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RATIONAL INVARIANTS ON THE SPACE OF ALL STRUCTURES OF ALGEBRAS ON A TWO-DIMENSIONAL VECTOR SPACE*

J. MUÑOZ MASQUÉ† AND M. EUGENIA ROSADO MARÍA‡

Abstract. Let V be a 2-dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic different from 2 and 3. A non-empty Zariski-open subset $O \subset \otimes^2 V^* \otimes V$ and four $GL(V)$-invariant rational functions $\mathcal{I}_i, \mathcal{F}_i \colon O \to \mathbb{F}$ for $i = 1, 2$ are proved to exist such that two bilinear maps $t, t' \in O$ are $GL(V)$-equivalent with respect to the tensorial representation of $GL(V)$ if and only if $\mathcal{I}_i(t) = \mathcal{I}_i(t')$ and $\mathcal{F}_i(t) = \mathcal{F}_i(t')$ for $i = 1, 2$. The matrix reducing $t \in O$ to normal form is also studied. As the computation of the invariants $\mathcal{F}_i$, $i = 1, 2$, is expensive, two new invariants $\mathcal{I}_j$ with $j = 3, 4$ are introduced, which are easy to be computed and have a geometric meaning. The invariants $\mathcal{F}_i$, $i = 1, 2$, are written in terms of $\mathcal{I}_i$, $i = 1, \ldots, 4$, on a suitable Zariski-open subset $O' \subset O$. Hence, they also solve the equivalence problem on O'.

Key words. Normal form, Rational invariant, Tensorial representation, Traceless tensor.

AMS subject classifications. 15A72, 14L24, 14L35, 15A21.

1. Introduction. Let $\mathbb{F}$ be an algebraically closed field of characteristic p with $p \notin \{2, 3\}$, and let V be a 2-dimensional $\mathbb{F}$-vector space. The basic goal of the present paper is to prove the existence of a non-empty Zariski-open subset $O \subset \otimes^2 V^* \otimes V$ (see Remark 4.4 for the definition) and four $GL(V)$-invariant functions $\mathcal{I}_i, \mathcal{F}_i \colon O \to \mathbb{F}$, $i = 1, 2$, such that two bilinear maps $t, t' \in O$ are $GL(V)$-equivalent if and only if $\mathcal{I}_i(t) = \mathcal{I}_i(t'), \mathcal{F}_i(t) = \mathcal{F}_i(t')$, $i = 1, 2$, (see Theorem 6.1), where the tensorial representation of the full linear group $GL(V)$ on $\otimes^2 V^* \otimes V$ is considered; namely, for $A \in GL(V)$, $(A \cdot t)(x, y) = A(t(A^{-1}x, A^{-1}y))$ for all tensors $t \in \otimes^2 V^* \otimes V$ and all $x, y \in V$.

The space $\otimes^2 V^* \otimes V$ has the following important simple interpretation: It is the space of all structures of algebras (not necessarily associative) on the two-dimensional vector space V, and G-orbits are precisely the classes of isomorphic algebras.

The functions $\mathcal{I}_i$ and $\mathcal{F}_i$ $(i = 1, 2)$ are rational invariants (cf. [4, 5]) which can be

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written as the quotient of semi-invariant homogeneous polynomials of degree 4 and weight -2 with respect to the character $\det\colon GL(V) \to \mathbb{F}^*$. A general result states that any action of an algebraic group on an irreducible algebraic variety admits a finite set of rational invariants that separates orbits in general position (e.g., see [4, Chapter 1, Proposition 3], [5, Section 2, Theorem 2.3]), but the difficulty of the problem of obtaining a specific set of invariants depends on the particular linear representation under consideration. On the other hand, as the classical invariant theory (e.g., see [3, Section 4.3.1]) proves, no polynomial $GL(V)$-invariant exists on mixed tensors of different degrees.

The initial motivation for studying invariants on $S^2V^* \otimes V$ under the tensor representation of $GL(V)$ came from continuous and discrete dynamical systems; see [1], [2]. Once symmetric bilinear maps have been classified, the problem of classifying arbitrary (not necessarily symmetric) bilinear maps $f\colon V \times V \to V$ arises naturally. The complexity of the structure of invariants on $\otimes^2 V^* \otimes V$ is rather unexpected. As taking account of the fact that $GL(V)$ acts transitively on $\wedge^2 V^* \otimes V \setminus \{0\}$, one could naively expect the difficulty of the problem to be similar to the symmetric case, which is not true at all. In fact, the existence of an alternating part in addition to the symmetric part on a $(2,1)$ tensor produces two new invariants.

In [2], a basis $I_i\colon R \to \mathbb{F}$ $(i = 1, 2)$ for $GL(V)$-invariant functions on a Zariski-open subset $R \subset S^2V^* \otimes V$ has been obtained. For the explicit definitions of I_1, I_2 and R, see the formulas (3.1), (3.2), (3.3), (3.4) and (3.5). These two invariants induce invariants on $\mathrm{sym}^{-1}(R) \subset \otimes^2 V^* \otimes V$ by setting $\mathcal{I}_i(t) = I_i(\mathrm{sym}t)$, $i = 1, 2$, where sym denotes the symmetrization operator.

The goal of the present paper is two-fold. First, we complete the results in [2] as follows: (1) Normal forms for symmetric tensors in an adequate Zariski-open subset are given and for every tensor $t \in O$, a unique matrix $C_t \in GL(V)$ transforming $\mathrm{sym}t$ to its normal form, is proved to exist (see Proposition 4.1) and (2) On a Zariski-open subset in O the entries of the matrix C_t are shown to belong to a quadratic extension of the field $\mathbb{F}_p(t^k_{ij})$, where t^k_{ij} are the components of t (see Proposition 4.5). Explicit formulas for these entries are provided.

Second, in Proposition 5.1, we define two new invariants $\mathcal{F}_1$ and $\mathcal{F}_2$ which control the skew-symmetric part of a tensor in $\otimes^2 V^* \otimes V$ and, in Theorem 6.1, we state our main result, namely: The four invariants $\mathcal{I}_i$, $\mathcal{F}_i$ $(i = 1, 2)$ solve the equivalence problem on O. As a consequence, in Proposition 6.3, we obtain generic normal forms for arbitrary (not necessarily symmetric) tensors in $\otimes^2 V^* \otimes V$.

Unfortunately, the invariants $\mathcal{F}_1$ and $\mathcal{F}_2$ are expensive of computing as the formulas in the proof of Proposition 4.5 show. Hence, in Section 7, we introduce two new invariants $\mathcal{I}_3$ and $\mathcal{I}_4$ which are much easier to be computed and, in addition, they

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have a simple geometric meaning. As proved in Proposition 7.5, the invariants $\mathcal{F}_1$ and $\mathcal{F}_2$ can be written as a function of $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ on a suitable Zariski-open subset $O' \subset O$, and, accordingly, the invariants $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ also solve the equivalence problem on O'.

The method of defining $\mathcal{I}_3$ and $\mathcal{I}_4$ is completely different from that used in [2]. Here, we use Theorem 2.1 which proves that the algebra of generic $GL(V)$-invariants on $\otimes^2 V^* \otimes V$ is isomorphic to the algebra of $GL(V, w_0)$-invariants on the subspace of traceless tensors with respect to the subgroup $GL(V, w_0) \subset GL(V)$ keeping $w_0 \in V^*$ fixed.

We would also like to remark that our results allow one to solve the generic equivalence problem efficiently from the computational point of view: If $t, t' \in O$ are two equivalent tensors, then the matrix $(C_{t'})^{-1} C_t$, transforming t into t', can be computed by means of a polynomial number of operations in the ground field and taking one square root.

2. Reduction to traceless tensors. Let V and V' be 2-dimensional vector spaces over a field $\mathbb{F}$. In the following, we use freely the isomorphism $\ell \colon V^* \otimes V' \to \mathrm{Hom}(V, V')$ between $V^* \otimes V'$ and the space of $\mathbb{F}$-linear maps from V into V', determined by $\ell(w \otimes v')(x) = w(x)v'$ $\forall x \in V$, $\forall v' \in V'$ and $\forall w \in V^*$.

The $\mathbb{F}$-algebra of all functions $f \colon X \to \mathbb{F}$ defined on a set X is denoted by $\mathcal{F}(X)$. If a group G acts on the left of X, then $\mathcal{F}(X)^G$ denotes the subalgebra of G-invariant functions in $\mathcal{F}(X)$, i.e., $f \in \mathcal{F}(X)^G$ if and only if $f(g \cdot x) = f(x)$ $\forall x \in X$ and $\forall g \in G$. Every map $\mu \colon X \to Y$ induces an $\mathbb{F}$-algebra homomorphism $\mu^\star \colon \mathcal{F}(Y) \to \mathcal{F}(X)$, given by $\mu^\star(f) = f \circ \mu$ for all $f \in \mathcal{F}(Y)$.

Theorem 2.1. *Let V be a 2-dimensional vector space over a field $\mathbb{F}$. The homomorphism* $\mathrm{tr} \colon \otimes^2 V^* \otimes V \to V^*$, *obtained by contracting the second covariant argument with the contravariant one, induces a split epimorphism of $GL(V)$-modules. In fact, the map $\sigma \colon V^* \to \otimes^2 V^* \otimes V$, defined by $\sigma(w)(x, y) = w(y)x$ for all $x, y \in V$ and $w \in V^*$, is a $GL(V)$-equivariant section of* tr, *and the map*

$$\varphi \colon \otimes^2 V^* \otimes V \xrightarrow{\cong} \ker \mathrm{tr} \oplus V^*, \quad \varphi(t) = (t - \sigma(\mathrm{tr} t), \mathrm{tr} t), \tag{2.1}$$

is an isomorphism of $GL(V)$-modules. Furthermore, let O^1 be the Zariski-open subset of the elements $t \in \otimes^2 V^* \otimes V$ such that $\mathrm{tr} t \neq 0$. *If*

$$GL(V, w_0) = \{A \in GL(V) : A \cdot w_0 = w_0\}, \ \ w_0 \in V^* \backslash \{0\},$$

then an isomorphism of $\mathbb{F}$-algebras holds

$$\phi \colon \mathcal{F}(O^1)^{GL(V)} \xrightarrow{\cong} \mathcal{F}(\ker \mathrm{tr})^{GL(V, w_0)}. \tag{2.2}$$

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Proof. As the trace map is $GL(V)$-equivariant, it suffices to prove that σ is a $GL(V)$-equivariant section of tr. If (v_1, v_2) is a basis for V with dual basis (v^1, v^2), i.e., $v^i(v_j) = \delta^i_j$, then tr$t = t^i_{ji} v^j$ with

$$t = \sum_{i,j,k=1}^{2} t^k_{ij} v^i \otimes v^j \otimes v_k. \tag{2.3}$$

From the definition of σ it follows $\sigma(w) = v^j \otimes w \otimes v_j$. Hence $\mathrm{tr}(\sigma(w)) = w(v_j)v^j = w$, and σ is a section of tr. Moreover, for all $A \in GL(V)$, $x, y \in V$ and $w \in V^*$, we have

$$\begin{aligned}(A \cdot \sigma(w))(x, y) &= A(\sigma(w)(A^{-1}x, A^{-1}y)) \\ &= (w \circ A^{-1})(y)x = (A \cdot w)(y)x \\ &= \sigma(A \cdot w)(x, y),\end{aligned}$$

i.e., $A \cdot \sigma(w) = \sigma(A \cdot w)$. Therefore, σ is $GL(V)$-equivariant. The isomorphism (2.1) induces a bijection $\varphi \colon O^1 \to \ker \mathrm{tr} \times (V^* \backslash \{0\})$. Let $\Phi \colon \mathcal{F}(O^1) \to \mathcal{F}(\ker \mathrm{tr})$ be the map defined by $\Phi(f)(t) = f(\varphi^{-1}(t, w_0))$ for every $t \in \ker \mathrm{tr}$, which is an $\mathbb{F}$-algebra homomorphism as $\Phi = \iota^\star \circ (\varphi^{-1})^\star$, where $\iota \colon \ker \mathrm{tr} \to \ker \mathrm{tr} \times (V^* \backslash \{0\})$ is the inclusion map $\iota(t) = (t, w_0)$. We claim that if $f \in \mathcal{F}(O^1)^{GL(V)}$, then $\Phi(f)$ belongs to $\mathcal{F}(\ker \mathrm{tr})^{GL(V, w_0)}$. In fact, as φ is $GL(V)$-equivariant, so is φ^{-1}, and for all $A \in GL(V, w_0)$ and $t \in \ker \mathrm{tr}$, we obtain

$$\begin{aligned}\Phi(f)(A \cdot t) &= f\left(\varphi^{-1}(A \cdot t, w_0)\right) \\ &= f\left(\varphi^{-1}(A \cdot t, A \cdot w_0)\right) \\ &= f\left(\varphi^{-1}\left(A \cdot (t, w_0)\right)\right) \\ &= f\left(A \cdot \varphi^{-1}\left(t, w_0\right)\right) \\ &= f\left(\varphi^{-1}\left(t, w_0\right)\right) \\ &= \Phi(f)(t).\end{aligned}$$

Hence, by restricting Φ to $\mathcal{F}(O^1)^{GL(V)}$, it induces an $\mathbb{F}$-algebra homomorphism from $\mathcal{F}(O^1)^{GL(V)}$ to $\mathcal{F}(\ker \mathrm{tr})^{GL(V, w_0)}$, which we prove to be bijective.

If $f \in \ker \phi$, then $f\left(\varphi^{-1}(t, w_0)\right) = 0$ for each $t \in \ker \mathrm{tr}$. As $GL(V)$ acts transitively on $V^* \backslash \{0\}$, given $(t, w) \in \ker \mathrm{tr} \times (V^* \backslash \{0\})$, there exists $A \in GL(V)$ such that $A \cdot w_0 = w$. By setting $t' = A^{-1} \cdot t$ and the fact that φ^{-1} is $GL(V)$-equivariant as well the $GL(V)$-invariance of f, from the hypothesis, we have

$$\begin{aligned}0 &= f(\varphi^{-1}(t', w_0)) \\ &= f(A \cdot \varphi^{-1}(t', w_0)) \\ &= f(\varphi^{-1}(A \cdot (t', w_0)))\end{aligned}$$

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$$
\begin{aligned}
&= f(\varphi^{-1}(A \cdot t', A \cdot w_0)) \\
&= f(\varphi^{-1}(t, w)).
\end{aligned}
$$

Hence, $f(\varphi^{-1}(t,w)) = 0$ for each $(t,w) \in \ker \mathrm{tr} \times (V^* \backslash \{0\})$, and $f = 0$ as φ is bijective. This proves that ϕ is injective. Next, we prove that ϕ is also surjective. Given a map $g \in \mathcal{F}(\ker \mathrm{tr})^{GL(V,w_0)}$, we define $f \in \mathcal{F}(O^1)$ by setting $f(t) = g(A^{-1} \cdot (t - \sigma(\mathrm{tr}t)))$ for all $t \in O^1$, and $A \in GL(V)$ being any matrix such that $\mathrm{tr}t = A \cdot w_0$. The definition makes sense as it does not depend on the matrix chosen, since for $B \in GL(V)$ with $\mathrm{tr}t = B \cdot w_0$, $A^{-1}B \in GL(V, w_0)$. Since g is $GL(V, w_0)$-invariant, we obtain

$$
\begin{aligned}
g\left(A^{-1} \cdot (t - \sigma(\mathrm{tr}t))\right) &= g\left(\left(A^{-1}B\right) \cdot \left(B^{-1} \cdot (t - \sigma(\mathrm{tr}t))\right)\right) \\
&= g\left(B^{-1} \cdot (t - \sigma(\mathrm{tr}t))\right).
\end{aligned}
$$

Moreover, f is $GL(V)$-invariant. In fact, given $B \in GL(V)$ and $t \in O^1$, from the definition of f, we have $f(B \cdot t) = g((BA)^{-1} \cdot (B \cdot t - \sigma(\mathrm{tr}(B \cdot t))))$, since $\mathrm{tr}(B \cdot t) = B \cdot \mathrm{tr}t = BA \cdot w_0$. Hence

$$
\begin{aligned}
f(B \cdot t) &= g(A^{-1}B^{-1} \cdot (B \cdot t - B \cdot \sigma(\mathrm{tr}t))) \\
&= g(A^{-1} \cdot (t - \sigma(\mathrm{tr}t))) \\
&= f(t).
\end{aligned}
$$

Finally, we prove $\Phi(f) = g$. By setting $t' = t + \sigma(w_0)$ for every $t \in \ker \mathrm{tr}$, we have $\mathrm{tr}t' = w_0$, and therefore $t' \in O^1$, $\varphi(t') = (t, w_0)$. Hence, from the definitions of Φ and f, we obtain $\Phi(f)(t) = f(\varphi^{-1}(t, w_0)) = f(t') = g(t' - \sigma(w_0)) = g(t)$. $\square$

REMARK 2.2. For an arbitrary $\mathbb{F}$-vector space V, the subgroup $GL(V, w_0)$ in Theorem 2.1 is isomorphic to the affine group $\mathcal{A}(V')$ of the hyperplane $V' = \ker w_0$. In fact, if $v_0 \in V$ is such that $w_0(v_0) = 1$, then a matrix $A \in GL(V)$ belongs to $GL(V, w_0)$ if and only if (i) $A(V') = V'$ and (ii) $A(v_0) - v_0 \in V'$. We can thus define a map $\beta\colon GL(V, w_0) \to GL(V') \times V'$ by $\beta(A) = (A|_{V'}, A(v_0) - v_0)$, which is bijective as $V = \mathbb{F}v_0 \oplus V'$. If $f \in \mathcal{A}(V')$ (resp. $\bar{f} \in \mathcal{A}(V')$), $f(x') = A'(x') + v_0'$ (resp. $\bar{f}(x') = \bar{A}'(x') + \bar{v}_0'$) for each $x' \in V'$ is the affine transformation associated to the pair $(A' = A|_{V'}, v_0' = A(v_0) - v_0)$ (resp. $(\bar{A}' = \bar{A}|_{V'}, \bar{v}_0' = \bar{A}(v_0) - v_0)$), then it follows that $(f \circ \bar{f})(x') = (A'\bar{A}')(x') + A'(\bar{v}_0') + v_0'$, and hence we obtain $(A\bar{A})(v_0) - v_0 = A'(\bar{v}_0') + v_0'$. Thus, β is a group isomorphism.

3. The invariants $\mathcal{I}_1$ and $\mathcal{I}_2$. If the characteristic of $\mathbb{F}$ is odd, then symmetrization (resp. anti-symmetrization) operator is given by

$$
\begin{aligned}
&\mathrm{sym}\colon \otimes^2 V^* \otimes V \to S^2V^* \otimes V, && \mathrm{alt}\colon \otimes^2 V^* \otimes V \to \wedge^2 V^* \otimes V, \\
&\mathrm{sym}(t)(x,y) = \tfrac{1}{2}(t(x,y) + t(y,x)), && \mathrm{alt}(t)(x,y) = \tfrac{1}{2}(t(x,y) - t(y,x)),
\end{aligned}
$$

for all $x, y \in V$. When the characteristic of $\mathbb{F}$ is also distinct from 3, a basis for generic $GL(V)$-invariant functions on $S^2V^* \otimes V$ has been obtained in [2]. The equivalence

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between the notations in [2] and those used here is as follows: $F = t$; $v_i^* = v^i$, $i = 1, 2$; $a_1 = t_{11}^1$, $a_2 = t_{11}^2$, $b_1 = t_{12}^1$, $b_2 = t_{12}^2$, $c_1 = t_{22}^1$, $c_2 = t_{22}^2$. Note that the invariants $I_i \colon R \to \mathbb{F}$ for $i \in \{1, 2\}$, defined in [2, Theorem 4–2], can be rewritten as follows:

$$I_i(t) = \frac{H_i(t)}{\det Q_{\bar{t}}}, \quad t \in R, \ i = 1, 2, \tag{3.1}$$

$$\begin{array}{ll} \bar{t} = t - \sigma_S(\mathrm{tr} t), & t \in S^2V^* \otimes V, \\ \sigma_S \colon V^* \to S^2V^* \otimes V, & \sigma_S(w)(x, y) = \frac{1}{3}(w(y)x + w(x)y), \end{array}$$

for all $x, y \in V$, $w \in V^*$, and Q and H_i are given by

$$Q_{\bar{t}} = \begin{pmatrix} \bar{t}_{11}^1 \bar{t}_{12}^2 - \bar{t}_{11}^2 \bar{t}_{12}^1 & \frac{1}{2}\left(\bar{t}_{11}^1 \bar{t}_{22}^2 - \bar{t}_{11}^2 \bar{t}_{22}^1\right) \\ \frac{1}{2}\left(\bar{t}_{11}^1 \bar{t}_{22}^2 - \bar{t}_{11}^2 \bar{t}_{22}^1\right) & \bar{t}_{12}^1 \bar{t}_{22}^2 - \bar{t}_{12}^2 \bar{t}_{22}^1 \end{pmatrix}, \tag{3.2}$$

$$R = \left\{ t \in S^2V^* \otimes V : \det Q_{\bar{t}} \neq 0 \right\}, \tag{3.3}$$

$$\begin{aligned} H_1(t) = \frac{1}{12} \Big\{ & \left(t_{11}^1 + t_{12}^2\right)^2 \left(\left(2t_{12}^1 - t_{22}^2\right)^2 + 3\left(2t_{12}^2 - t_{11}^1\right) t_{22}^1 \right) \\ & + \left(t_{11}^1 + t_{12}^2\right)\left(t_{12}^1 + t_{22}^2\right) \left(\left(2t_{12}^2 - t_{11}^1\right)\left(2t_{12}^1 - t_{22}^2\right) - 9t_{11}^2 t_{22}^1 \right) \\ & + \left(t_{12}^1 + t_{22}^2\right)^2 \left(\left(2t_{12}^2 - t_{11}^1\right)^2 + 3\left(2t_{12}^1 - t_{22}^2\right) t_{11}^2 \right) \Big\}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} H_2(t) = \frac{1}{4} \Big\{ & -t_{22}^1 \left(t_{11}^1 + t_{12}^2\right)^3 + \left(t_{11}^1 + t_{12}^2\right)^2 \left(t_{12}^1 + t_{22}^2\right)\left(2t_{12}^1 - t_{22}^2\right) \\ & + \left(t_{11}^1 + t_{12}^2\right)\left(t_{12}^1 + t_{22}^2\right)^2 \left(2t_{12}^2 - t_{11}^1\right) - t_{11}^2 \left(t_{12}^1 + t_{22}^2\right)^3 \Big\}, \end{aligned} \tag{3.5}$$

and where t is a symmetric tensor as in (2.3) with $t_{ij}^k = t_{ji}^k$.

$$\begin{aligned} t = {} & t_{11}^1 v^1 \otimes v^1 \otimes v_1 + t_{12}^1 \left(v^1 \otimes v^2 + v^2 \otimes v^1\right) \otimes v_1 + t_{22}^1 v^2 \otimes v^2 \otimes v_1 \\ & + t_{11}^2 v^1 \otimes v^1 \otimes v_2 + t_{12}^2 \left(v^1 \otimes v^2 + v^2 \otimes v^1\right) \otimes v_2 + t_{22}^2 v^2 \otimes v^2 \otimes v_2. \end{aligned}$$

Using I_1 and I_2, four functions can be defined as follows:

$$\mathcal{I}_i \colon \mathrm{sym}^{-1}(R) \to \mathbb{F}, \quad \mathcal{I}_i(t) = I_i\left(\mathrm{sym} t\right), \ i = 1, 2, \tag{3.6}$$

$$J_i \colon \ker \mathrm{tr} \cap \mathrm{sym}^{-1}(R) \to \mathbb{F}, \quad J_i(\bar{t}) = I_i\left(\mathrm{sym} \bar{t}\right), \ i = 1, 2. \tag{3.7}$$

The functions $\mathcal{I}_1, \mathcal{I}_2, J_1$ and J_2 are $GL(V)$-invariant, and the pairs (I_1, I_2) and (J_1, J_2) determine each other as in the 2-dimensional case where the symmetrization operator induces an isomorphism of $GL(V)$-modules $\mathrm{sym} \colon \ker \mathrm{tr} \xrightarrow{\approx} S^2V^* \otimes V$. By applying the isomorphism ϕ in the formula (2.2) to $\mathcal{I}_1$ and $\mathcal{I}_2$, we obtain four invariants $\phi(\mathcal{I}_1|_{O^1 \cap \mathrm{sym}^{-1}(R)})$, $\phi(\mathcal{I}_2|_{O^1 \cap \mathrm{sym}^{-1}(R)})$, J_1 and J_2 defined on some Zariski-open subsets in $\ker \mathrm{tr}$, but they are not independent. In fact, we have the following result.

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PROPOSITION 3.1. *Suppose that characteristic*$(\mathbb{F}) \neq 2,3,5$. *Then, for every* $\bar{t} \in \ker \operatorname{tr} \cap \operatorname{sym}^{-1}(R)$ *such that* $a\bar{t} + \sigma(w_0) \in O^1 \cap \operatorname{sym}^{-1}(R)$, $a \in \{\frac{1}{2}, \frac{1}{3}, 1, 2, 3\}$, *the following holds:*

$$J_i(\bar{t}) = 3\phi\left(\mathcal{I}_i\right)(\bar{t}) + \frac{81}{10}\phi\left(\mathcal{I}_i\right)(3\bar{t}) - \frac{3}{5}\phi\left(\mathcal{I}_i\right)\left(\frac{1}{2}\bar{t}\right) + \frac{1}{10}\phi\left(\mathcal{I}_i\right)\left(\frac{1}{3}\bar{t}\right) - \frac{48}{5}\phi\left(\mathcal{I}_i\right)(2\bar{t}).$$

Proof. Let W be an $\mathbb{F}$-vector space. As is well known, there is a bijective mapping between the homogeneous polynomials $Q\colon W \to \mathbb{F}$ of degree d and the d-multilinear symmetric functions $q\colon S^4W \to \mathbb{F}$. For $d = 4$, from the polarization identity (e.g., see [3, Lemma B.2.5]), we deduce that this correspondence is given by the following formulas:

$$\begin{aligned}
Q(x) &= q(x,x,x,x),\\
q(x,y,z,t) &= \frac{1}{24}\left\{Q(x+y+z+t) - Q(x+y+z) - Q(x+y+t)\right.\\
&\quad -Q(x+z+t) - Q(y+z+t) + Q(x+y) + Q(x+z) + Q(x+t)\\
&\quad \left.+Q(y+z) + Q(y+t) + Q(z+t) - Q(x) - Q(y) - Q(z) - Q(t)\right\}.
\end{aligned}$$

As the functions H_i $(i = 1,2)$ defined in the formulas (3.4) and (3.5) are homogeneous polynomials of degree $d = 4$ on $W = S^2V^* \otimes V$, from the previous formula, we obtain the following equation, after a simple—but rather long—computation:

$$\begin{aligned}
H_i(\operatorname{sym}\bar{t} + t_0) &= \frac{1}{3}H_i(\operatorname{sym}\bar{t}) - \frac{1}{30}H_i\left(\operatorname{sym}(3\bar{t}) + t_0\right) + \frac{16}{5}H_i\left(\operatorname{sym}\left(\frac{1}{2}\bar{t}\right) + t_0\right)\\
&\quad -\frac{27}{10}H_i\left(\operatorname{sym}\left(\frac{1}{3}\bar{t}\right) + t_0\right) + \frac{1}{5}H_i\left(\operatorname{sym}(2\bar{t}) + t_0\right),
\end{aligned}$$

where $t_0 = \operatorname{sym}\sigma(w_0)$ and we have used the identity $H_i(t_0) = 0$. Moreover, as a computation shows, we have $\overline{a\operatorname{sym}(\bar{t}) + t_0} = \overline{a\operatorname{sym}(\bar{t})}$ for each $a \in \mathbb{F}$. Hence, from the formula (3.2), we obtain

$$\begin{aligned}
\det Q_{\overline{a\operatorname{sym}(\bar{t})+t_0}} &= \det Q_{\overline{a\operatorname{sym}(\bar{t})}}\\
&= a^4 \det Q_{\overline{\operatorname{sym}(\bar{t})}}\\
&= a^4 \det Q_{\overline{\operatorname{sym}(\bar{t})+t_0}}.
\end{aligned}$$

Taking these formulas into account, we conclude

$$\begin{aligned}
I_i(\operatorname{sym}\bar{t} + t_0) &= \frac{1}{3}I_i(\operatorname{sym}\bar{t}) - \frac{27}{10}I_i\left(\operatorname{sym}(3\bar{t}) + t_0\right) + \frac{1}{5}I_i\left(\operatorname{sym}\left(\frac{1}{2}\bar{t}\right) + t_0\right)\\
&\quad -\frac{1}{30}I_i\left(\operatorname{sym}\left(\frac{1}{3}\bar{t}\right) + t_0\right) + \frac{16}{5}I_i\left(\operatorname{sym}(2\bar{t}) + t_0\right).
\end{aligned}$$

The formula in the statement now follows from the formulas (3.6) and (3.7). $\square$

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4. Symmetric normal forms.

4.1. The symmetric normal forms defined. We first introduce the normal forms for a symmetric bilinear map.

PROPOSITION 4.1. *We set*

$$\begin{aligned}
O^2 &= \left\{ t \in \mathrm{sym}^{-1}(R) : \mathcal{I}_1(t) + 2\mathcal{I}_2(t) \neq 0, \mathcal{I}_2(t) \neq 0 \right\}, \\
O'^2 &= \left\{ t \in O^2 : 4\mathcal{I}_1(t)^3 \neq 27\mathcal{I}_2(t)^2 \right\}, \\
X^1 &= \left\{ t \in \mathrm{sym}^{-1}(R) : \mathcal{I}_1(t) + 2\mathcal{I}_2(t) = 0, \mathcal{I}_2(t) \neq 0 \right\}, \\
X'^1 &= \left\{ t \in X^1 : \mathcal{I}_1(t) \neq \frac{27}{16} \right\}, \\
X^2 &= \left\{ t \in \mathrm{sym}^{-1}(R) : \mathcal{I}_1(t) \neq 0, 9, \mathcal{I}_2(t) = 0 \right\}, \\
C^1 &= \left\{ t \in \mathrm{sym}^{-1}(R) : \mathcal{I}_1(t) = 9, \mathcal{I}_2(t) = 0 \right\}, \\
C^2 &= \left\{ t \in \mathrm{sym}^{-1}(R) : \mathcal{I}_1(t) = 0, \mathcal{I}_2(t) = 0 \right\},
\end{aligned}$$

$$\xi_i = \mathcal{I}_i(t), \ i = 1, 2, \quad \forall t \in \mathrm{sym}^{-1}(R), \tag{4.1}$$

$$\Xi_1 = \frac{\xi_1 - \xi_2}{\xi_1 + 2\xi_2}, \quad \Xi_2 = \left(\frac{4(\xi_1)^3 - 27(\xi_2)^2}{(\xi_1 + 2\xi_2)^3} \right)^{\frac{1}{2}}, \quad \forall t \in O^2, \tag{4.2}$$

$$\Xi_3 = \left(16 - \frac{27}{\xi_1} \right)^{\frac{1}{2}}, \quad \forall t \in X^1, \tag{4.3}$$

$$\Xi_4 = 2\left(\frac{3\xi_1}{9 - \xi_1} \right)^{\frac{1}{2}}, \quad \forall t \in X^2. \tag{4.4}$$

Given a linear form $w_0 \in V^* \backslash \{0\}$, let (v_1, v_2) be a basis for V such that its dual basis (v^1, v^2) satisfies $v^1 = w_0$. Let $\tau_t \in S^2V^* \otimes V$ be the tensor whose components in the basis (v_1, v_2) are given as follows:

i) *If $t \in O^2$, then*

$$\begin{aligned}
&(\tau_t)^1_{11} = 0, \quad (\tau_t)^1_{12} = \Xi_1, \quad (\tau_t)^1_{21} = \Xi_1, \quad (\tau_t)^1_{22} = \Xi_2, \\
&(\tau_t)^2_{11} = 1, \quad (\tau_t)^2_{12} = 0, \quad (\tau_t)^2_{21} = 0, \quad (\tau_t)^2_{22} = -1.
\end{aligned}$$

ii) *If $t \in X^1$, then*

$$\begin{aligned}
&(\tau_t)^1_{11} = 0, \quad (\tau_t)^1_{12} = 3 \cdot 2^{-\frac{1}{3}}, \quad (\tau_t)^1_{21} = 3 \cdot 2^{-\frac{1}{3}}, \quad (\tau_t)^1_{22} = \Xi_3, \\
&(\tau_t)^2_{11} = 1, \quad (\tau_t)^2_{12} = 0, \quad (\tau_t)^2_{21} = 0, \quad (\tau_t)^2_{22} = 0.
\end{aligned}$$

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iii) *If $t \in X^2$, then*

$$(\tau_t)^1_{11} = \Xi_4, \quad (\tau_t)^1_{12} = 1, \quad (\tau_t)^1_{21} = 1, \quad (\tau_t)^1_{22} = 0,$$
$$(\tau_t)^2_{11} = 1, \quad (\tau_t)^2_{12} = 0, \quad (\tau_t)^2_{21} = 0, \quad (\tau_t)^2_{22} = -1.$$

iv) *If $t \in C^1$, then*

$$(\tau_t)^1_{11} = -\tfrac{1}{2}, \quad (\tau_t)^1_{12} = 1, \quad (\tau_t)^1_{21} = 1, \quad (\tau_t)^1_{22} = 0,$$
$$(\tau_t)^2_{11} = 0, \quad (\tau_t)^2_{12} = -1, \quad (\tau_t)^2_{21} = -1, \quad (\tau_t)^2_{22} = -1.$$

v) *If $t \in C^2$, then*

$$(\tau_t)^1_{11} = 1, \quad (\tau_t)^1_{12} = 1, \quad (\tau_t)^1_{21} = 1, \quad (\tau_t)^1_{22} = 0,$$
$$(\tau_t)^2_{11} = 0, \quad (\tau_t)^2_{12} = -1, \quad (\tau_t)^2_{21} = -1, \quad (\tau_t)^2_{22} = -1.$$

With these notations, $\mathcal{I}_i(\tau_t) = \mathcal{I}_i(t)$, $i = 1, 2$, and a matrix $C_t \in GL(V)$ exists such that $C_t \cdot \mathrm{sym}t = \tau_t$, which is unique for $t \in O'^2 \cup X'^1 \cup X^2 \cup C^1$. If $t \in (O^2 \setminus O'^2) \cup (X^1 \setminus X'^1)$, then the isotropy group of τ_t is $\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Remark 4.2. In the definition of O^2, the condition $\xi_2 \neq 0$ is needed as the denominator of $I_i(\tau_t)$, $i = 1, 2$, is $\frac{81(\xi_2)^2}{(\xi_1+2\xi_2)^3}$.

Remark 4.3. In the cases iv) and v), the tensor τ_t does not depend on t. In fact, in the case v), τ_t coincides with the tensor denoted by F_0 in [2, formula (8)]. Hence the isotropy group of τ_t in $GL(V)$ coincides with the finite group $\mathcal{G}$ in [2, formula (9)]. Consequently, the matrix C_t is not unique in the fifth case.

Remark 4.4. The subsets O^2, X^1, X^2, C^1 and C^2 form a partition of the Zariski-open subset $\mathrm{sym}^{-1}(R)$, and from their definitions we have

$$O'^2 \cup X'^1 \cup X^2 \cup C^1 = \{t \in \mathrm{sym}^{-1}(R) : 4\mathcal{I}_1(t)^3 \neq 27\mathcal{I}_2(t)^2\}.$$

Hence, $O = O'^2 \cup X'^1 \cup X^2 \cup C^1$ is a Zariski-open subset in $\mathrm{sym}^{-1}(R)$.

Proof. [Proof of Proposition 4.1] The equations $\mathcal{I}_i(\tau_t) = \mathcal{I}_i(t)$, $i = 1, 2$, follow from the computation of the invariants I_i, $i = 1, 2$, for the symmetric tensor τ_t by using the formulas (3.1). The existence of the matrix C_t thus follows from the definition of the invariants $\mathcal{I}_i$, $i = 1, 2$ in the formula (3.6) by [2, Theorem 4–2].

In the cases $t \in O'^2$, $t \in X'^1$, $t \in X^2$ and $t \in C^1$, the uniqueness of C_t is equivalent to saying that the isotropy group of τ_t in $GL(V)$ reduces to the identity matrix. To prove this, we proceed as follows. Assume the matrix $A \in GL(V)$ given by $A(v_1) = av_1 + bv_2$, $A(v_2) = cv_1 + dv_2$, transforms τ_t into itself. From the transformation formulas

$$(A^{-1} \cdot \tau_t)^q_{rs} = a^j_r a^k_s (\tau_t)^i_{jk} (A^{-1})^q_i, \tag{4.5}$$

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we obtain the following systems of equations for the isotropy group of τ_t:

In the case i), i.e., if $t \in O^2$, then

$$0 = -2\Xi_1 abd - \Xi_2 b^2 d + a^2 c - b^2 c, \tag{4.6}$$
$$0 = \Xi_1 ad - \Xi_1 bc - \Xi_1 bcd - \Xi_1 ad^2 - \Xi_2 bd^2 + ac^2 - bcd, \tag{4.7}$$
$$0 = \Xi_2 ad - \Xi_2 bc - 2\Xi_1 cd^2 - \Xi_2 d^3 + c^3 - cd^2, \tag{4.8}$$
$$0 = ad - bc + 2\Xi_1 ab^2 + \Xi_2 b^3 - a^3 + ab^2, \tag{4.9}$$
$$0 = \Xi_1 b^2 c + \Xi_1 abd + \Xi_2 b^2 d - a^2 c + abd, \tag{4.10}$$
$$0 = -ad + bc + 2\Xi_1 bcd + \Xi_2 bd^2 - ac^2 + ad^2. \tag{4.11}$$

If $b \neq 0$, by summing up the equations (4.6) and (4.10), it follows $\Xi_1 = 1$, which is not possible as this equation implies $\xi_2 = 0$, but $\xi_2 \neq 0$ on O^2 according to the definition of such a set. Hence $b = 0$, and consequently $a \neq 0$ and $d \neq 0$ as the matrix A is not singular. The formulas (4.6)–(4.11) thus imply the following: $0 = c$, $0 = \Xi_2(a - d^2)$, $d = a^2$, $d = 1$. If $t \in O'^2$, then $\Xi_2 \neq 0$, and from these equations we obtain $a = 1$. If $t \in O^2 \backslash O'^2$, then $\Xi_2 = 0$ and hence, $a = \pm 1$.

In the case ii), i.e., if $t \in X^1$, then we obtain the following equations for the isotropy group of τ_t:

$$0 = 3 \cdot 2^{\frac{2}{3}} abd - a^2 c + b^2 d \Xi_3, \tag{4.12}$$
$$0 = 3 \cdot 2^{\frac{2}{3}} (ad - bc) - 3 \cdot 2^{\frac{2}{3}} bcd - 3 \cdot 2^{\frac{2}{3}} ad^2 + 2ac^2 - 2bd^2 \Xi_3, \tag{4.13}$$
$$0 = c^3 - 3 \cdot 2^{\frac{2}{3}} cd^2 + \Xi_3 \left(ad - bc - d^3\right), \tag{4.14}$$
$$0 = ad - bc + 3 \cdot 2^{\frac{2}{3}} ab^2 - a^3 + \Xi_3 b^3, \tag{4.15}$$
$$0 = 3 \cdot 2^{\frac{2}{3}} b^2 c + 3 \cdot 2^{\frac{2}{3}} abd - 2a^2 c + 2b^2 d \Xi_3, \tag{4.16}$$
$$0 = 3 \cdot 2^{\frac{2}{3}} bcd - ac^2 + bd^2 \Xi_3. \tag{4.17}$$

The system of (4.12) and (4.15) is linear with respect to the unknowns c, d, and its determinant $3 \cdot 2^{\frac{2}{3}} ab^2 - a^3 + \Xi_3 b^3$ does not vanish by virtue of the equation (4.15) as A is not singular. Hence,

$$c = b\left(3 \cdot 2^{\frac{2}{3}} a + b\Xi_3\right), \quad d = a^2. \tag{4.18}$$

If $b \neq 0$, by substituting the values for c and d given in (4.18), the equation (4.16) is written as follows $0 = 3 \cdot 2^{\frac{2}{3}} b(3 \cdot 2^{\frac{2}{3}} ab^2 + b^3 \Xi_3 - a^3)$, which leads us to a contradiction as $3 \cdot 2^{\frac{2}{3}} ab^2 + b^3 \Xi_3 - a^3 \neq 0$. Hence, $b = 0$ and the system of equations (4.12)–(4.17) is readily seen to imply $0 = c$, $d = 1$, $\Xi_3(a - d^2) = 0$ and $a^2 = 1$.

If $t \in X'^1$, then $\Xi_3 \neq 0$, and from these equations we obtain $a = 1$.

If $t \in (X^1 \backslash X'^1)$, then $\Xi_3 = 0$ and hence $a = \pm 1$.

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In the case iii) ($t \in X^2$), we obtain the following equations for the isotropy group of τ_t:

$$0 = \Xi_4(ad - bc) - a^2 d\Xi_4 - 2abd + a^2 c - b^2 c, \tag{4.19}$$

$$0 = ad - bc - acd\Xi_4 - 2bcd - ad^2 + ac^2, \tag{4.20}$$

$$0 = c(cd\Xi_4 + 3d^2 - c^2), \tag{4.21}$$

$$0 = ad - bc + a(ab\Xi_4 + 3b^2 - a^2), \tag{4.22}$$

$$0 = abc\Xi_4 + b^2 c + 2abd - a^2 c, \tag{4.23}$$

$$0 = -ad + bc + bc^2\Xi_4 + 2bcd - ac^2 + ad^2. \tag{4.24}$$

Summing up (4.19) and (4.23) (resp. (4.20) and (4.24)), it follows $0 = \Xi_4(ad-bc)(1-a)$, (resp. $0 = \Xi_4 c(ad - bc)$). As $\Xi_4 \neq 0$ in this case, from the equations above we obtain $a = 1$ and $c = 0$. The formulas (4.19)–(4.22) thus imply $0 = -2bd$, $0 = d(1-d)$, $0 = d + b\Xi_4 + 3b^2 - 1$. Hence, $d = 1$ and $b = 0$, as A is not singular.

In the case iv) ($t \in C^1$), we obtain the following equations for the isotropy group of τ_t:

$$0 = ad - cb - a^2 d + 4abd + 4abc + 2b^2 c, \tag{4.25}$$

$$0 = -2ad + 2cb + acd + 4bcd + 2ad^2 + 2bc^2, \tag{4.26}$$

$$0 = cd(c + 2d), \tag{4.27}$$

$$0 = -ba(a + 2b), \tag{4.28}$$

$$0 = 2ad - 2bc - abc - 2b^2 c - 4abd - 2a^2 d, \tag{4.29}$$

$$0 = 2ad - 2bc + bc^2 - 4bcd - 4acd - 2ad^2. \tag{4.30}$$

If $a = 0$, then $b \neq 0$ and $c \neq 0$ as the matrix A is not singular. In this case, (4.25) and (4.29) imply $0 = 1 - 2b$ and $0 = 1 + b$, which cannot occur. If $a = -2b$, then the equation (4.29) becomes $0 = 2b(c + 2d)$, which leads us to a contradiction as in this case $\det A = -b(c + 2d)$. Therefore, from the equation (4.28), we deduce $b = 0$ and consequently $a \neq 0$ and $d \neq 0$ as the matrix A is not singular. The formulas (4.25)–(4.30) thus imply $0 = ad(1 - a)$, $0 = -2 + c + 2d$, $0 = c(c+2d)$, $0 = 1 - 2c - d$. From these equations we obtain $a = 1$, $c = 0$, and $d = 1$. ∎

4.2. The matrix C_t computed. The main aim of this subsection is to prove the following

PROPOSITION 4.5. *Let p be the characteristic of $\mathbb{F}$ and let t_{ij}^k be the components of the tensor $t \in O'^2 \cup X'^1 \cup X^2$ as in* (2.3) *in a basis (v_1, v_2) such that $v^1 = w_0$, (v^1, v^2) being the dual basis. If $t \in O'^2$ ($t \in X'^1$ and $t \in X^2$, respectively), then the entries in (v_1, v_2) of the only matrix such that $C_t \cdot \mathrm{symt} = \tau_t$ (see* Proposition 4.1*) belong to the subfield $\mathbb{F}_p\left(t_{ij}^k, \Xi_2\right)_{i,j,k=1,2}$ ($\mathbb{F}_p\left(t_{ij}^k, \Xi_3\right)_{i,j,k=1,2}$ and $\mathbb{F}_p\left(t_{ij}^k, \Xi_4\right)_{i,j,k=1,2}$, respectively).*

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Proof. We set $C_t(v_1) = av_1 + bv_2$ and $C_t(v_2) = cv_1 + dv_2$. If $t \in O'^2$, then the equation $C_t \cdot \mathrm{sym} t = \tau_t$ is equivalent to the following system of equations:

$$0 = \left[-t^1_{11}b + a^2 - b^2\right] c + \left[t^1_{11}a - 2\Xi_1 ab - \Xi_2 b^2\right] d, \tag{4.31}$$

$$\begin{aligned} 0 = {} & \left[2c^2 - 2\Xi_1 d^2 + \left(t^1_{21} + t^1_{12}\right) d\right] a \\ & - \left[2\left(\Xi_1 + 1\right) cd + \left(t^1_{12} + t^1_{21}\right) c + 2\Xi_2 d^2\right] b, \end{aligned} \tag{4.32}$$

$$0 = t^1_{22}\left(ad - cb\right) + c^3 - \left(2\Xi_1 + 1\right) cd^2 - \Xi_2 d^3, \tag{4.33}$$

$$0 = t^2_{11}\left(ad - cb\right) - a^3 + \left(2\Xi_1 + 1\right) ab^2 + \Xi_2 b^3, \tag{4.34}$$

$$\begin{aligned} 0 = {} & \left[2\Xi_1 b^2 - \left(t^2_{21} + t^2_{12}\right) b - 2a^2\right] c \\ & + \left[\left(t^2_{12} + t^2_{21}\right) a + 2\left(\Xi_1 + 1\right) ab + 2\Xi_2 b^2\right] d, \end{aligned} \tag{4.35}$$

$$0 = \left[t^2_{22}d - c^2 + d^2\right] a + \left[-t^2_{22}c + 2\Xi_1 cd + \Xi_2 d^2\right] b. \tag{4.36}$$

The equations (4.31) and (4.35) (resp. (4.32) and (4.36)) constitute a homogeneous linear system for c and d (resp. a and b). Let U (resp. V) be the matrix of coefficients of (4.31) and (4.35) (resp. (4.32) and (4.36)). As $\det C_t = ad - bc \neq 0$, each of these systems admits a non-trivial solution. Hence, taking the equations (4.33) and (4.34) into account, we obtain

$$\begin{aligned} 0 &= \det U = -t^2_{11}(ad - cb)\left(-t^2_{21} - 2t^1_{11} - t^2_{12} + 2\left(\Xi_1 - 1\right) b\right), \\ 0 &= \det V = t^1_{22}(ad - cb)\left(t^1_{12} + t^1_{21} + 2t^2_{22} - 2\left(\Xi_1 - 1\right) d\right). \end{aligned}$$

The general solution to this pair of equations is

$$b = \frac{2t^1_{11} + t^2_{12} + t^2_{21}}{2\left(\Xi_1 - 1\right)}, \quad d = \frac{2t^2_{22} + t^1_{12} + t^1_{21}}{2\left(\Xi_1 - 1\right)}. \tag{4.37}$$

(Recall that $\Xi_1 \neq 1$, as $t \in O'^2$ in the present case.)

Furthermore, the equations (4.35) and (4.36) can be omitted, as they are proportional to (4.31) and (4.32), respectively. By substituting the values for b and d obtained in (4.37) into the system (4.31)–(4.33), we obtain

$$\begin{aligned} 0 = {} & -8\left(\Xi_1 - 1\right)^3 a^2 c + \Xi_2\left(2t^1_{11} + t^2_{21} + t^2_{12}\right)^2\left(2t^2_{22} + t^1_{12} + t^1_{21}\right) \\ & +4\left(\Xi_1 - 1\right)\left(\Xi_1 t^2_{12} + \Xi_1 t^2_{21} + \Xi_1 t^1_{11} + t^1_{11}\right)\left(2t^2_{22} + t^1_{12} + t^1_{21}\right) a \\ & +2\left(\Xi_1 - 1\right)\left(t^2_{21} + 2t^1_{11} + t^2_{12}\right)\left(2\Xi_1 t^1_{11} + t^2_{12} + t^2_{21}\right) c, \end{aligned} \tag{4.38}$$

$$\begin{aligned} 0 = {} & -8\left(\Xi_1 - 1\right)^3 ac^2 + 2\left(\Xi_1 - 1\right)\left(2t^2_{22} + t^1_{12} + t^1_{21}\right)\left(2t^2_{22}\Xi_1 + t^1_{12} + t^1_{21}\right) a \\ & +4\left(\Xi_1 - 1\right)\left(\Xi_1 t^1_{12} + \Xi_1 t^1_{21} + \Xi_1 t^2_{22} + t^2_{22}\right)(2t^1_{11} + t^2_{21} + t^2_{12})c \\ & +\Xi_2(2t^1_{11} + t^2_{21} + t^2_{12})(2t^2_{22} + t^1_{12} + t^1_{21})^2, \end{aligned} \tag{4.39}$$

$$\begin{aligned} 0 = {} & -8(\Xi_1 - 1)^3 c^3 + 2(\Xi_1 - 1)\Big\{\left(2\Xi_1 + 1\right)\left(\left(t^1_{21} + t^1_{12}\right)^2 + 4\left(t^2_{22}\right)^2\right) \\ & +2\left(\Xi_1 - 1\right)\left(2t^1_{11}t^1_{22} + t^1_{22}t^2_{12} + t^1_{22}t^2_{21}\right) + 4\left(2\Xi_1 + 1\right)\left(t^1_{21}t^2_{22} + t^1_{12}t^2_{22}\right)\Big\} c \\ & -4(\Xi_1 - 1)^2 t^1_{22}(2t^2_{22} + t^1_{12} + t^1_{21})a + \Xi_2(2t^2_{22} + t^1_{12} + t^1_{21})^3. \end{aligned} \tag{4.40}$$

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If $c = 0$, then $2t^2_{22} + t^1_{12} + t^1_{21} \neq 0$ by virtue of the formula (4.37) as c and d cannot simultaneously vanish, C_t being non-singular. Hence the equation (4.40) transforms into $0 = 4(\Xi_1-1)^2 t^1_{22} a - \Xi_2(2t^2_{22}+t^1_{12}+t^1_{21})^2$, and $t^1_{22} \neq 0$ in this case, since otherwise we would obtain $2t^2_{22}+t^1_{12}+t^1_{21} = 0$. Recall that $\Xi_2 \neq 0$ as $t \in O'^2$. From the previous equation and (4.37) we then obtain $a = \Xi_2(t^1_{22})^{-1}d^2$.

If $c \neq 0$, by subtracting a times the equation (4.39) from c times the equation (4.38), we have

$$
\begin{aligned}
(4.41)\qquad 0 = & \left[\left(2t^2_{22}+t^1_{12}+t^1_{21}\right)a - \left(2t^1_{11}+t^2_{12}+t^2_{21}\right)c\right] \\
& \cdot \left[\Xi_2(2t^1_{11}+t^2_{12}+t^2_{21})(2t^2_{22}+t^1_{12}+t^1_{21}) \right. \\
& \left. +2\left(\Xi_1-1\right)\left\{\left(2\Xi_1 t^2_{22}+t^1_{21}+t^1_{12}\right)a + \left(2\Xi_1 t^1_{11}+t^2_{21}+t^2_{12}\right)c\right\}\right].
\end{aligned}
$$

On the right-hand side of (4.41) the first factor cannot vanish as, from (4.37), we obtain $2(\Xi_1-1)\det C_t = \left(2t^2_{22}+t^1_{12}+t^1_{21}\right)a - \left(2t^1_{11}+t^2_{12}+t^2_{21}\right)c$. Hence

$$
(4.42)\ a = -\frac{\frac{\Xi_2}{2(\Xi_1-1)}(2t^1_{11}+t^2_{21}+t^2_{12})(2t^2_{22}+t^1_{21}+t^1_{12}) + (2t^1_{11}\Xi_1+t^2_{21}+t^2_{12})c}{2t^2_{22}\Xi_1+t^1_{21}+t^1_{12}}.
$$

Substituting (4.42) into (4.39) and into (4.40) we respectively obtain the following: $0 = cp(c)$ with $p(c) = p_2c^2+p_1c+p_0$ and $0 = q(c) = q_3c^3+q_1c+q_0$, where

$$
\begin{aligned}
p_2 &= 4\left(\Xi_1-1\right)^2\left(2\Xi_1 t^1_{11}+t^2_{21}+t^2_{12}\right),\\
p_1 &= 2\Xi_2\left(\Xi_1-1\right)\left(2t^1_{11}+t^2_{12}+t^2_{21}\right)\left(2t^2_{22}+t^1_{12}+t^1_{21}\right),\\
p_0 &= -2(2(\Xi_1-1)t^2_{22}+2t^2_{22}+t^1_{12}+t^1_{21})\left[4t^1_{11}t^2_{22}-(t^1_{12}+t^1_{21})(t^2_{21}+t^2_{12})\right.\\
&\quad \left. +(t^1_{12}+t^1_{21})(2t^1_{11}+t^2_{21}+t^2_{12})+(t^2_{21}+t^2_{12})(t^1_{12}+2t^2_{22}+t^1_{21})\Xi_1\right],
\end{aligned}
$$

$$
\begin{aligned}
q_3 &= 8\left(\Xi_1-1\right)^3\left(2t^2_{22}+t^1_{12}+t^1_{21}+2\left(\Xi_1-1\right)t^2_{22}\right),\\
q_1 &= -2(\Xi_1-1)\left[4\left(\Xi_1-1\right)\left(\Xi_1+1\right)t^1_{11}\left(t^1_{12}+t^1_{21}\right)t^1_{22}\right.\\
&\quad +16\Xi_1(\Xi_1-1)t^1_{11}t^1_{22}t^2_{22}+4(\Xi_1+2)(2\Xi_1+1)t^1_{12}t^1_{21}t^2_{22}\\
&\quad +4(\Xi_1-1)\left(t^1_{12}+t^1_{21}\right)t^1_{22}t^2_{12}+4(\Xi_1-1)\left(t^1_{12}+t^1_{21}\right)t^1_{22}t^2_{21}\\
&\quad +4(2\Xi_1+1)^2\left(t^1_{12}+t^1_{21}\right)(t^2_{22})^2+(2\Xi_1+1)\left((t^1_{12})^3+(t^1_{21})^3\right)\\
&\quad +2(\Xi_1+2)(2\Xi_1+1)\left((t^1_{12})^2+(t^1_{21})^2\right)t^2_{22}\\
&\quad +4\left(\Xi_1-1\right)\left(\Xi_1+1\right)t^1_{22}\left(t^2_{12}+t^2_{21}\right)t^2_{22}\\
&\quad \left. +3\left(2\Xi_1+1\right)\left(t^1_{12}+t^1_{21}\right)t^1_{12}t^1_{21}+8\Xi_1(2\Xi_1+1)(t^2_{22})^3\right],\\
q_0 &= -\Xi_2\left(t^1_{12}+t^1_{21}+2t^2_{22}\right)^2\left[(2\Xi_1 t^2_{22}+t^1_{12}+t^1_{21})(t^1_{12}+t^1_{21}+2t^2_{22})\right.\\
&\quad \left. +2(\Xi_1-1)t^1_{22}(2t^1_{11}+t^2_{21}+t^2_{12})\right].
\end{aligned}
$$

The remainder of $q(c)$ divided by $p(c)$ must vanish. Hence we finally obtain

$$
c = \frac{p_0p_1q_3+(p_2)^2q_0}{p_0p_2q_3-(p_1)^2q_3-(p_2)^2q_1}.
$$

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The remaining cases, i.e., when $t \in X'^1$ or $t \in X^2$, are dealt with similarly. ∎

COROLLARY 4.6. *With the same notations and assumptions as in* Proposition 4.5*, if* $C_t(v_1) = av_1 + bv_2$*,* $C_t(v_2) = cv_1 + dv_2$*, and all the components* t^k_{ij} *of* t *belong to a subfield* $F \subseteq \mathbb{F}$*, then* $\dim_F F(a, b, c, d) \leq 2$.

Proof. This follows directly from Proposition 4.5 by taking the formula for Ξ_2 in Proposition 4.1 into account. ∎

REMARK 4.7. For every $t \in O^2$, by (4.2), (4.1) and the definition of the invariants $\mathcal{I}_i(t)$, $i = 1, 2$ in the formulas (3.6), we have

$$\begin{aligned}(\Xi_2)^2 \left(\mathcal{I}_1(t) + 2\mathcal{I}_2(t)\right)^2 &= \frac{4 \left(\mathcal{I}_1(t)\right)^3 - 27 \left(\mathcal{I}_2(t)\right)^2}{\mathcal{I}_1(t) + 2\mathcal{I}_2(t)} \\ &= \frac{4 \left(H_1(\mathrm{sym}t)\right)^3 - 27 \left(H_2(\mathrm{sym}t)\right)^2 \det Q_{\mathrm{sym}\bar{t}}}{\left(\det Q_{\mathrm{sym}\bar{t}}\right)^2 \left(H_1(\mathrm{sym}t) + 2H_2(\mathrm{sym}t)\right)}.\end{aligned}$$

Hence,

$$\left\{\det Q_{\mathrm{sym}\bar{t}} \Xi_2(\mathcal{I}_1(t) + 2\mathcal{I}_2(t))\right\}^2 = \frac{4(H_1(\mathrm{sym}t))^3 - 27(H_2(\mathrm{sym}t))^2 \det Q_{\mathrm{sym}\bar{t}}}{H_1(\mathrm{sym}t) + 2H_2(\mathrm{sym}t)}.$$

Taking the formulas (3.4), (3.5) for sym$t$ and (3.2) for $\bar{t}$ into account, a simple computation shows the following:

$$4H_1(\mathrm{sym}t)^3 - 27 \det Q_{\mathrm{sym}\bar{t}} H_2(\mathrm{sym}t)^2 = P(t)^2, \tag{4.43}$$

where $P(t)$ is a polynomial in the components t^i_{jk} of the tensor t. Therefore, Ξ_2 belongs to $\mathbb{F}_p\left(t^k_{ij}\right)_{i,j,k=1,2}$ if and only if $H_1(\mathrm{sym}t) + 2H_2(\mathrm{sym}t)$ is a square. From the formulas (3.4) and (3.5) for sym$t$ we obtain

$$\begin{aligned}&2^4 \cdot 3 \left\{H_1(\mathrm{sym}t) + 2H_2(\mathrm{sym}t)\right\} = \\ &\left(2t^1_{11} + t^2_{12} + t^2_{21}\right)^2 (t^1_{12} + t^1_{21} - t^2_{22})^2 + \left(t^1_{12} + t^1_{21} + 2t^2_{22}\right)^2 (t^2_{12} + t^2_{21} - t^1_{11})^2 \\ &+ \left(2t^1_{11} + t^2_{12} + t^2_{21}\right) \left(t^1_{12} + t^1_{21} + 2t^2_{22}\right) \left(t^2_{12} + t^2_{21} - t^1_{11}\right)(t^1_{12} + t^1_{21} - t^2_{22}) \\ &-9t^1_{11}t^1_{22} \left(2t^1_{11} + t^2_{12} + t^2_{21}\right)^2 - 9t^2_{11}t^2_{22} \left(t^1_{12} + t^1_{21} + 2t^2_{22}\right)^2 \\ &-9t^2_{11}t^1_{22} \left(2t^1_{11} + t^2_{12} + t^2_{21}\right) \left(t^1_{12} + t^1_{21} + 2t^2_{22}\right) \\ &+3 \left(2t^1_{11} + t^2_{12} + t^2_{21}\right) \left(t^1_{12} + t^1_{21} + 2t^2_{22}\right) \left[t^1_{11}(t^1_{12} + t^1_{21}) + t^2_{22}(t^2_{12} + t^2_{21})\right. \\ &\left.-4t^2_{22}t^1_{11} + 2 \left(t^1_{12} + t^1_{21}\right) \left(t^2_{12} + t^2_{21}\right)\right].\end{aligned}$$

For every $t \in X^1$, we have $H_1(\mathrm{sym}t) + 2H_2(\mathrm{sym}t) = 0$, as $\mathcal{I}_1(t) + 2\mathcal{I}_2(t) = 0$. From the formula (4.43) and recalling that $H_2(\mathrm{sym}t) = -\frac{1}{2}H_1(\mathrm{sym}t)$, we obtain

$$P(t)^2 = \frac{1}{4}H_1(\mathrm{sym}t)^2 \left(16H_1(\mathrm{sym}t) - 27 \det Q_{\mathrm{sym}\bar{t}}\right).$$

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Hence, taking (4.3), (4.1), and the definition of the invariant $\mathcal{I}_1$ into account, we have

$$(\Xi_3)^2 = 16 - \frac{27}{\mathcal{I}_1(t)} = \frac{4P(t)^2}{H_1(\mathrm{sym}t)^2}\frac{1}{H_1(\mathrm{sym}t)}.$$

Therefore, Ξ_3 belongs to $\mathbb{F}_p\left(t_{ij}^k\right)_{i,j,k=1,2}$ if and only if $H_1(\mathrm{sym}t)$ is a square. Finally, for every $t \in X^2$ we have $H_2(\mathrm{sym}t) = 0$, as $\mathcal{I}_2(t) = 0$. From (4.43) and taking the formula $H_2(\mathrm{sym}t) = 0$ into account, we obtain $4(H_1(\mathrm{sym}t))^3 = P(t)^2$. Hence, from this equation, taking the formulas (4.4), (4.1), and the definition of the invariant $\mathcal{I}_1$ into account, we have

$$(\Xi_4)^2 = \frac{12\mathcal{I}_1(t)}{9 - \mathcal{I}_1(t)} = \frac{3}{9\det Q_{\mathrm{sym}\bar{t}} - H_1(\mathrm{sym}t)}\frac{P(t)^2}{H_1(\mathrm{sym}t)^2}.$$

Therefore, $\Xi_4 \in \mathbb{F}_p(t_{ij}^k)_{i,j,k=1,2}$ if and only if $9\det Q_{\mathrm{sym}\bar{t}} - H_1(\mathrm{sym}t)$ is a square.

5. The invariants $\mathcal{F}_1$ and $\mathcal{F}_2$.

The invariants $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined in the following proposition.

Proposition 5.1. *Let (v_1, v_2) be a basis for V such that $v^1 = w_0$. The functions*

$$\mathcal{F}_i\colon O \to \mathbb{F},\ \ \mathcal{F}_i(t) = v^i\left(\left(C_t \cdot t\right)(v_1, v_2)\right), \quad i = 1, 2, \tag{5.1}$$

where O is the Zariski-open subset defined in Remark 4.4 *and $C_t \in GL(V)$ is the only matrix satisfying $C_t \cdot \mathrm{sym}t = \tau_t$ according to* Proposition 4.1, *are $GL(V)$-invariant and do not depend on the basis chosen.*

Proof. From the results in [2] we have that the Zariski-open subset R defined in the formula (3.3) (and hence O) is $GL(V)$-invariant, i.e., $A \cdot R = R$ for each $A \in GL(V)$. As $\mathcal{I}_i(t) = \mathcal{I}_i(A \cdot t)$ $(i = 1, 2)$ for every $A \in GL(V)$ by the definition of τ_t, it follows $\tau_{A\cdot t} = \tau_t$. Consequently,

$$\begin{aligned} C_{A\cdot t} \cdot \mathrm{sym}(A \cdot t) &= \tau_{A\cdot t} \\ &= \tau_t \\ &= C_t \cdot \mathrm{sym}t, \\ C_{A\cdot t} \cdot (A \cdot \mathrm{sym}t) &= (C_{A\cdot t}A) \cdot \mathrm{sym}t, \end{aligned}$$

and from the uniqueness of the matrix C_t we deduce $C_{A\cdot t} = C_t A^{-1}$. Hence

$$\begin{aligned} \mathcal{F}_i\left(A \cdot t\right) &= v^i\left(\left(C_{A\cdot t} \cdot (A \cdot t)\right)(v_1, v_2)\right) \\ &= v^i\left(\left(\left(C_t A^{-1}\right) \cdot (A \cdot t)\right)(v_1, v_2)\right) \\ &= v^i\left(\left(C_t \cdot t\right)(v_1, v_2)\right) = \mathcal{F}_i(t). \end{aligned}$$

Moreover, if $(\bar{v}_1, \bar{v}_2)$ is another basis such that $\bar{v}^1 = w_0$, and $A \in GL(V, w_0)$ is the automorphism defined by $A(v_1) = \bar{v}_1$, $A(v_2) = \bar{v}_2$, then, with the obvious notations

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for the new basis $(\bar{v}_1, \bar{v}_2)$, we have $\bar{\tau}_t = A \cdot \tau_t$. Hence $\left(A^{-1}\bar{C}_t\right) \cdot \mathrm{sym}t = A^{-1} \cdot \bar{\tau}_t = \tau_t$. Accordingly, $\bar{C}_t = AC_t$ and hence, for $i = 1, 2$,

$$\begin{aligned}\bar{\mathcal{F}}_i(t) &= \bar{v}^i\left(\left(\bar{C}_t \cdot t\right)(\bar{v}_1, \bar{v}_2)\right)\\ &= \left(v^i \circ A^{-1}\right)\{(A \cdot (C_t \cdot t))(A(v_1), A(v_2))\}\\ &= \left(v^i \circ A^{-1}\right)\{A((C_t \cdot t)(v_1, v_2))\}\\ &= \mathcal{F}_i(t). \ \square\end{aligned}$$

6. The equivalence problem. This equivalence problem is solved in the following statement.

THEOREM 6.1. *The four $GL(V)$-invariant functions $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{F}_1$ and $\mathcal{F}_2$ defined in the formulas* (3.6) *and in* Proposition 5.1 *solve generically the equivalence problem in dimension* 2 *on the Zariski-open subset O defined in* Remark 4.4*, namely, if two tensors $t, t' \in O$ satisfy $\mathcal{I}_i(t) = \mathcal{I}_i(t')$ and $\mathcal{F}_i(t) = \mathcal{F}_i(t')$ for $i = 1, 2$, then t and t' are $GL(V)$-equivalent.*

