2012

Spectral properties for a new composition of a matrix and a complex representation

Zhiping Lin
Yongzhi Liu
Giuseppe Molteni
giuseppe.molteni1@unimi.it
Dongye Zhang

Follow this and additional works at: http://repository.uwyo.edu/ela

Recommended Citation
Lin, Zhiping; Liu, Yongzhi; Molteni, Giuseppe; and Zhang, Dongye. (2012), 'Spectral properties for a new composition of a matrix and a complex representation', Electronic Journal of Linear Algebra, Volume 23.
DOI: https://doi.org/10.13001/1081-3810.1538
SPECTRAL PROPERTIES FOR A NEW COMPOSITION OF A MATRIX AND A COMPLEX REPRESENTATION∗

ZHIPING LIN†, YONGZHI LIU‡, GIUSEPPE MOLTENI§, AND DONGYE ZHANG¶

Abstract. A way to compose a matrix $P$ and a finite dimensional representation $\rho$ of $\mathbb{C}$ via a map $h$ into a new matrix $P \ast_h \rho$ is defined. Several results about the spectrum, eigenvectors, kernel and rank of $P \ast_h \rho$ are proved.

Key words. Spectrum, Complex representation of $\mathbb{C}$.

AMS subject classifications. 15A18, 15A99.

1. Introduction. In two recent papers [4, 5], it has been pointed out the interest of the symmetric matrix

$$P(\omega) := \begin{bmatrix} P_1(\omega) & P_2(\omega) \\ P_2^T(\omega) & P_1(\omega) \end{bmatrix}$$

for the design of some signal filters, where $P_1(\omega)$ and $P_2(\omega)$ are the square matrices of order $N$ whose entries are

$$(P_1(\omega))_{i,j=1}^N := (i + j - 2) \cos((i - j)\omega), \quad (P_2(\omega))_{i,j=1}^N := (i + j - 2) \sin((i - j)\omega).$$

In particular, it has been conjectured that the spectrum of $P(\omega)$, i.e., its eigenvalues and their multiplicities, is actually independent of $\omega$. In this paper we prove this fact as a consequence of a more general result (Theorem 2.5 and Proposition 2.8 below). Indeed, we introduce a procedure which generalizes the construction of $P(\omega)$ and we prove the conjecture for each matrix we obtain in this way. Several other results exploring the connection of the new operation with other ways to combine matrices into a new matrix are given.

∗Received by the editors on December 29, 2011. Accepted for publication on May 20, 2012. Handling Editor: Tin-Yau Tam.
†School of Electrical and Electronic Engineering, Nanyang Technological University, Block S2, Nanyang Avenue, Singapore 639798, Singapore (ezplin@ntu.edu.sg).
‡GE China Technology Center, Shangai 201203, China (yongzhi.liu@ge.com).
§Dipartimento di Matematica, Università di Milano, via Saldini 50, I-20133 Milano, Italy (giuseppe.molteni1@unimi.it).
¶School of Electrical and Electronic Engineering, Nanyang Technological University, Block S2, Nanyang Avenue, Singapore 639798, Singapore (zh0002ye@e.ntu.edu.sg).
2. Results. Let \( \rho \) be an \( r \) dimensional linear representation of \( \mathbb{C} \), i.e., a map \( \mathbb{C} \rightarrow \text{GL}(r, \mathbb{C}) \) satisfying the condition

\[
\rho(x + y) = \rho(x) \cdot \rho(y) \quad \forall x, y \in \mathbb{C},
\]

let \( h : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{C} \) be any function for which

\[
h(i, k) + h(k, j) = h(i, j) \quad \forall i, j, k,
\]

and finally let \( P \) be any complex \( n \times m \) matrix. With these ingredients we define a new matrix \( P \ast_h \rho \) in \( \mathcal{M}(nr \times mr, \mathbb{C}) \) as

\[
P \ast_h \rho := \begin{bmatrix}
(P \ast_h \rho)_{(1,1)} & \cdots & (P \ast_h \rho)_{(1,r)} \\
\vdots & \ddots & \vdots \\
(P \ast_h \rho)_{(r,1)} & \cdots & (P \ast_h \rho)_{(r,r)}
\end{bmatrix},
\]

where each block \( (P \ast_h \rho)_{(I,J)} \) is itself an \( n \times m \) matrix and is defined as

\[
(P \ast_h \rho)_{(I,J)}(i,j) := P_{i,j} \rho_{I,J}(h(i,j)), \quad i = 1, \ldots, n, \quad j = 1, \ldots, m.
\]

Remark 2.1. There is a canonic way to build \( \rho \): let \( T \) be an arbitrary square matrix in \( \mathcal{M}(r, \mathbb{C}) \), and take

\[
\rho(x) := \exp(xT) := \sum_{k=0}^{\infty} \frac{x^k}{k!} T^k, \quad \forall x \in \mathbb{C}.
\]

Every regular (analytic) representation is of this form (see [1, Ch. 6 Appendix A] and [3, Ch. 8]).

Remark 2.2. It is immediate to realize that (2.2) is satisfied if and only if \( h(i, j) = g(i) - g(j) \) for some map \( g : \mathbb{N} \rightarrow \mathbb{C} \).

Remark 2.3. The matrix \( P(\omega) \) is of the form \( P \ast_h \rho \) with

\[
P \in \mathcal{M}(N, \mathbb{C}): \quad (P)_{i,j} := i + j - 2,
\]

\[
\rho : \mathbb{C} \rightarrow \text{GL}(2, \mathbb{C}): \quad \rho(x) := \begin{bmatrix}
\cos x & \sin x \\
-\sin x & \cos x
\end{bmatrix},
\]

\[
h(i, j) := (i - j)\omega.
\]

The following theorem gives a first set of properties for \( P \ast_h \rho \) in terms of analogous properties for \( P \).

Theorem 2.4.

1. \( (\mu P + \nu Q) \ast_h \rho = \mu P \ast_h \rho + \nu Q \ast_h \rho \) for every \( P, Q \in \mathcal{M}(n \times m, \mathbb{C}) \) and for every \( \mu, \nu \in \mathbb{C} \). Moreover, \( P \ast_h \rho = 0 \) if and only if \( P = 0 \).
2. Let $P$ be diagonal, then $P_{h} \rho$ is the direct sum of $r$ copies of $P$ and therefore it is diagonal too. In particular, $I_{n} P_{h} \rho = I_{n} P_{h} \rho$.

3. $(P_{h} \rho)^{*} = P^{*} \rho_{-h}^{*}$. In particular, if $P$ is a square matrix and the restriction of $\rho$ on the range of $h$ is unitary (i.e., if $\rho^{*}(x) = \rho(-x)$ for every $x$ in the range of $h$), then $P_{h} \rho$ is self-adjoint if and only if $P$ is self-adjoint.

4. Let $P \in M(n \times l, \mathbb{C})$ and $Q \in M(l \times m, \mathbb{C})$, then

$$
(P_{h} \rho) \cdot (Q_{h} \rho) = (P \cdot Q)_{h} \rho.
$$

5. Let $P$ be a square matrix. The minimal polynomials of $P$ and $P_{h} \rho$ are equal.

6. $P_{h} \rho$ is diagonalizable if and only if $P$ is diagonalizable, and a complex number $\lambda$ is an eigenvalue for $P_{h} \rho$ if and only if it is an eigenvalue for $P$.

7. $P \in GL(n, \mathbb{C})$ if and only if $P_{h} \rho \in GL(n r, \mathbb{C})$, with $(P_{h} \rho)^{-1} = P^{-1} P_{h} \rho$.

**Proof.**

1. The linearity of $P_{h} \rho$ as a function of $P$ is evident; it implies that $0_{h} \rho = 0$. Suppose that $P_{i,j} \rho_{i,j}(h(i, j)) = 0$ for every $I, J = 1, \ldots, r$ and every $i = 1, \ldots, n, j = 1, \ldots, m$, and that by absurd $P_{i,j} \neq 0$ for a couple of indexes $i, j$. Then $\rho_{i,j}(h(i, j)) = 0$ for every $I$ and $J$, which is impossible because $\rho(h(i, j)) \in GL(r, \mathbb{C})$.

2. Let $P$ be diagonal, so that $P_{i,j} = a_{i} \delta_{i,j}$, then:

$$
(P_{h} \rho)_{(i,j,i,j)}(i, j, i, j) = a_{i} \delta_{i,j} \rho_{i,j}(h(i, j)) = a_{i} \delta_{i,j} \rho_{i,j}(h(i, i)) = a_{i} \delta_{i,j} \rho_{i,j}(0)
$$

because $h$ is an odd map, and this is $a_{i} \delta_{i,j} \rho_{i,j}$, because $\rho(0) = I_{r}$.

3. The equality $(P_{h} \rho)^{*} = P^{*} \rho_{-h}^{*}$ is an immediate consequence of the definition of the $*_{h}$-product. Now suppose that $\rho^{*}(-h(i, j)) = \rho(h(i, j))$, then $(P_{h} \rho)^{*} = P^{*} P_{h} \rho$ so that this is equal to $P_{h} \rho$ if and only if $(P^{*} - P)_{h} \rho = 0$, i.e., if and only if $P^{*} = P$, by Item 1.

4. The proof is a direct consequence of Relations (2.1) and (2.2). In fact, for every couple of indexes $I, J = 1, \ldots, r$ and $i = 1, \ldots, n, j = 1, \ldots, m$ we have

$$
((P_{h} \rho) \cdot (Q_{h} \rho))_{(i,j,i,j)} = \sum_{k,k,K} P_{i,k} \rho_{i,k}(h(i, k)) Q_{k,j} \rho_{K,J}(h(k, j))
$$

$$
= \sum_{k} P_{i,k} Q_{k,j} \sum_{K} \rho_{i,K}(h(i, k)) \rho_{K,J}(h(k, j)).
$$
Spectral Properties

By (2.1) the inner sum is \( \rho_{I,J}(h(i,k) + h(k,j)), \) i.e., \( \rho_{I,J}(h(i,j))) \), by (2.2). Thus

\[
(P \ast h \rho) \cdot (Q \ast h \rho))_{(I,J,i,j)} = \sum_k P_{i,k}Q_{k,j}\rho_{I,J}(h(i,j)),
\]

which is the claim.

5. The formula in Item 4 implies that \( f(P \ast h \rho) = f(P) \ast h \rho \) for every polynomial \( f \in \mathbb{C}[x] \), so that \( f(P \ast h \rho) \) is null if and only if \( f(P) \) is null as well, by Item 1. The claim follows by the definition of the minimal polynomial of a matrix \( A \) as the monic generator of the ideal of complex polynomials \( f \) for which \( f(A) = 0 \).

6. A matrix is diagonalizable if and only if its minimal polynomial \( f \in \mathbb{C}[x] \) factorizes in \( \mathbb{C}[x] \) as product of distinct linear polynomials. Therefore the first claim follows by Item 5. Moreover, the eigenvalues coincide with the roots of the minimal polynomial, therefore Item 5 implies also the second part of this claim.

7. A matrix is invertible if and only if 0 is not an eigenvalue. Therefore the first part of the claim follows by Item 6. The formula for \( (P \ast h \rho)^{-1} \) is an immediate consequence of Items 2 and 4.

Item 6 of the previous theorem already shows that the spectrum of \( P \ast h \rho \) and that one of \( P \) contain the same points, but their spectral structures are even more strictly related. In fact, the next two theorems prove that also the eigenvectors of \( P \ast h \rho \) can be easily deduced by those ones of \( P \). We start with a general result which is of some independent interest.

**Theorem 2.5.** Let \( P \in \mathcal{M}(n \times m, \mathbb{C}) \). Then \( \dim \ker(P \ast h \rho) = r \dim \ker(P) \) and \( \operatorname{rank}(P \ast h \rho) = r \operatorname{rank}(P) \).

**Proof.** Let \( s \) denote the rank of \( P \). The definition of rank implies the existence of a permutation \( \mathcal{P} \in \operatorname{GL}(n, \mathbb{C}) \) and a permutation \( \mathcal{P}' \in \operatorname{GL}(m, \mathbb{C}) \) such that

\[
P' := \mathcal{P} PP' = \begin{bmatrix} P'' & * \\ * & * \end{bmatrix}
\]

with \( P'' \in \operatorname{GL}(s, \mathbb{C}) \). By Theorem 2.3 Item 7 the matrices \( \mathcal{P} \ast h \rho \) and \( \mathcal{P}' \ast h \rho \) are invertible, so that the rank of \( P' \ast h \rho = (P \ast h \rho)(P \ast h \rho)(P' \ast h \rho) \) (by Item 4) is equal to that one of \( P \ast h \rho \). Moreover, the matrix \( P'' \ast h \rho \) is a submatrix in \( P' \ast h \rho \), therefore \( \operatorname{rank}(P' \ast h \rho) \geq \operatorname{rank}(P'' \ast h \rho) \) and the rank of \( P'' \ast h \rho \) is \( rs \) because it is in \( \operatorname{GL}(sr, \mathbb{C}) \).

As a consequence we have proved that

\[
(2.3) \quad \operatorname{rank}(P \ast h \rho) \geq r \operatorname{rank}(P).
\]

Let \( v_1, \ldots, v_k \) be a basis for the kernel of \( P \). Let \( V := [v_1 | \ldots | v_k] \) be the matrix having the vectors \( v_j \) for \( j = 1, \ldots, k \) as columns. By (2.3) applied to \( V \) we get that
rank($V \ast h \rho$) \geq kr. The columns in $V \ast h \rho$ belong to the kernel of $P \ast h \rho$, by Item 4 in Theorem 2.4, this proves that
\begin{equation}
(2.4) \quad \dim \ker(P \ast h \rho) \geq r \dim \ker(P).
\end{equation}

Adding (2.3) and (2.4) and recalling the rank-nullity theorem we conclude that
\[
mr = \text{rank}(P \ast h \rho) + \dim \ker(P \ast h \rho) \geq r \text{rank}(P) + r \dim \ker(P) = mr
\]
which proves that the equality holds in (2.3) and (2.4). \[\square\]

Let $P$ be a square matrix. For each $\lambda \in \mathbb{C}$ let $E_\lambda$ denote the kernel of $P - \lambda \mathbb{I}_n$ (i.e., the $\lambda$-eigenspace of $P$ when $\lambda$ belongs to the spectrum of $P$), and analogously let $E_{\lambda,\rho}$ denote the kernel of $P \ast h \rho - \mathbb{I}_{nr}$.

**Proposition 2.6.** For every $\lambda \in \mathbb{C}$, $\dim E_{\lambda,\rho} = r \dim E_\lambda$. In particular, $P$ and $P \ast h \rho$ have the same eigenvalues, and the multiplicity of every $\lambda$ as eigenvalue for $P \ast h \rho$ is $r$ times its multiplicity as eigenvalue for $P$. Moreover, if the columns in $V \in \mathcal{M}(m \times \dim E_\lambda, \mathbb{C})$ are a basis for $E_\lambda$, then the columns of $V \ast h \rho$ are a basis for $E_{\lambda,\rho}$.

**Proof.** In fact, $E_{\lambda,\rho} = \ker(P \ast h \rho - \lambda \mathbb{I}_{nr}) = \ker((P - \lambda \mathbb{I}_n) \ast h \rho)$ so that the claims follow by the previous theorem. \[\square\]

**Remark 2.7.** We can rephrase the claims of Proposition 2.6 by saying that the spectrum (i.e., the eigenvalues and the dimension of each eigenspace) of $P \ast h \rho$ is independent of $h$ and depends on $\rho$ only via its dimension; this claim already suffices to completely determine the spectrum of $P \ast h \rho$ since one sees immediately that $P \ast h \rho$ collapses to the direct sum of $r$ copies of $P$ when $h$ is taken equal to 0 identically. The conjectured independence of the spectrum of $P(\omega)$ of $\omega$ in \[5\], therefore, is evidently only a special case of the independence of the spectrum of $P \ast h \rho$ on $h$ claimed in Proposition 2.6 when it is restated in this way.

Let $V$ and $W$ be two matrices, respectively in $\mathcal{M}(n \times v, \mathbb{C})$ and $\mathcal{M}(n \times w, \mathbb{C})$. Then we can form the new matrix $[V \mid W]$ in $\mathcal{M}(n \times (v + w), \mathbb{C})$ by joining the columns of $W$ to those ones of $V$. In general, the matrices $[V \ast h \rho \mid W \ast h \rho]$ and $[V \mid W] \ast h \rho$ are distinct, but they are quite strictly related. We begin with a simple computation, which is useful in applications. Suppose that the columns of $V$ and $W$ be eigenvectors for a matrix $P$, so that $PV = V D_V$ and $PW = W D_W$ with $D_V$ and $D_W$ diagonal matrices. Then, using two times the multiplicativity property for the $\ast h$-product (Item 4 of Theorem 2.4), we get
\[
(P \ast h \rho) \cdot [V \mid W] \ast h \rho = (P \cdot [V \mid W]) \ast h \rho = ([PV \mid PW]) \ast h \rho
= ([VD_V \mid WD_W]) \ast h \rho = ([V \mid W] \cdot (D_V \oplus D_W)) \ast h \rho
= ([V \mid W] \ast h \rho) \cdot ((D_V \oplus D_W) \ast h \rho).
\]
The matrix \((D_V \oplus D_W) \ast h \rho\) is diagonal and in particular is the direct sum of \(r\) copies of \(D_V \oplus D_W\) (by Item 2 of Theorem 2.4 because \(D_V \oplus D_W\) is diagonal by the assumption), hence we have proved the following claim.

**Proposition 2.8.** With the previous notations, the columns of \([V \mid W] \ast h \rho\) are eigenvectors for \(P \ast h \rho\), and if \(\lambda_V\) and \(\lambda_W\) denote the eigenvalues for the columns of \(V\) and \(W\) (i.e., the main diagonals of \(D_V\) and \(D_W\)), then the eigenvalues corresponding to the columns of \([V \mid W] \ast h \rho\) are the sequence \(\lambda_V, \lambda_W, \lambda_V, \lambda_W, \ldots, \lambda_V, \lambda_W\) (\(r\) couples).

Probably, the typical use of this computation will be in ‘tandem’ with Proposition 2.6, to produce a set of eigenvectors for \(P \ast h \rho\). We illustrate this through a simple example as follows. Consider \(P\) with \(N = 2\), \(\rho\), \(h\) and \(P(\omega)\) as given in Remark 2.3, thus

\[
P = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad P(\omega) = \begin{bmatrix} 0 & \cos \omega & 0 & -\sin \omega \\ \cos \omega & 2 & \sin \omega & 0 \\ 0 & \sin \omega & 0 & \cos \omega \\ -\sin \omega & 0 & \cos \omega & 2 \end{bmatrix}.
\]

The eigenvalues of \(P\) (and hence also \(P(\omega)\), with multiplicity 2) are \(\lambda_{\pm} := 1 \pm \sqrt{2}\) with \(v_{\pm} := \begin{bmatrix} 1 \\ \pm \sqrt{2} \end{bmatrix}\) as corresponding eigenvectors. By extending each \(v_{\pm}\) through \(v_{\pm} \ast h \rho\) we form the matrix

\[
Q(\omega) := [v_+ \ast h \rho \mid v_- \ast h \rho] = \begin{bmatrix} 1 & 0 & (1 + \sqrt{2}) \cos \omega & (1 + \sqrt{2}) \sin \omega \\ 0 & 1 & (1 - \sqrt{2}) \cos \omega & (1 - \sqrt{2}) \sin \omega \\ -1 & (1 + \sqrt{2}) \sin \omega & 1 & 0 \\ 0 & 1 & (1 - \sqrt{2}) \cos \omega & (1 - \sqrt{2}) \sin \omega \end{bmatrix}
\]

for which \(P(\omega)Q(\omega) = Q(\omega) \text{diag}\{\lambda_+, \lambda_+, \lambda_-, \lambda_-\}\).

Analogously, we can form the other matrix

\[
Q'(\omega) := [v_+ \mid v_-] \ast h \rho = \begin{bmatrix} 1 & \cos \omega & 0 & -\sin \omega \\ (1 + \sqrt{2}) \cos \omega & 1 - \sqrt{2} & (1 + \sqrt{2}) \sin \omega & 0 \\ 0 & \sin \omega & 1 & \cos \omega \\ -1 & (1 + \sqrt{2}) \sin \omega & 0 & (1 + \sqrt{2}) \cos \omega \end{bmatrix},
\]

for which \(P(\omega)Q'(\omega) = Q'(\omega) \text{diag}\{\lambda_+, \lambda_-, \lambda_+, \lambda_-\}\). Both the procedures give a basis of eigenvectors for \(P(\omega)\) from a basis of eigenvectors for \(P\); for \(Q(\omega)\) this is a consequence of Proposition 2.6 for \(Q'(\omega)\) it is a consequence of Theorem 2.5 (or even by Item 7 in Theorem 2.4). These approaches to the construction of eigenvectors for \(P(\omega)\) are in general convenient for applications: to obtain the eigenvectors directly
from \( \text{ker}(P \ast \rho - \lambda I_4) \) one would need to solve a system of equations involving polynomials in \( \sin \omega \) and \( \cos \omega \) and this could be a computationally quite difficult task. In fact, although there are computational algebraic methods for solving a system of equations in a polynomial ring with several independent variables \( z_1, \ldots, z_k \), e.g., Gröbner bases [2], there is not a constructive method for solving a system of equations in the polynomial ring in \( \sin \omega \) and \( \cos \omega \) because now the variables \( z_1 := \sin \omega \) and \( z_2 := \cos \omega \) are algebraically dependent.

In the previous example it is easy to check that \( \det Q(\omega) = -8 \), \( \det Q'(\omega) = 8 \) and \( \det([v_+ | v_-]) = \sqrt{8} \): the simple relation between these determinants is not casual, but it is a consequence of a general relation which we explore now.

Let \( V, W \) be generic \( n \times v \) and \( n \times w \) matrices. Then,

\[
([V | W] \ast \rho)_{(I,J;i,j)} = \begin{cases} V_{i,j} \rho_{I,J}(h(i,j)) & \text{if } j \leq v, \\ W_{i,j-v} \rho_{I,J}(h(i,j)) & \text{if } j > v. \end{cases}
\]

This proves that there exists a permutation \( \mathcal{P} \) such that

\[
(2.5) \quad [V | W] \ast \rho \cdot \mathcal{P}^{-1} = [V \ast \rho | S],
\]

where

\[
S \in \mathcal{M}(nr \times wr, \mathbb{C}), \quad \text{with} \quad S_{(I,J;i,j)} := W_{i,j} \rho_{I,J}(h(i,j+v)).
\]

Noting that \( h(i,j+v) \) can be written as \( h(i,j) + h(j,j+v) \) and using the multiplicativity of the representation \( \rho \), we get

\[
S_{(I,J;i,j)} = W_{i,j} \rho_{I,J}(h(i,j) + h(j,j+v)) = \sum_K W_{i,j} \rho_{I,K}(h(i,j)) \rho_{K,J}(h(j,j+v)) = \sum_K (W \ast \rho)_{(I,K;i,j)} \rho_{K,J}(h(j,j+v)).
\]

In matricial form this equality can be written as

\[
S = (W \ast \rho) \cdot B,
\]

where

\[
B = \begin{bmatrix} B_{1,1} & \ldots & B_{1,r} \\ \ldots & & \ldots \\ B_{r,1} & \ldots & B_{r,r} \end{bmatrix}, \quad B_{I,J} := \text{diag}(\rho_{I,J}(h(1,v+1)), \ldots, \rho_{I,J}(h(w,v+w))).
\]

This structure proves the existence of two permutations \( \mathcal{Q}, \mathcal{Q}' \) in \( \text{GL}(rw, \mathbb{C}) \) such that

\[
B = \bigoplus_{j=1}^{w} \rho(h(j,j+v)) \mathcal{Q}'.
\]
Their definition makes evident that $\mathcal{Q}$ and $\mathcal{Q}'$ depend on $\rho$ only via its order; when $\rho$ is the trivial representation both $\mathcal{B}$ and $\bigoplus_{j=1}^{w} \rho(h(j, j + v))$ collapse to the identity, thus proving that $\mathcal{Q}' = \mathcal{Q}^{-1}$. As a consequence we have proved that
\[
S = W \ast_{h} \rho \cdot \bigoplus_{j=1}^{w} \rho(h(j, j + v))\mathcal{Q}^{-1}.
\]
With (2.5), this equality proves the following formula.

**Theorem 2.9.** With the previous notations, we have
\[
(V \mid W) \ast_{h} \rho = (V \ast_{h} \rho \mid W \ast_{h} \rho) \cdot (1_{r} \oplus \bigoplus_{j=1}^{w} \rho(h(j, j + v))\mathcal{Q}^{-1}) \cdot \mathcal{P}.
\]
In particular, the ranks of $(V \mid W) \ast_{h} \rho$ and of $(V \ast_{h} \rho \mid W \ast_{h} \rho)$ are equal and when $v + w = n$, i.e., when $(V \mid W)$ is a square matrix, we have
\[
\det((V \ast_{h} \rho \mid W \ast_{h} \rho)) = (-1)^{vw(\frac{r}{2})} \det((V \mid W) \ast_{h} \rho) \det\left(\rho(\sum_{j=1}^{w} h(j + v, j))\right).
\]

**Proof.** The formula for $\det((V \ast_{h} \rho \mid W \ast_{h} \rho))$ is a direct consequence of (2.6), apart the computation of the determinant of $\mathcal{P}$, for which we need the following explicit description coming directly from its definition in (2.5). Split the integers $\{1, \ldots, vr\}$ in $r$ consecutive blocks denoted as $n_{1}, \ldots, n_{r}$ having $v$ integers each one, and analogously split the integers $\{vr + 1, \ldots, vr + w\}$ in $r$ consecutive blocks denoted as $m_{1}, \ldots, m_{r}$, having $w$ integers each one. Then $\mathcal{P}$ is the ‘shuffle’ permutation which moves the blocks according to the following rule:
\[
\mathcal{P} : (n_{1}, n_{2}, \ldots, n_{r}, m_{1}, m_{2}, \ldots, m_{r}) \rightarrow (n_{1}, m_{1}, n_{2}, m_{2}, \ldots, n_{r}, m_{r}).
\]
It is now easy to verify that $\det(\mathcal{P}) = (-1)^{vw(\frac{r}{2})}$. □

**Remark 2.10.** The permutation $\mathcal{Q}$ in Theorem 2.9 can be concretely described as follows. Each integer $n$ in $\{0, \ldots, wr - 1\}$ can be uniquely written both as $a + bw$ and as $a' + b'r$, with $0 \leq a, b' < w$ and $0 \leq a', b < r$. The map $a + bw \rightarrow b + ar$ is therefore a well defined bijection of $\{0, \ldots, wr - 1\}$ in itself: $\mathcal{Q}$ is the matrix representing this permutation.

Theorem 2.9 here above explains the equality $\det Q(\omega) = -\det Q'(\omega)$ in our previous example. As we will see now, the other equality $\det Q'(\omega) = (\det([v_{+} \mid v_{-}]))^{2}$ is a consequence of a general formula relating the characteristic polynomial of $P \ast_{h} \rho$ to that one of $P$ (see next Theorem 2.13). We will deduce this formula via the Jordan decomposition of $P$ and using the following proposition describing the behavior of the $\ast_{h}$-product with respect to a direct sum in its first argument.
Proposition 2.11. Let $P \in \mathcal{M}(p, \mathbb{C})$ and $Q \in \mathcal{M}(q, \mathbb{C})$. Then, there is a permutation $\mathcal{P} \in \text{GL}((p+q)r, \mathbb{C})$ such that

$$
\mathcal{P} \cdot ((P \oplus Q) *_{h \rho}) \cdot \mathcal{P}^{-1} = (P *_{h \rho} \rho) \oplus (Q *_{h \rho} \rho),
$$

where $h_P(i,j) := h(i+p,j+p)$.

Proof. We have

$$(P \oplus Q)_{i,j} *_{h \rho} I,J(h(i,j)) =
\begin{cases}
P_{i,j} \rho I,J(h(i,j)) & \text{if } i,j \leq p, \\
Q_{i-p,j-p} \rho I,J(h_P(i-p,j-p)) & \text{if } i,j > p, \\
0 & \text{otherwise},
\end{cases}
$$

for every $I$ and $J$. Thus, according to the definition of the $*_{h \rho}$-product, we see that $(P \oplus Q) *_{h \rho}$ can be obtained by permuting columns and rows of $(P *_{h \rho} \rho) \oplus (Q *_{h \rho} \rho)$, i.e.,

$$(2.7) \quad \mathcal{P}((P \oplus Q) *_{h \rho})\mathcal{P}' = (P *_{h \rho} \rho) \oplus (Q *_{h \rho} \rho)$$

for two suitable permutations $\mathcal{P}$ and $\mathcal{P}'$. The formula also shows that these permutations depend on $P$ and $Q$ only via their orders, thus substituting $P$ and $Q$ with the identities of the same order and using the conclusion in Item 2 of Theorem 2.4, we get that

$$
\mathcal{P} \mathcal{P}' = \mathcal{P}((I_{p+q} *_{h \rho} \rho) \mathcal{P}' = \mathcal{P}((I_p \oplus I_q) *_{h \rho} \rho) \mathcal{P}'
= (I_p *_{h \rho} \rho) \oplus (I_q *_{h \rho} \rho) = I_{pr} \oplus I_{qr} = I_{(p+q)r},
$$

thus proving that $\mathcal{P}' = \mathcal{P}^{-1}$ in (2.7). \(\square\)

Remark 2.12. The argument in the proof of Proposition 2.11 also shows that $\mathcal{P}$ coincides with the permutation having the same name already described in the proof of Theorem 2.9,

$$
\mathcal{P} : (n_1, n_2, \ldots, n_r, m_1, m_2, \ldots, m_r) \rightarrow (n_1, m_1, n_2, m_2, \ldots, n_r, m_r),
$$

where $n_1, \ldots, n_r$ are a partition of $\{1, \ldots, pr\}$ in blocks of consecutive integers having $p$ integers each one, and $m_1, \ldots, m_r$ a partition of $\{pr+1, \ldots, pr+qr\}$ in blocks of consecutive integers having $q$ integers each one.

Theorem 2.13. The characteristic polynomial of $P *_{h \rho}$ is the $r$th power of that one of $P$.

Proof. Let $\oplus_l \oplus_m (\lambda_l I_{n_1, m} + J_{n_2, m})$ be the Jordan canonical decomposition of $P$, where $\lambda_l$ are the distinct eigenvalues of $P$ and $\{J_{n_1, m}\}_m$ are the Jordan blocks corresponding to the eigenvalue $\lambda_l$. Then, $P$ is similar to $\oplus_l \oplus_m (\lambda_l I_{n_1, m} + J_{n_2, m})$.
and by Items 4 and 7 of Theorem 2.4

\[ P \ast_h \rho \text{ is similar to } (\oplus_l \oplus_m (\lambda_l I_{n_l,m} + J_{n_l,m})) \ast_h \rho. \]

By Proposition 2.11

\[ P \ast_h \rho \text{ is similar to } \oplus_l \oplus_m (\lambda_l I_{n_l,m} + J_{n_l,m}) \ast_{h_{l,m}} \rho, \]

where each \( h_{l,m} \) is a suitable map satisfying (2.2). Since

\[ I_{r \rho} = \oplus_l \oplus_m (I_{n_{l,m}} \ast_{h_{l,m}} \rho), \]

(by Theorem 2.4 Item 2) we get that

\[ x I_{r \rho} - P \ast_h \rho \text{ is similar to } \oplus_l \oplus_m ((x - \lambda_l) I_{n_{l,m}} - J_{n_{l,m}}) \ast_{h_{l,m}} \rho). \]

Consider a matrix of the form \((\lambda I_n + J_n) \ast_h \rho\). By Item 5 of Theorem 2.4 its minimal polynomial is \((x - \lambda)^n\). Thus, its characteristic polynomial must be a power of \((x - \lambda)\), and hence is \((x - \lambda)^{nr}\), i.e., the \(r\)-th power of the characteristic polynomial of \(\lambda I_n + J_n\). The claim now follows by (2.8), by multiplicativity.

REFERENCES