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THE (SIGNLESS) LAPLACIAN SPECTRAL RADII OF C-CYCLIC GRAPHS WITH N VERTICES AND K PENDANT VERTICES*

MUHUO LIU†

Abstract. A connected graph is called a c-cyclic graph if it contains n vertices and $n + c - 1$ edges. Let $\mathbb{C}(n, k, c)$ denote the class of connected c-cyclic graphs with n vertices and k pendant vertices. Recently, the unique extremal graph, which has greatest (respectively, signless) Laplacian spectral radius, in $\mathbb{C}(n, k, c)$ has been determined for $0 \le c \le 3$, $k \ge 1$ and $n \ge 2c + k + 1$. In this paper, the unique graph with greatest (respectively, signless) Laplacian spectral radius in $\mathbb{C}(n, k, c)$ is determined for $c \ge 0$, $k \ge 1$ and $n \ge 2c + k + 1$.

Key words. (Signless) Laplacian spectral radius, c-Cyclic graph, Pendant vertex.

AMS subject classifications. 05C50, 05C75, 05C05.

1. Introduction. Throughout the paper, $G = (V, E)$ is a connected undirected simple graph with $V = \{v_1, v_2, \ldots, v_n\}$. Let $N(v)$ be the neighbor set of vertex v, and let $d(v)$ be the degree of v. When $d(v) = 1$, we call v a *pendant vertex* of G. In the following, we enumerate the degrees of G in non-increasing order, i.e., $d_1 \ge d_2 \ge \cdots \ge d_n$, where $d_i = d(v_i)$.

If G contains n vertices and $n + c - 1$ edges, then G is called a *c-cyclic graph*. In particular, G is called a tree, unicyclic graph, bicyclic graph or a tricyclic graph if $c = 0$, 1, 2 or 3, respectively. In the coming discussion, n and k are two positive integers, and c is a nonnegative integer. Let $\mathbb{C}(n, k, c)$ denote the class of connected c-cyclic graphs with n vertices and k pendant vertices.

Let P_n and C_n be a path and a cycle on n vertices, respectively. Generally, C_3 is called a *triangle* and C_4 is called a *quadrilateral*. Suppose u is a vertex of a graph G. Suppose $P_s = w_1 w_2 \cdots w_s$ and $C_q = v_1 v_2 \cdots v_q v_1$, where $w_i \notin V(G)$ for $1 \le i \le s$ and $v_j \notin V(G)$ for $1 \le j \le q$. If we obtain G' by identifying the vertex u with w_1, then we say that G' is obtained from G by *attaching the path P_s to u of G*. Similarly,

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if we obtain G' by identifying the vertex u with v_1, then we say that G' is obtained from G by *attaching the cycle* C_q *to* u *of* G.

Paths $P_{l_1}, \ldots, P_{l_k}$ are said to have *almost equal lengths* if $l_1, \ldots, l_k$ satisfy $|l_i - l_j| \leq 1$ for $1 \leq i \leq j \leq k$. Denote by $F_n(k, C_4^{(t)}, C_3^{(c-t)})$ the unique connected c-cyclic graph on n vertices obtained by attaching t quadrilaterals, $c-t$ triangles, and k paths of almost equal lengths, respectively, to a common vertex. Let $F_n(k, C_3^{(c)})$ be the c-cyclic graph on n vertices obtained by attaching k paths of almost equal lengths and c triangles, respectively, to a common vertex, and let $F_n(k, C_4^{(c)})$ define the c-cyclic graph on n vertices obtained by attaching k paths of almost equal lengths and c quadrilaterals, respectively, to a common vertex. It follows that $F_n(k, C_3^{(c)}) = F_n(k, C_4^{(0)}, C_3^{(c)})$ and $F_n(k, C_4^{(c)}) = F_n(k, C_4^{(c)}, C_3^{(0)})$.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$ and the *signless Laplacian matrix* of G is $Q(G) = D(G) + A(G)$. Denote by $\lambda(G)$ and $\mu(G)$, respectively, the Laplacian spectral radius and signless Laplacian spectral radius of G. Thus, $\lambda(G)$ and $\mu(G)$ are equal to the largest eigenvalues of $L(G)$ and $Q(G)$, respectively. It is well-known that $Q(G)$ is positive semidefinite and nonnegative, and, when G is connected, it is irreducible [9]. Thus, when G is connected, by the famous Perron-Frobenius Theorem of non-negative matrices (see e.g. [5]), it follows that $\mu(H) < \mu(G)$ holds for any proper subgraph H of G.

We call G an *extremal graph in* $\mathbb{C}(n, k, c)$ *of first* (*respectively, second*) *type* if G has greatest signless Laplacian (respectively, Laplacian) spectral radius in $\mathbb{C}(n, k, c)$.

Recently, the extremal graphs, which have greatest (signless) Laplacian spectral radii, in $\mathbb{C}(n, k, c)$ has been studied [1, 3, 4, 8, 10, 11, 14, 15]. From these recent results, we can conclude that: $F_n(k, C_3^{(0)})$ is the unique extremal tree of $\mathbb{C}(n, k, 0)$ [8, 14]; $F_n(k, C_3^{(1)})$ is the unique extremal unicyclic graph in $\mathbb{C}(n, k, 1)$ of first type [10, 15] when $n \geq k + 3$ and $F_n(k, C_4^{(1)})$ is the unique extremal unicyclic graph in $\mathbb{C}(n, k, 1)$ of second type [3, 10] when $n \geq k + 4$; $F_n(k, C_3^{(2)})$ is the unique extremal bicyclic graph in $\mathbb{C}(n, k, 2)$ of first type [1, 11] when $n \geq k + 5$ and $F_n(k, C_4^{(2)})$ is the unique extremal bicyclic graph in $\mathbb{C}(n, k, 2)$ of second type [3, 10] when $n \geq k + 7$; $F_n(k, C_3^{(3)})$ is the unique extremal tricyclic graph in $\mathbb{C}(n, k, 3)$ of first type [8, 11] when $n \geq k + 7$ and $F_n(k, C_4^{(3)})$ is the unique extremal tricyclic graph in $\mathbb{C}(n, k, 3)$ of second type [4] when $n \geq k + 10$.

In this paper, the unique extremal graph in $\mathbb{C}(n, k, c)$ of first (respectively, second) type is identified for $c \geq 0$, $k \geq 1$ and $n \geq 2c + k + 1$. Thus, the main results of [1, 3, 4, 8, 10, 11] immediately follow from our results. Our main results can be stated

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as follows.

THEOREM 1.1. *If $k \geq 1$, $c \geq 0$ and $n \geq 2c+k+1$, then $F_n(k, C_3^{(c)})$ is the unique extremal graph in $\mathbb{C}(n,k,c)$ of first type.*

THEOREM 1.2. *Suppose $k \geq 1$, $c \geq 0$ and $n \geq 2c+k+1$.*

- *(i) If $n \geq 3c+k+1$, then $F_n(k, C_4^{(c)})$ is the unique extremal graph in $\mathbb{C}(n,k,c)$ of second type.*
- *(ii) If $n = 2c+k+1+t$ and $0 \leq t \leq c-1$, then $F_n(k, C_4^{(t)}, C_3^{(c-t)})$ is the unique extremal graph in $\mathbb{C}(n,k,c)$ of second type.*

2. Some preliminaries. The graph $W_G(uv)$ is obtained from G by subdividing the edge uv, i.e., adding a new vertex w and edges wu, wv in $G-uv$, where $uv \in E(G)$. An *internal path*, say $P = v_1 \cdots v_{s+1}$ $(s \geq 1)$, is a path joining v_1 and v_{s+1} (which need not be distinct) such that the degrees of v_1 and v_{s+1} are greater than 2, while all other vertices $v_2, \ldots, v_s$ are of degree 2.

LEMMA 2.1. [10] *Let uv be an edge in an internal path of a connected graph G. Then, $\mu(G) > \mu(W_G(uv))$.*

Suppose v is a vertex of G with at least two vertices. Let $G_{t,l}$ $(l \geq t \geq 2)$ be the graph obtained from G by attaching two new paths $P_t = v_1v_2 \cdots v_t$ and $P_l = u_1u_2 \cdots u_l$, respectively, to v of G. Let $G_{t-1,l+1} = G_{t,l} - v_{t-1}v_t + u_lv_t$.

LEMMA 2.2. [10] *Let G be a connected graph with at least two vertices. If $l \geq t \geq 2$, then $\mu(G_{t,l}) > \mu(G_{t-1,l+1})$.*

Let $m(v)$ denote the average of the degrees of the vertices adjacent to v, i.e., $m(v) = \sum_{u \in N(v)} d(u)/d(v)$. