l : v_i) \cap V(C_r)| = |N^+(G_r^l : v_r) \cap V(C_r)| = l + 1. \]

If \( 1 \leq i \leq r - 1 \), by Lemma 2.1,

\[ |N^+(G_r^l : v_i) \cap V(C_r)| = l + 1. \]

For \( 2 \leq m \leq r - 1 \), and any \( v_i, v_j \in V(G_r) \), noticing that \( \frac{r+1}{2} \leq \left\lfloor \frac{r+m-1}{2} \right\rfloor \leq r - 1 \), we have

\[ |N^+(G_r^{\left\lfloor \frac{r+m-1}{2} \right\rfloor} : v_i, v_j)| \geq |N^+(G_r^{\left\lfloor \frac{r+m-1}{2} \right\rfloor} : v_i, v_j) \cap V(C_r)| \geq |N^+(G_r^{\left\lfloor \frac{r+m-1}{2} \right\rfloor} : v_i) \cap V(C_r)| + |N^+(G_r^{\left\lfloor \frac{r+m-1}{2} \right\rfloor} : v_i) \cap V(C_r)| - r \]

\[ = 2\left(\left\lfloor \frac{r + m - 1}{2} \right\rfloor + 1\right) - r \geq m, \]

and so \( k_m(G_r) \leq \left\lfloor \frac{r+m-1}{2} \right\rfloor \).
Now, we will show that $k_m(G_r) > \left\lfloor \frac{r+m-1}{2} \right\rfloor - 1$. If $\left\lfloor \frac{r+m-1}{2} \right\rfloor - 1$ is even, then

$$N^+(G_r^\left\lfloor \frac{r+m-1}{2} \right\rfloor : v_r) = \{v_2, v_4, \ldots, v_{\left\lfloor \frac{r+m-1}{2} \right\rfloor - 1}, v_{r-2}, v_{r-4}, \ldots, v_{r-\left\lfloor \frac{r+m-1}{2} \right\rfloor + 1} \} \cup \{v_r, v_{r+1}, \ldots, v_n\},$$

$$N^+(G_r^\left\lfloor \frac{r+m-1}{2} \right\rfloor : v_1) = \{v_1, v_3, \ldots, v_{\left\lfloor \frac{r+m-1}{2} \right\rfloor}, v_{r-1}, v_{r-3}, \ldots, v_{r-\left\lfloor \frac{r+m-1}{2} \right\rfloor + 2}\}.$$

If $\left\lfloor \frac{r+m-1}{2} \right\rfloor - 1$ is odd, then

$$N^+(G_r^\left\lfloor \frac{r+m-1}{2} \right\rfloor : v_r) = \{v_1, v_3, \ldots, v_{\left\lfloor \frac{r+m-1}{2} \right\rfloor - 1}, v_{r-1}, v_{r-3}, \ldots, v_{r-\left\lfloor \frac{r+m-1}{2} \right\rfloor + 1}\},$$

$$N^+(G_r^\left\lfloor \frac{r+m-1}{2} \right\rfloor : v_1) = \{v_2, v_4, \ldots, v_{\left\lfloor \frac{r+m-1}{2} \right\rfloor}, v_{r-2}, v_{r-4}, \ldots, v_{r-\left\lfloor \frac{r+m-1}{2} \right\rfloor + 2}\} \cup \{v_r, v_{r+1}, \ldots, v_n\}.$$

Then

$$N^+(G_r^\left\lfloor \frac{r+m-1}{2} \right\rfloor : v_r) = N^+(G_r^\left\lfloor \frac{r+m-1}{2} \right\rfloor : v_r) \cap N^+(G_r^\left\lfloor \frac{r+m-1}{2} \right\rfloor : v_1) = \{v_{r-\left\lceil \frac{r+m-1}{2} \right\rceil + 1}, v_{r-\left\lfloor \frac{r+m-1}{2} \right\rfloor + 2}, \ldots, v_{\left\lceil \frac{r+m-1}{2} \right\rceil}\},$$

and

$$|N^+(G_r^\left\lfloor \frac{r+m-1}{2} \right\rfloor : v_r, v_1)| = 2 \left\lfloor \frac{r+m-1}{2} \right\rfloor - r \leq m - 1.$$

The fact that $|N^+(G_r^\left\lfloor \frac{r+m-1}{2} \right\rfloor : v_r, v_1)| \leq m - 1$ implies that $k_m(G_r) > \left\lfloor \frac{r+m-1}{2} \right\rfloor - 1$. Therefore, $k_m(G_r) = \left\lfloor \frac{r+m-1}{2} \right\rfloor$, for $2 \leq m \leq r - 1$.

Case 2. $r \leq m \leq n - 1$.

It is easy to see that for each vertex $v_i$, $N^+(G_r^{r-1} : v_i) = V(G_r)$. So $r - 1 = k_{r-1}(G_r) \leq k_m(G_r) \leq r - 1$, and we have $k_m(G_r) = r - 1$.

The theorem follows. □

Now, we consider the graph $\overline{G_r}$ as shown in Figure 4, where $r$ is odd with $3 \leq r \leq n - 1$. 
We also characterize the graphs in \( S_n(\{8\}) \), and 
\[
\left\{ \begin{array}{ll}
-2 & \text{if } 2 \leq m \leq r - 1,
-1 & \text{if } r \leq m \leq n - 1.
\end{array} \right.
\]

**Theorem 3.1.** For any graph \( G \in S_n(r) \), \( 3 \leq r \leq n - 1 \), and \( 2 \leq m \leq n - 1 \),

\[
k_m(G) \leq k_m(G_{r,n-r}) = \left\{ \begin{array}{ll}
n - r + \left\lfloor \frac{r+m-2}{2} \right\rfloor & , \text{ if } 2 \leq m \leq r - 1,
n + m - r - 1 & , \text{ if } r \leq m \leq n - 1.
\end{array} \right.
\]

**Proof.** Let \( C_r \) be the cycle in \( G \) of length \( r \). For any \( v_i, v_j \in V(G) \), let \( P_i \) be the shortest path from \( v_j \) to \( C_r \), \( P_j \) the shortest path from \( v_j \) to \( C_r \), \( V(P_i) \cap V(C_r) = \{v_i\} \)
and $V(P_j) \cap V(C_r) = \{ u_j \}$. Then $l(P_i) \leq n - r$, and $l(P_j) \leq n - r$. Consider the following cases.

Case 1. $2 \leq m \leq r - 1$.

For any $v_i, v_j \in V(G)$, it is clear that $l(P_i) \leq n - r - 1$ or $l(P_j) \leq n - r - 1$. Without loss of generality, we assume that $l(P_j) \leq n - r - 1$ and $l(P_j) \leq n - r$. By Lemma 2.4,

$$|N^+(G^{n-r+\lceil \frac{r+m-2}{2} \rceil} : v_i) \cap V(C_r)|$$
$$\geq |N^+(G^{n-r+\lceil \frac{r+m-2}{2} \rceil} \setminus l(P_i) : u_i) \cap V(C_r)|$$
$$= \min\{ n - r + \left\lfloor \frac{r + m - 2}{2} \right\rfloor - (n - r) + 1, r \}$$
$$\geq \min\{ n - r + \left\lfloor \frac{r + m - 2}{2} \right\rfloor, n - r + 1, r \}$$
$$= \min\{ \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 1, r \} = \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 1.$$

Similarly,

$$|N^+(G^{n-r+\lceil \frac{r+m-2}{2} \rceil} : v_j) \cap V(C_r)| \geq \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 1.$$

Therefore,

$$|N^+(G^{n-r+\lceil \frac{r+m-2}{2} \rceil} : v_i, v_j)|$$
$$\geq |N^+(G^{n-r+\lceil \frac{r+m-2}{2} \rceil} : v_i, v_j) \cap V(C_r)|$$
$$\geq |N^+(G^{n-r+\lceil \frac{r+m-2}{2} \rceil} : v_i) \cap V(C_r)| + |N^+(G^{n-r+\lceil \frac{r+m-2}{2} \rceil} : v_j) \cap V(C_r)| - r$$
$$\geq 2 \left\lfloor \frac{r + m - 2}{2} \right\rfloor + 3 - r \geq m,$$

and so $k_m(G) \leq n - r + \left\lfloor \frac{r+m-2}{2} \right\rfloor$.

Case 2. $r \leq m \leq n - 1$.

For a vertex $x \in V(G)$, if $d(x, C_r) \leq n - r - 1$, then from $x$ to each vertex $v \in V(C_r)$, there exist two walks of lengths $l_1$ and $l_2$, respectively, such that $l_1$ and $l_2$ have different parity and $\max\{l_1, l_2\} \leq n - 1$. So $V(C_r) \subseteq N^+(G^{n-1} : x)$.

If $d(x, C_r) = n - r$, denoting by $P_x$ the shortest path from $x$ to $C_r$, $V(P_x) \cap V(C_r) = \{ u_x \}$, then from $x$ to each vertex $v \neq u_x \in V(C_r)$, there exist two walks of lengths $l_1$ and $l_2$, respectively, such that $l_1$ and $l_2$ have different parity and $\max\{l_1, l_2\} \leq n - 1$. So $V(C_r) \setminus \{ u_x \} \subseteq N^+(G^{n-1} : x)$. Note that $d(x, u_x) = n - r < n - 1$, and $n - r$ and $n - 1$ have the same parity. Hence, $\{ u_x \} \subseteq N^+(G^{n-1} : x)$, and we have $V(C_r) \subseteq N^+(G^{n-1} : x)$. 
Therefore, for any $v_i, v_j \in V(G)$ and any integer $k$ with $k \geq n - 1$,

$$v_i \mapsto^k u, \ v_j \mapsto^k u, \text{ for each vertex } u \text{ in } C_r.$$

So, for $m = r$, $k_m(G) \leq n - 1 = n + m - r - 1$. For $r + 1 \leq m \leq n - 1$, since $G$ is connected, there exist $m - r$ vertices $u_1, u_2, \ldots, u_{m-r} \in V(G)\setminus V(C_r)$ such that $d(u_t, C_r) \leq m - r$ for $t = 1, 2, \ldots, m - r$. Then

$$v_i \mapsto^{n+m-r-1} u_t, \ v_j \mapsto^{n+m-r-1} u_t, \text{ for } t = 1, 2, \ldots, m - r.$$

Thus, $k_m(G) \leq n + m - r - 1$.

By the above discussions, we see that Theorem 3.1 holds.

**Theorem 3.2.** For any $G \in S^n_0$, and $2 \leq m \leq n - 1$,

$$k_m(G) \leq n + m - 4.$$

The equality holds if and only if the graph $G$ is isomorphic to $G_{3,n-3}$.

**Proof.** Let $G \in S^n_0$. Then there exists an odd number $r$ with $3 \leq r \leq n$ such that $G \in S_n(r)$.

Case 1. $r = 3$.

By Theorem 3.1, we have $k_m(G) \leq n + m - 4$ for $2 \leq m \leq n - 1$.

Now, we assume that $G$ is not isomorphic to $G_{3,n-3}$. We will show that $k_m(G) \leq n + m - 5$ for $2 \leq m \leq n - 1$.

Take $C_3$ to be a 3-cycle in $G$ such that $\max_{u \in V(G)} d(u, C_3)$ is the smallest. Since $G$ is not isomorphic to $G_{3,n-3}$, we have that $\max_{u \in V(G)} d(u, C_3) \leq n - 4$.

For any $v_i, v_j \in V(G)$, let $P_i$ be the shortest path from $v_i$ to $C_r$, $P_j$ the shortest path from $v_j$ to $C_r$, $V(P_i) \cap V(C_r) = \{u_i\}$ and $V(P_j) \cap V(C_r) = \{u_j\}$. Then $l(P_i) \leq n - 4$, and $l(P_j) \leq n - 4$. Consider the following cases.

Subcase 1.1. $m = 2$.

If $l(P_i) = l(P_j) = n - 4$, then $G$ is isomorphic to $G_{3,n-4}$, and $k_m(G) = n - 3$.

If $l(P_i) \leq n - 5$ or $l(P_j) \leq n - 5$, then, without loss of generality, we assume $l(P_i) \leq n - 5$ and $l(P_j) \leq n - 4$. By Lemma 2.1

$$|N^+(G^{n-3} : v_i) \cap V(C_3)|$$

$$\geq |N^+(G^{n-3-l(P_i)} : u_i) \cap V(C_3)|$$

$$= \min\{n - 3 - l(P_i) + 1, 3\}$$

$$\geq \min\{n - 2 - (n - 4), 3\} = 2.$$
Similarly, $|N^+(G^{n-3} : v_j) \cap V(C_3)| = 3$. Therefore,

$$\left|N^+(G^{n-3} : v_i, v_j)\right| \geq \left|N^+(G^{n-3} : v_i, v_j) \cap V(C_3)\right| \geq 2,$$

and so $k_m(G) \leq n - 3$.

Subcase 1.2. $3 \leq m \leq n - 1$.

For a vertex $x \in V(G)$, if $d(x, C_r) \leq n - 5$, then from $x$ to each vertex $v \in V(C_r)$, there exist two walks of lengths $l_1$ and $l_2$, respectively, such that $l_1$ and $l_2$ have different parity and $\max\{l_1, l_2\} \leq n - 2$. So $V(C_r) \subseteq N^+(G^{n-2} : x)$.

If $d(x, C_3) = n - 4$, denoting by $P_x$ the shortest path from $x$ to $C_3$, with $V(P_x) \cap V(C_3) = \{u_x\}$, then from $x$ to each vertex $v \neq u_x \in V(C_3)$, there exist walks of length $n - 2$. So $V(C_r) \setminus \{u_x\} \subseteq N^+(G^{n-2} : x)$. Noting that $d(x, u_x) = n - 4 < n - 2$ and $n - 4$ and $n - 2$ have the same parity, so $\{u_x\} \subseteq N^+(G^{n-2} : x)$, and we have $V(C_3) \subseteq N^+(G^{n-2} : x)$.

Therefore, for any $v_i, v_j \in V(G)$ and any integer $k$ with $k \geq n - 2$,

$$v_i \xrightarrow{k} u, \quad v_j \xrightarrow{k} u,$$

for each vertex $u$ in $C_3$.

So, for $m = 3$, $k_m(G) \leq n - 2 = n + m - 5$. For $r + 1 \leq m \leq n - 1$, since $G$ is connected, there exist $m - 3$ vertices $u_1, u_2, \ldots, u_{m-3} \in V(G) \setminus V(C_3)$, such that $d(C_3, u_t) \leq m - 3$ for $t = 1, 2, \ldots, m - 3$. Then

$$v_i \xrightarrow{n+m-5} u_t, \quad v_j \xrightarrow{n+m-5} u_t,$$

for $t = 1, 2, \ldots, m - 3$.

Thus, $k_m(G) \leq n + m - 5$.

Case 2. $r \geq 5$.

If $r \leq n - 1$, then by Theorem 3.1, $k_m(G) \leq n - r + \left\lfloor \frac{r + m - 2}{2} \right\rfloor < n + m - 4$ when $2 \leq m \leq r - 1$, and $k_m(G) \leq n + m - r - 1 < n + m - 4$ when $r \leq m \leq n - 1$.

If $n$ is odd and $r = n$, by Theorem 2.5, $k_m(G) = \left\lfloor \frac{n + m - 1}{2} \right\rfloor < n + m - 4$.

The proof is now complete. □

Lemma 3.3. Let $r$ be odd with $3 \leq r \leq n - 1$. For $2 \leq m \leq n - 1$,

$$E_m(r) \geq \left\{ \left\lfloor \frac{r + m - 1}{2} \right\rfloor, \left\lfloor \frac{r + m - 1}{2} \right\rfloor + 1, \ldots, n - r + \left\lfloor \frac{r + m - 2}{2} \right\rfloor \right\},$$

if $2 \leq m \leq r - 1$,

$$\left\{ r - 1, r, \ldots, n + m - r - 1 \right\},$$

if $r \leq m \leq n - 1$.  

Proof. By Theorem 2.2, taking 1 \leq l \leq n - r, we have

\[ E_m(r) \supseteq \begin{cases} 
1 + \left\lfloor \frac{r + m - 2}{2} \right\rfloor, & \text{if } 2 \leq m \leq r - 1, \\
\{m, m + 1, \ldots, m + n - r - 1\}, & \text{if } m = r, \\
\{m + 1, \ldots, m + n - r - 1\} \cup \{r + 1\}, & \text{if } m = r + 1, \\
\{m + 2, \ldots, m + n - r - 1\} \cup \{r + 1, r + 3\}, & \text{if } m = r + 2, \\
\ldots \\
\{m + n - r - 2, m + n - r - 1\} \cup \{r + 1, r + 3, \ldots, 2n - r - 5\}, & \text{if } m = n - 2, \\
\{m + n - r - 1\} \cup \{r + 1, r + 3, \ldots, 2n - r - 3\}, & \text{if } m = n - 1.
\end{cases} \]

By Theorem 2.4, taking 1 \leq l \leq n - r - 2, we have

\[ E_m(r) \supseteq \begin{cases} 
1 + \left\lfloor \frac{r + m - 2}{2} \right\rfloor, & \text{if } 2 \leq m \leq r - 1, \\
\{m + 1, \ldots, m + n - r - 2\}, & \text{if } m = r, \\
\{m + 2, \ldots, m + n - r - 2\} \cup \{r + 2\}, & \text{if } m = r + 1, \\
\{m + 3, \ldots, m + n - r - 2\} \cup \{r + 2, r + 4\}, & \text{if } m = r + 2, \\
\ldots \\
\{r + 2, r + 4, \ldots, 2n - r - 4\}, & \text{if } m = n - 2, n - 1.
\end{cases} \]

By Theorem 2.5,

\[ E_m(r) \supseteq \begin{cases} 
\left\lfloor \frac{r + m - 2}{2} \right\rfloor, & \text{if } 2 \leq m \leq r - 1, \\
r - 1, & \text{if } r \leq m \leq n - 1.
\end{cases} \]

By Theorem 2.6 for r \leq m \leq n - 1, \( E_m(r) \supseteq \{r\} \).

By the above discussions, we have

\[ E_m(r) \supseteq \begin{cases} 
\left\lfloor \frac{r + m - 2}{2} \right\rfloor, & \text{if } 2 \leq m \leq r - 1, \\
\{r - 1, r, \ldots, m + n - r - 1\}, & \text{if } r \leq m \leq n - 1.
\end{cases} \]

**Theorem 3.4.**

\[ E_m = \begin{cases} 
\{1, 2, \ldots, n + m - 4\}, & \text{if } 2 \leq m \leq n - 2, \\
\{2, 3, \ldots, n + m - 4\}, & \text{if } m = n - 1.
\end{cases} \]

Proof. By Lemma 3.3 with \( r = 3 \), we know \( E_m \supseteq \{2, 3, \ldots, n + m - 4\} \) for \( 2 \leq m \leq n - 1 \). For the complete graph \( K_n \), it is clear that

\[ k_m(K_n) = \begin{cases} 
1, & \text{if } 2 \leq m \leq n - 2, \\
2, & \text{if } m = n - 1.
\end{cases} \]

Note that for any \( G \in S^0_n \), \( k_{n-1}(G) \geq 2 \). By Theorem 3.2, the theorem follows. \( \square \)
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