Proof. As $\mathcal{I}_i(t) = \mathcal{I}_i(t')$ for $i = 1, 2$, we have $\tau_t = \tau_{t'}$, where τ_t is given by the formula in the item i) (resp. ii), resp. iii), resp. iv)) of Proposition 4.1 if $t, t' \in O'^2$ (resp. $t, t' \in X'^1$, resp. $t, t' \in X^2$, resp. $t, t' \in C^1$). We set $\tilde{t} = C_t \cdot t$ (resp. $\tilde{t}' = C_{t'} \cdot t'$), where C_t (resp. $C_{t'}$) is the only matrix such that $C_t \cdot \mathrm{sym}t = \tau_t$ (resp. $C_{t'} \cdot \mathrm{sym}t' = \tau_t$) according to Proposition 4.1. Hence $\mathrm{sym}\tilde{t} = \mathrm{sym}\tilde{t}' = \tau_t$ as $\mathrm{sym}\tilde{t} = \mathrm{sym}(C_t \cdot t) = C_t \cdot \mathrm{sym}t$, and similarly for t'. The difference $\tilde{t}' - \tilde{t}$ is thus alternate. By using the notations in the formulas (4.2) (resp. (4.3), resp. (4.4)), we obtain

$$\tilde{t}^1_{11} = 0,\ \tilde{t}^1_{21} = 2\Xi_1 - \tilde{t}^1_{12},\ \tilde{t}^1_{22} = \Xi_2,\ \tilde{t}^2_{11} = 1,\ \tilde{t}^2_{21} = -\tilde{t}^2_{12},\ \tilde{t}^2_{22} = -1, \tag{6.1}$$

$$(\text{resp. } \tilde{t}^1_{11} = 0,\ \tilde{t}^1_{21} = 3 \cdot 4^{\frac{1}{3}} - \tilde{t}^1_{12},\ \tilde{t}^1_{22} = \Xi_3,\ \tilde{t}^2_{11} = 1,\ \tilde{t}^2_{21} = -\tilde{t}^2_{12},\ \tilde{t}^2_{22} = 0, \tag{6.2}$$

$$\text{resp. } \tilde{t}^1_{11} = \Xi_4,\ \tilde{t}^1_{21} = 2 - \tilde{t}^1_{12},\ \tilde{t}^1_{22} = 0,\ \tilde{t}^2_{11} = 1,\ \tilde{t}^2_{21} = -\tilde{t}^2_{12},\ \tilde{t}^2_{22} = -1, \tag{6.3}$$

$$\text{resp. } \tilde{t}^1_{11} = -\frac{1}{2},\ \tilde{t}^1_{21} = 2 - \tilde{t}^1_{12},\ \tilde{t}^1_{22} = 0,\ \tilde{t}^2_{11} = 0,\ \tilde{t}^2_{21} = -2 - \tilde{t}^2_{12},\ \tilde{t}^2_{22} = -1), \tag{6.4}$$

and similarly for $\tilde{t}'$. Furthermore, from the definition of $\mathcal{F}_i$ in the formula (5.1), it follows $\mathcal{F}_i(t) = \tilde{t}^i_{12}$ and $\mathcal{F}_i(t') = \tilde{t}'^i_{12}$ for $i = 1, 2$. As $\mathcal{F}_i(t) = \mathcal{F}_i(t')$ $(i = 1, 2)$, by the hypothesis, we conclude $\tilde{t}^i_{12} = \tilde{t}'^i_{12}$, $i = 1, 2$. The formulas (6.1)–(6.4) and the corresponding ones for t' then prove that $\tilde{t} = \tilde{t}'$, i.e., $C_t \cdot t = C_{t'} \cdot t'$. Hence, $t' = ((C_{t'})^{-1}C_t) \cdot t$, thus concluding the proof of the theorem. $\square$

REMARK 6.2. The invariants $\mathcal{F}_1$ and $\mathcal{F}_2$ are unsatisfactory from the computational point of view, because they are defined in terms of the matrix C_t. This matrix requires a rather big number of operations in the ground field to be computed, as the formulas in the proof of Proposition 4.5 show. In the next section, two new invariants are introduced, which enjoy the double advantage of being easily computable and having a simple intrinsic meaning.

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PROPOSITION 6.3 (Normal forms). *Let $w_0 \in V^*\backslash\{0\}$ be a linear form and let (v_1, v_2) be a basis for V such that its dual basis (v^1, v^2) satisfies $v^1 = w_0$. We set:*

- i) *If $t \in O'^2$, then $\alpha_t = v^1 \wedge v^2 \otimes ((\mathcal{F}_1(t) - \Xi_1)v_1 + \mathcal{F}_2(t)v_2)$, where Ξ_1 is defined as in the formula (4.2).*
- ii) *If $t \in X'^1$, then $\alpha_t = v^1 \wedge v^2 \otimes ((\mathcal{F}_1(t) - 3 \cdot 2^{-\frac{1}{3}})v_1 + \mathcal{F}_2(t)v_2)$.*
- iii) *If $t \in X^2$, then $\alpha_t = v^1 \wedge v^2 \otimes ((\mathcal{F}_1(t) - 1)v_1 + \mathcal{F}_2(t)v_2)$.*
- iv) *If $t \in C^1$, then $\alpha_t = v^1 \wedge v^2 \otimes ((\mathcal{F}_1(t) - 1) + (\mathcal{F}_2(t) + 1)v_2)$.*

In the formulas above, $\mathcal{F}_1, \mathcal{F}_2$ denote the functions defined in the formula (5.1). *With such notations, for the matrix $C_t \in GL(V)$ in* Proposition 4.1, *$C_t \cdot t = \tau_t + \alpha_t$ for each $t \in O$.*

Proof. If $t \in O'^2$, then from the formulas (6.1) and the definition of $\mathcal{F}_1, \mathcal{F}_2$ we obtain

$$\begin{aligned} C_t \cdot \mathrm{alt}t = \mathrm{alt}(C_t \cdot t) = \mathrm{alt}\tilde{t} &= \frac{1}{2} v^1 \wedge v^2 \otimes \left((\tilde{t}^1_{12} - \tilde{t}^1_{21})v_1 + (\tilde{t}^2_{12} - \tilde{t}^2_{21})v_2\right) \\ &= v^1 \wedge v^2 \otimes \left(\left(\tilde{t}^1_{12} - \Xi_1\right) v_1 + \tilde{t}^2_{12} v_2\right) = \alpha_t. \end{aligned}$$

For the other cases, their proofs are similar. ∎

7. The invariants $\mathcal{I}_3$ and $\mathcal{I}_4$.

We first state two auxiliary lemmas.

LEMMA 7.1. *The set of tensors $\bar{t} \in \ker \mathrm{tr}$, where the metric $g_{\bar{t}} \in S^2V^*$ defined by $g_{\bar{t}} = w_0 \circ \mathrm{sym}\bar{t}$ is not degenerate, is a non-empty Zariski-open subset O_{w_0}.*

Proof. Let (v_1, v_2) be basis for V with dual basis (v^1, v^2), $w_0 = v^1$, and let $\bar{t}^k_{ij}$ be the components of $\bar{t}$ in such a basis. As $\mathrm{tr}\bar{t} = (\bar{t}^1_{11} + \bar{t}^2_{12})v^1 + (\bar{t}^1_{21} + \bar{t}^2_{22})v^2 = 0$, we obtain $\bar{t}^2_{12} = -\bar{t}^1_{11}$ and $\bar{t}^1_{21} = -\bar{t}^2_{22}$. From the definition of $g_{\bar{t}}$ in the statement, we have $g_{\bar{t}}(v_1, v_1) = \bar{t}^1_{11}$, $g_{\bar{t}}(v_1, v_2) = g_{\bar{t}}(v_2, v_1) = \frac{1}{2}(\bar{t}^1_{12} - \bar{t}^2_{22})$ and $g_{\bar{t}}(v_2, v_2) = \bar{t}^1_{22}$. Hence, $\det(g_{\bar{t}}(v_i, v_j))^2_{i,j=1} = \bar{t}^1_{11}\bar{t}^1_{22} - \frac{1}{4}(\bar{t}^1_{12} - \bar{t}^2_{22})^2$ does not vanish identically. ∎

LEMMA 7.2. *For every $\bar{t} \in \ker \mathrm{tr}$, let $h_{\bar{t}} \in \wedge^2 V^*$ be the alternating metric defined by $h_{\bar{t}} = w_0 \circ \mathrm{alt}\bar{t}$. If $\bar{t} \in O_{w_0}$, then a unique linear mapping $L_{\bar{t}}: V \to V$ exists such that*

$$g_{\bar{t}}(x, L_{\bar{t}}y) = h_{\bar{t}}(x, y), \quad \forall x, y \in V, \tag{7.1}$$

and the following formulas hold:

$$g_{A\cdot\bar{t}} = A \cdot g_{\bar{t}}, \quad h_{A\cdot\bar{t}} = A \cdot h_{\bar{t}}, \quad \forall A \in GL(V, w_0), \tag{7.2}$$

$$L_{A\cdot\bar{t}} = A \circ L_{\bar{t}} \circ A^{-1}, \quad \forall A \in GL(V, w_0). \tag{7.3}$$

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Proof. The existence and uniqueness of $L_{\bar{t}}$ follows directly from the fact of $g_{\bar{t}}$ being non-degenerate. Moreover, from the very definition of $GL(V, w_0)$, we have $w_0 \circ A = w_0$ for every matrix $A \in GL(V, w_0)$. Hence

$$\begin{aligned} g_{A\cdot\bar{t}}(x,y) &= \frac{1}{2} w_0 \left\{ \left(A \cdot \bar{t}\right)(x,y) + \left(A \cdot \bar{t}\right)(y,x) \right\} \\ &= \frac{1}{2} w_0 \left\{ A\left[\bar{t}(A^{-1}x, A^{-1}y)\right] + A\left[\bar{t}(A^{-1}y, A^{-1}x)\right] \right\} \\ &= \frac{1}{2} \left(w_0 \circ A\right) \left\{ \bar{t}(A^{-1}x, A^{-1}y) + \bar{t}(A^{-1}y, A^{-1}x) \right\} \\ &= \frac{1}{2} w_0 \left\{ \bar{t}(A^{-1}x, A^{-1}y) + \bar{t}(A^{-1}y, A^{-1}x) \right\} \\ &= g_{\bar{t}}\left(A^{-1}x, A^{-1}y\right) = \left(A \cdot g_{\bar{t}}\right)(x,y) \end{aligned}$$

for each $A \in GL(V, w_0)$. Similarly, the case for $h_{\bar{t}}$ can be proved. This proves the formula (7.2).

From the formula (7.1) we obtain $g_{A\cdot\bar{t}}(x, L_{A\cdot\bar{t}}y) = h_{A\cdot\bar{t}}(x,y)\ \forall x, y \in V$, and by applying (7.2) we obtain $(A \cdot g_{\bar{t}})(x, L_{A\cdot\bar{t}}y) = (A \cdot h_{\bar{t}})(x,y)$. By expanding this,

$$\begin{aligned} g_{\bar{t}}(A^{-1}x, A^{-1}(L_{A\cdot\bar{t}}y)) &= h_{\bar{t}}(A^{-1}x, A^{-1}y) \\ &= g_{\bar{t}}(A^{-1}x, L_{\bar{t}}(A^{-1}y)). \end{aligned}$$

As $g_{\bar{t}}$ is non-degenerate, the previous equation implies $A^{-1} \circ L_{A\cdot\bar{t}} = L_{\bar{t}} \circ A^{-1}$, thus proving (7.3). □

Notations. If $\bar{t} \in O_{w_0}$, then the linear map $g_{\bar{t}}^{\flat} \colon V \to V^*$, defined by $g_{\bar{t}}^{\flat}(x)(y) = g_{\bar{t}}(x,y)$ for all $x, y \in V$, is an isomorphism; its inverse map is denoted by $g_{\bar{t}}^{\sharp} \colon V^* \to V$ and the contravariant metric induced by $g_{\bar{t}}$ is denoted by ${}^{\sharp}g_{\bar{t}}$; i.e., ${}^{\sharp}g_{\bar{t}} = S^2(g_{\bar{t}}^{\sharp})(g_{\bar{t}})$, $S^2(g_{\bar{t}}^{\sharp}) \colon S^2(V^*) \to S^2(V)$ being the extension of $g_{\bar{t}}^{\sharp}$ to the 2nd symmetric power.

Proposition 7.3. *The functions*

$$I_3 \colon O_{w_0} \to \mathbb{F}, \quad I_3(\bar{t}) = \det(L_{\bar{t}}), \tag{7.4}$$

$$I_4 \colon O_{w_0} \to \mathbb{F}, \quad I_4(\bar{t}) = \frac{1}{4}{}^{\sharp}g_{\bar{t}}(w_0, w_0), \tag{7.5}$$

are $GL(V, w_0)$-invariant. If

$$\begin{aligned} \bar{t} = \bar{t}^1_{11}\left(v^1 \otimes v^1 \otimes v_1 - v^1 \otimes v^2 \otimes v_2\right) + \bar{t}^1_{12} v^1 \otimes v^2 \otimes v_1 + \bar{t}^1_{22} v^2 \otimes v^2 \otimes v_1 \\ + \bar{t}^2_{11} v^1 \otimes v^1 \otimes v_2 + \bar{t}^2_{21} v^2 \otimes v^1 \otimes v_2 + \bar{t}^2_{22}\left(v^2 \otimes v^2 \otimes v_2 - v^2 \otimes v^1 \otimes v_1\right) \end{aligned}$$

is any tensor in O_{w_0}, where (v_1, v_2) is a basis for V such that $v^1 = w_0$, then

$$I_3(\bar{t}) = \frac{\left(\bar{t}^1_{12} + \bar{t}^2_{22}\right)^2}{4\bar{t}^1_{11}\bar{t}^1_{22} - \left(\bar{t}^1_{12} - \bar{t}^2_{22}\right)^2}, \tag{7.6}$$

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$$I_4(\bar{t}) = \frac{\bar{t}^1_{22}}{4\bar{t}^1_{11}\bar{t}^1_{22} - \left(\bar{t}^1_{12} - \bar{t}^2_{22}\right)^2}. \tag{7.7}$$

Proof. The invariance of I_3 is an immediate consequence of the formula (7.3). As for I_4, we claim the following equation holds: $g^\flat_{A\cdot\bar{t}} = (A^*)^{-1} \circ g^\flat_{\bar{t}} \circ A^{-1}$ $\forall A \in GL(V)$ and $\forall \bar{t} \in O_{w_0}$. In fact, from (7.2) and the definition of $g^\flat_{\bar{t}}$, we obtain

$$\begin{aligned} g^\flat_{A\cdot\bar{t}}(x)(y) &= (A \cdot g_{\bar{t}})(x, y) \\ &= g_{\bar{t}}(A^{-1}x, A^{-1}y) \\ &= g^\flat_{\bar{t}}(A^{-1}x)(A^{-1}y) \\ &= (g^\flat_{\bar{t}}(A^{-1}x) \circ A^{-1})(y) \end{aligned}$$

for all $x, y \in V$. Therefore,

$$\begin{aligned} g^\flat_{A\cdot\bar{t}}(x) &= g^\flat_{\bar{t}}(A^{-1}x) \circ A^{-1} \\ &= (g^\flat_{\bar{t}} \circ A^{-1})(x) \circ A^{-1} \\ &= (A^*)^{-1}((g^\flat_{\bar{t}} \circ A^{-1})(x)) \\ &= ((A^*)^{-1} \circ g^\flat_{\bar{t}} \circ A^{-1})(x). \end{aligned}$$

Moreover, ${}^\sharp g_{\bar{t}}$ can be computed by the formula ${}^\sharp g_{\bar{t}}(w_1, w_2) = g_{\bar{t}}(g^\sharp_{\bar{t}}(w_1), g^\sharp_{\bar{t}}(w_2))$ for every $w_1, w_2 \in V^*$. Hence, for every $A \in GL(V, w_0)$ we obtain

$$\begin{aligned} I_4(A \cdot \bar{t}) &= {}^\sharp g_{A\cdot\bar{t}}(w_0, w_0) = g_{A\cdot\bar{t}}\left(g^\sharp_{A\cdot\bar{t}}(w_0), g^\sharp_{A\cdot\bar{t}}(w_0)\right) \\ &= (A \cdot g_{\bar{t}})\left(\left(A \circ g^\sharp_{\bar{t}} \circ A^*\right)(w_0), \left(A \circ g^\sharp_{\bar{t}} \circ A^*\right)(w_0)\right) \\ &= g_{\bar{t}}\left(\left(g^\sharp_{\bar{t}} \circ A^*\right)(w_0), \left(g^\sharp_{\bar{t}} \circ A^*\right)(w_0)\right) = g_{\bar{t}}\left(g^\sharp_{\bar{t}}(w_0), g^\sharp_{\bar{t}}(w_0)\right) \\ &= I_4(\bar{t}). \end{aligned}$$

From the definitions of $g_{\bar{t}}$ and $h_{\bar{t}}$ in Lemmas 7.1 and 7.2, respectively, we obtain

$$\begin{aligned} g_{\bar{t}} &= \bar{t}^1_{11} v^1 \otimes v^1 + \frac{1}{2}\left(\bar{t}^1_{12} - \bar{t}^2_{22}\right)\left(v^1 \otimes v^2 + v^2 \otimes v^1\right) + \bar{t}^1_{22} v^2 \otimes v^2, \\ h_{\bar{t}} &= \frac{1}{2}\left(\bar{t}^1_{12} + \bar{t}^2_{22}\right)\left(v^1 \otimes v^2 - v^2 \otimes v^1\right). \end{aligned}$$

The formulas (7.6) and (7.7) now follow by means of a simple calculation from the definitions of I_3 and I_4 in (7.4) and (7.5), respectively. $\square$

Proposition 7.4. *Let O'_{w_0} be the non-empty Zariski-open subset of all the tensors $t \in O^1$ such that the metric $g_{A^{-1}\cdot(t-\sigma(\mathrm{trt}))} = w_0 \circ \mathrm{sym}(A^{-1} \cdot (t - \sigma(\mathrm{trt})))$, where $A \in GL(V)$ satisfies $A \cdot w_0 = \mathrm{trt}$, is not degenerate. Let $\mathcal{I}_i \colon O'_{w_0} \to \mathbb{F}$, $i = 3, 4$, be the*

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functions defined by $\mathcal{I}_i = \phi^{-1}(I_i)$*, where* I_i *are the functions in the formulas* (7.4), (7.5)*, and* ϕ *is the isomorphism in* (2.2)*. Then*

$$\mathcal{I}_3(t) = \frac{N_3(t)}{D(t)}, \quad \mathcal{I}_4(t) = \frac{N_4(t)}{D(t)}, \tag{7.8}$$

where (v_1, v_2) *is a basis such that* $w_0 = v^1$*, the expression for* t *is as in* (2.3)*, and*

$$\begin{aligned} D(t) = {} & \left((t^1_{12} - t^2_{22})(t^1_{11} + t^2_{12}) + (t^2_{21} - t^1_{11})(t^1_{21} + t^2_{22})\right)^2 \\ & -4(t^2_{12}t^1_{21} + t^1_{22}t^2_{11})(t^1_{11} + t^2_{12})(t^1_{21} + t^2_{22}) \\ & +4(t^1_{11} + t^2_{12})^2 t^1_{22}t^2_{12} + 4(t^1_{21} + t^2_{22})^2 t^1_{21}t^2_{11}, \end{aligned} \tag{7.9}$$

$$N_3(t) = -\left((t^1_{12} - t^1_{21})(t^1_{11} + t^2_{12}) + (t^2_{12} - t^2_{21})(t^1_{21} + t^2_{22})\right)^2, \tag{7.10}$$

$$\begin{aligned} N_4(t) = {} & -(t^1_{11} + t^2_{12})^3 t^1_{22} + (t^1_{21} + t^1_{12} - t^2_{22})(t^1_{21} + t^2_{22})(t^1_{11} + t^2_{12})^2 \\ & +(t^2_{12} + t^2_{21} - t^1_{11})(t^1_{21} + t^2_{22})^2(t^1_{11} + t^2_{12}) - (t^1_{21} + t^2_{22})^3 t^2_{11}. \end{aligned} \tag{7.11}$$

Proof. The existence of a matrix $A \in GL(V)$ such that $A \cdot w_0 = \mathrm{tr}t$ follows from the fact that $GL(V)$ acts transitively on $V^*\backslash\{0\}$ and the definition of O'_{w_0} makes sense as it does not depend on the matrix A chosen. In fact, if $A' \in GL(V)$ is another matrix such that $\mathrm{tr}t = A' \cdot w_0$, then $B = A^{-1}A'$ belongs to $GL(V, w_0)$ as $B \cdot w_0 = A^{-1} \cdot (A' \cdot w_0) = A^{-1} \cdot \mathrm{tr}t = w_0$. Hence

$$\begin{aligned} g_{A'^{-1}\cdot(t-\sigma(\mathrm{tr}t))} &= g_{(AB)^{-1}\cdot(t-\sigma(\mathrm{tr}t))} \\ &= g_{(B^{-1}A^{-1})\cdot(t-\sigma(\mathrm{tr}t))} \\ &= g_{B^{-1}\cdot(A^{-1}\cdot(t-\sigma(\mathrm{tr}t)))} \\ &= B^{-1} \cdot g_{A^{-1}\cdot(t-\sigma(\mathrm{tr}t))}, \end{aligned}$$

where the last equality follows from (7.2). Hence, $g_{A'^{-1}\cdot(t-\sigma(\mathrm{tr}t))}$ is non-degenerate if and only if $g_{A^{-1}\cdot(t-\sigma(\mathrm{tr}t))}$ is non-degenerate.

From the proof of Theorem 2.1 we know that the function $\mathcal{I}_i$ is defined by setting $\mathcal{I}_i(t) = I_i(A^{-1} \cdot (t - \sigma(\mathrm{tr}t)))$, $i = 3, 4$, $t \in O^1$, where A is any matrix such that $\mathrm{tr}t = A \cdot w_0$. Letting $T = t - \sigma(\mathrm{tr}t)$, from the expressions for t and $\mathrm{tr}t$ above and the very definition of σ in Theorem 2.1, we obtain

$$\begin{aligned} T = {} & -t^2_{12}v^1 \otimes v^1 \otimes v_1 + t^2_{11}v^1 \otimes v^1 \otimes v_2 + \left(t^1_{12} - t^1_{21} - t^2_{22}\right) v^1 \otimes v^2 \otimes v_1 \\ & +t^2_{12}v^1 \otimes v^2 \otimes v_2 + t^1_{21}v^2 \otimes v^1 \otimes v_1 + \left(t^2_{21} - t^1_{11} - t^2_{12}\right) v^2 \otimes v^1 \otimes v_2 \\ & +t^1_{22}v^2 \otimes v^2 \otimes v_1 - t^1_{21}v^2 \otimes v^2 \otimes v_2. \end{aligned}$$

If $t \in O'_{w_0}$ is as in (2.3), then $\mathrm{tr}t = (t^1_{11} + t^2_{12})v^1 + (t^1_{21} + t^2_{22})v^2 \neq 0$ and we are led to distinguish two cases:

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1. If $t^1_{11}+t^2_{12}\neq 0$, then we can take
$$A=\begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix}=\begin{pmatrix} \frac{1}{t^1_{11}+t^2_{12}} & -\left(t^1_{21}+t^2_{22}\right) \\ 0 & t^1_{11}+t^2_{12} \end{pmatrix}.$$
2. If $t^1_{11}+t^2_{12}=0$ but $t^1_{21}+t^2_{22}\neq 0$, then we can take
$$A=\begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix}=\begin{pmatrix} 0 & -\left(t^1_{21}+t^2_{22}\right) \\ \frac{1}{t^1_{21}+t^2_{22}} & 0 \end{pmatrix}.$$

From the transformation formulas (4.5), (7.6) and (7.7), we obtain the expressions in the statement after a computation. □

Proposition 7.5. *On the non-empty Zariski-open subset*
$$O'=\left(O'_{w_0}\backslash(\mathcal{I}_4)^{-1}(0)\right)\cap\left(O'^2\cup\left(X'^1\backslash(\mathcal{I}_3)^{-1}(0)\right)\cup X^2\right)$$
the invariants $\mathcal{F}_1,\mathcal{F}_2$ can be written as a function of $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$. Hence, on this subset, the invariants $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$ and $\mathcal{I}_4$ also solve the equivalence problem.

Proof. We set $\xi_i=\mathcal{I}_i(t)$ for $i=3,4$. By applying the formulas (7.8)–(7.11) we have

$$\xi_3 D(t)-N_3(t)=0, \tag{7.12}$$
$$\xi_4 D(t)-N_4(t)=0. \tag{7.13}$$

As $\mathcal{F}_i$ is $GL(V)$-invariant, in order to compute $\mathcal{F}_i(t)$, we can assume sym$t=\tau_t$ by simply replacing t by $C_t\cdot t$.

(I) Assume $t\in(O'_{w_0}\backslash(\mathcal{I}_4)^{-1}(0))\cap O'^2$.

From the expression for τ_t in Proposition 4.1-i) we obtain $t^1_{11}=0$, $t^1_{21}=2\Xi_1-t^1_{12}$, $t^1_{22}=\Xi_2$, $t^2_{11}=1$, $t^2_{21}=-t^2_{12}$, $t^2_{22}=-1$, and from the very definition of $\mathcal{F}_i$ in the formula (5.1) we deduce $\mathcal{F}_i(t)=t^i_{12}$ for $i=1,2$. As $t\in O'_{w_0}$, either $t^1_{11}+t^2_{12}=t^2_{12}\neq 0$ or $t^1_{21}+t^2_{22}=2\Xi_1-t^1_{12}-1\neq 0$. Therefore, the case $t^1_{12}=2\Xi_1-1$, $t^2_{12}=0$ is excluded. Furthermore, as $t\in O'^2$, we have $\Xi_1\neq 1$ and $\Xi_2\neq 0$. Next, we are led to consider

1. If $t^2_{12}=0$, then $t^1_{12}\neq 2\Xi_1-1$, $D(t)=4(t^1_{12}-2\Xi_1)(t^1_{12}-2\Xi_1+1)^2\neq 0$, and from (7.10), (7.11) we obtain $\xi_3=0$, $\xi_4=-\frac{t^1_{12}+1-2\Xi_1}{4(t^1_{12}-2\Xi_1)}$. From the previous formula for ξ_4 we conclude $\xi_4\neq -\frac{1}{4}$, and hence, $t^1_{12}=2\Xi_1-(4\xi_4+1)^{-1}$.
2. If $t^2_{12}\neq 0$, then $\xi_3\neq 0$ as $N_3(t)=4(t^2_{12})^2(\Xi_1-1)^2$ in this case. By subtracting ξ_3 times the equation (7.13) from $4\xi_4+1$ times the equation (7.12), the following second-degree equation for t^1_{12} and t^2_{12} is obtained:
$$\begin{aligned}0=&-\xi_3(t^1_{12})^2-\Xi_2\xi_3t^1_{12}t^2_{12}-\left(\xi_3(\Xi_1)^2+(\Xi_1-1)^2(4\xi_4+1)\right)(t^2_{12})^2\\&+2(2\Xi_1-1)\xi_3t^1_{12}+(2\Xi_1-1)\Xi_2\xi_3t^2_{12}-(2\Xi_1-1)^2\xi_3,\end{aligned}\tag{7.14}$$

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and solving (7.14) with respect to t^1_{12}, we obtain

$$t^{1\pm}_{12} = 2\Xi_1 - 1 - \frac{\xi_3\Xi_2 \pm \rho(t)^{\frac{1}{2}}}{2\xi_3} t^2_{12}, \qquad (7.15)$$
$$\rho = \xi_3 \left[\xi_3((\Xi_2)^2 - 4(\Xi_1)^2) - 4(4\xi_4 + 1)(\Xi_1 - 1)^2\right].$$

Substituting the formula (7.15) for $t^{1\pm}_{12}$ into (7.12) we obtain

$$\left(D^0 \pm D^1\rho^{\frac{1}{2}}\right) t^2_{12} = -8\xi_3\xi_4(\Xi_1 - 1)^2, \qquad (7.16)$$
$$D^0 = \Xi_2\xi_3 \left(\xi_3 \left((\Xi_2)^2 - 4(\Xi_1)^2\right) + (\xi_3 - 3(4\xi_4 + 1))(\Xi_1 - 1)^2\right),$$
$$D^1 = \xi_3(\Xi_2)^2 - \xi_3(\Xi_1 + 1)^2 - (4\xi_4 + 1)(\Xi_1 - 1)^2.$$

As $\xi_3\xi_4(\Xi_1 - 1)^2 \neq 0$, from the equation (7.16) we deduce $D^0 \pm D^1\rho^{\frac{1}{2}} \neq 0$. Therefore, $t^{2\pm}_{12} = -8\xi_3\xi_4(\Xi_1 - 1)^2/(D^0 \pm D^1\rho^{\frac{1}{2}})$.

(II) Assume $t \in (O'_{w_0}\backslash(\mathcal{I}_4)^{-1}(0)) \cap (X'^1\backslash(\mathcal{I}_3)^{-1}(0))$. In this case, from the expression for τ_t in Proposition 4.1-ii) we obtain $t^1_{11} = 0$, $t^2_{11} = 1$, $t^2_{21} = -t^2_{12}$, $t^2_{22} = 0$, $t^1_{21} = 3\cdot 2^{-\frac{1}{3}} - t^1_{12}$, $t^1_{22} = \Xi_3$, and as in the previous case we deduce $\mathcal{F}_i(t) = t^i_{12}$, $i = 1, 2$. As $t \in O'_{w_0} \cap X'^1$, either $t^1_{11} + t^2_{12} = t^2_{12} \neq 0$ or $t^1_{21} + t^2_{22} = 3\cdot 2^{-\frac{1}{3}} - t^1_{12} \neq 0$. Therefore, the case $t^2_{12} = 0$, $t^1_{12} = 3\cdot 2^{-\frac{1}{3}}$ is excluded. Next, we consider

1. If $t^2_{12} = 0$, then $\xi_3 = 0$, which cannot occur as $t \in X'^1\backslash(\mathcal{I}_3)^{-1}(0)$.
2. If $t^2_{12} \neq 0$, then $\xi_3 \neq 0$ as $N_3(t) = 9\cdot 2^{\frac{4}{3}}(t^2_{12})^2$ in this case. By substracting $4\xi_3$ times the equation (7.13) from $4\xi_4 + 1$ times the equation (7.12), the following first-degree equation for t^1_{12} and t^2_{12} is obtained:

$$0 = 2^{\frac{2}{3}}\xi_3\Xi_3 t^1_{12} + 9(1 + \xi_3 + 4\xi_4)t^2_{12} - 6\cdot 2^{\frac{1}{3}}\xi_3\Xi_3.$$

Hence, $t^1_{12} = 3\cdot 2^{\frac{2}{3}} - 3^2\cdot 2^{-\frac{2}{3}}(1 + \xi_3 + 4\xi_4)t^2_{12}\xi_3^{-1}\Xi_3^{-1}$. By replacing t^1_{12} by its value in the previous formula into (7.12) we obtain

$$\sigma t^2_{12} = 9\cdot 2^{\frac{10}{3}}(\xi_3)^2\xi_4(\Xi_3)^3, \qquad (7.17)$$

where

$$\sigma = 3^6\left(1 + 3\xi_3 + (\xi_3)^3 + 2^6(\xi_4)^3\right) + 4((\Xi_3)^2 - 3^3)(\xi_3)^3(\Xi_3)^2$$
$$+3^3(3^4 - 2^2(\Xi_3)^2)(1 + 4\xi_4)(\xi_3)^2 + 2^2 3^7\xi_4(1 + \xi_3)(1 + 4\xi_4).$$

As $(\xi_3)^2\xi_4(\Xi_3)^3 \neq 0$, the equation (7.17) implies $\sigma \neq 0$.
Therefore, $t^2_{12} = 3^2\cdot 2^{\frac{10}{3}}(\xi_3)^2\xi_4(\Xi_3)^3/\sigma$.

(III) Assume $t \in (O'_{w_0}\backslash(\mathcal{I}_4)^{-1}(0)) \cap X^2$. In this case, from the expression for τ_t in Proposition 4.1-iii) we obtain $t^1_{11} = \Xi_4$, $t^2_{11} = 1$, $t^2_{21} = -t^2_{12}$, $t^2_{22} = -1$, $t^1_{21} = 2 - t^1_{12}$, $t^1_{22} = 0$, and as in the previous cases we deduce $\mathcal{F}_i(t) = t^i_{12}$, $i = 1, 2$. As $t \in O'_{w_0} \cap X^2$

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either $t^1_{11}+t^2_{12}=\Xi_4+t^2_{12}\neq 0$ or $t^1_{21}+t^2_{22}=1-t^1_{12}\neq 0$. Therefore, the case $t^1_{12}=1$, $t^2_{12}=-\Xi_4$ is excluded. If $t^1_{12}=1$, then $\xi_4=0$, which cannot occur because of the hypothesis. Therefore, $t^1_{12}\neq 1$.

By substracting $4\xi_3$ times the equation (7.13) from $4\xi_4+1$ times the equation (7.12), the following second-degree equation for t^1_{12} and t^2_{12} is obtained:

$$\begin{aligned}(7.18)\qquad 0 &= \xi_3(t^2_{12})^2-\xi_3\Xi_4t^1_{12}t^2_{12}+3\xi_3\Xi_4t^2_{12}+2\xi_3(\Xi_4)^2-\xi_3(\Xi_4)^2t^1_{12}\\ &\quad+\left(4\xi_4(\Xi_4)^2+(\Xi_4)^2+\xi_3\right)(t^1_{12}-1)^2.\end{aligned}$$

As $\xi_3\neq 0$, from the equation (7.18), we obtain $t^{2\pm}_{12}=\frac{\xi_3\Xi_4(t^1_{12}-3)+(t^1_{12}-1)\varsigma^{\frac{1}{2}}}{2\xi_3}$, where $\varsigma=\xi_3\left(\xi_3-16\xi_4-4\right)(\Xi_4)^2-4\left(\xi_3\right)^2$. By replacing the value for $t^{2\pm}_{12}$ above into the equation (7.12) we obtain

$$\begin{aligned}0 = 4(t^1_{12}-1)^2\Big(&\Big(2\xi_3(\Xi_4)^2-3(\Xi_4)^2-4\xi_3+2\Xi_4\varsigma(t)^{\frac{1}{2}}-12\xi_4(\Xi_4)^2\Big)t^1_{12}\\ &-2\xi_3(\Xi_4)^2+3(\Xi_4)^2+4\xi_3-2\Xi_4\varsigma(t)^{\frac{1}{2}}+8\xi_4(\Xi_4)^2\Big).\end{aligned}$$

As $t^1_{12}\neq 1$, from the previous equation and taking the following identity into account:

$$N_4(t)=\frac{(t^1_{12}-1)^3(2\xi_3(\Xi_4)^2-3(\Xi_4)^2-4\xi_3+2\Xi_4\varsigma(t)^{\frac{1}{2}}-12\xi_4(\Xi_4)^2}{\xi_3}\neq 0,$$

we deduce $t^1_{12}=1+\frac{4\xi_4(\Xi_4)^2}{(\Xi_4)^2(-3+2\xi_3-12\xi_4)-4\xi_3+2\Xi_4\varsigma(t)^{\frac{1}{2}}}$. □

In summary,

(I) If $t\in\left(O'_{w_0}\backslash(\mathcal{I}_4)^{-1}(0)\right)\cap O'^2$, then
 (I.1) If $\mathcal{F}_2(t)=0$, then $\mathcal{F}_1(t)=2\frac{\mathcal{I}_1(t)-\mathcal{I}_2(t)}{\mathcal{I}_1(t)+2\mathcal{I}_2(t)}-\frac{1}{4\mathcal{I}_4(t)+1}$.
 (I.2) If $\mathcal{F}_2(t)\neq 0$, then

$$\begin{aligned}\mathcal{F}_1(t) &= \frac{2\mathcal{I}_1(t)-2\mathcal{I}_2(t)-1}{\mathcal{I}_1(t)+2\mathcal{I}_2(t)}-\frac{1}{2}\mathcal{F}_2(t)\left(\frac{4\mathcal{I}_1(t)^3-27\mathcal{I}_2(t)^2}{\left(\mathcal{I}_1(t)+2\mathcal{I}_2(t)\right)^3}\right)^{\frac{1}{2}}\\ &\quad\pm\frac{\mathcal{F}_2(t)\rho(t)^{\frac{1}{2}}}{2\mathcal{I}_3(t)},\\ \mathcal{F}_2(t) &= -\frac{2^33^2\mathcal{I}_2(t)\mathcal{I}_3(t)\mathcal{I}_4(t)}{\tilde{D}^0\pm\tilde{D}^1\rho(t)^{\frac{1}{2}}},\end{aligned}$$

$$\rho(t)=\frac{\mathcal{I}_2(t)^2\mathcal{I}_3(t)}{\left(\mathcal{I}_1(t)+2\mathcal{I}_2(t)\right)^2}\left[\mathcal{I}_3(t)\frac{12\mathcal{I}_1(t)-8\mathcal{I}_2(t)-27}{\mathcal{I}_1(t)+2\mathcal{I}_2(t)}-36(4\mathcal{I}_4(t)+1)\right],$$

$$\tilde{D}^0(t)=\frac{\left(\mathcal{I}_1(t)+2\mathcal{I}_2(t)\right)^2}{\mathcal{I}_2(t)}D^0$$

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$$
\begin{aligned}
&= \mathcal{I}_2(t)\mathcal{I}_3(t)\left(\frac{4\mathcal{I}_1(t)^3 - 27\mathcal{I}_2(t)^2}{\mathcal{I}_1(t)+2\mathcal{I}_2(t)}\right)^{\frac{1}{2}} \\
&\quad \cdot \left[\mathcal{I}_3(t)\frac{12\mathcal{I}_1(t) - 8\mathcal{I}_2(t) - 27}{\mathcal{I}_1(t)+2\mathcal{I}_2(t)} + 9\left(\mathcal{I}_3(t) - 12\mathcal{I}_4(t) - 3\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{D}^1(t) &= \frac{\left(\mathcal{I}_1(t)+2\mathcal{I}_2(t)\right)^2}{\mathcal{I}_2(t)} D^1 \\
&= -\mathcal{I}_3(t)\frac{12\mathcal{I}_1(t)^2 + 9\mathcal{I}_1(t)\mathcal{I}_2(t) + 2\mathcal{I}_2(t)^2 + 27\mathcal{I}_2(t)}{\mathcal{I}_1(t)+2\mathcal{I}_2(t)} \\
&\qquad -9\mathcal{I}_2(t)\left(4\mathcal{I}_4(t)+1\right).
\end{aligned}
$$

(II) If $t \in \left(O'_{w_0}\backslash(\mathcal{I}_4)^{-1}(0)\right) \cap \left(X'^1\backslash(\mathcal{I}_3)^{-1}(0)\right)$, then

$$
\begin{aligned}
\mathcal{F}_1(t) &= 3\cdot 2^{\frac{2}{3}} - \frac{3^4 2^{\frac{8}{3}}\mathcal{I}_3(t)\mathcal{I}_4(t)(16\mathcal{I}_1(t)-27)[1+\mathcal{I}_3(t)+4\mathcal{I}_4(t)]}{\mathcal{I}_1(t)\sigma(t)}, \\
\mathcal{F}_2(t) &= \frac{9\cdot 2^{\frac{10}{3}}\mathcal{I}_3(t)^2\mathcal{I}_4(t)(16\mathcal{I}_1(t)-27)^{\frac{3}{2}}}{\mathcal{I}_1(t)^{\frac{3}{2}}\sigma(t)},
\end{aligned}
$$

with $\sigma(t) = 3^6(1+3\mathcal{I}_3(t)+\mathcal{I}_3(t)^3+2^6\mathcal{I}_4(t)^3) - 4\left(11+\frac{27}{\mathcal{I}_1(t)}\right)\left(16-\frac{27}{\mathcal{I}_1(t)}\right)\cdot$ $\mathcal{I}_3(t)^3 + 3^3\left(3^4-2^2\left(16-\frac{27}{\mathcal{I}_1(t)}\right)\right)\mathcal{I}_3(t)^2\left(1+4\mathcal{I}_4(t)\right) + 2^2 3^7\left(1+\mathcal{I}_3(t)\right)\cdot$ $\left(1+4\mathcal{I}_4(t)\right)\mathcal{I}_4(t)$.

(III) If $t \in \left(O'_{w_0}\backslash(\mathcal{I}_4)^{-1}(0)\right) \cap X^2$, then

$$
\begin{aligned}
\mathcal{F}_1(t) &= 1 + \frac{12\mathcal{I}_1(t)\mathcal{I}_4(t)}{7\mathcal{I}_1(t)\mathcal{I}_3(t) - 9\mathcal{I}_1(t)\left(4\mathcal{I}_4(t)+1\right) - 9\mathcal{I}_3(t) + 6\mathcal{I}_1(t)\varsigma(t)^{\frac{1}{2}}}, \\
\mathcal{F}_2(t) &= \left(\frac{3\mathcal{I}_1(t)}{9-\mathcal{I}_1(t)}\right)^{\frac{1}{2}}\left(\mathcal{F}_1(t)-3\right) + \frac{\mathcal{F}_1(t)-1}{2\mathcal{I}_3(t)}\varsigma(t)^{\frac{1}{2}},
\end{aligned}
$$

where $\varsigma(t) = \frac{4\mathcal{I}_3(t)}{9-\mathcal{I}_1(t)}\left[\mathcal{I}_3(t)\left(4\mathcal{I}_1(t)-9\right) - 12\mathcal{I}_1(t)\left(1+4\mathcal{I}_4(t)\right)\right]$.

Corollary 7.6. *Let O' be the Zariski-open susbet defined in* Proposition 7.5. *The map $(\mathcal{I}_1,\ldots,\mathcal{I}_4)\colon O'/GL(V) \to \mathbb{F}^4$ is a two-sheet covering ramifying on the union of the following two sets:*

$$
\left\{\begin{array}{l}
\mathcal{I}_3(t)\left(12\mathcal{I}_1(t) - 8\mathcal{I}_2(t) - 27\right) - 36(4\mathcal{I}_4(t)+1)\left(\mathcal{I}_1(t)+2\mathcal{I}_2(t)\right) = 0 \\
: t \in \left(O'_{w_0}\backslash(\mathcal{I}_4)^{-1}(0)\right) \cap O'^2
\end{array}\right\},
$$

$$
\left\{\begin{array}{l}
\mathcal{I}_3(t)\left(4\mathcal{I}_1(t) - 9\right) - 12\mathcal{I}_1(t)\left(1+4\mathcal{I}_4(t)\right) = 0 \\
: t \in \left(O'_{w_0}\backslash(\mathcal{I}_4)^{-1}(0)\right) \cap X^2
\end{array}\right\}.
$$

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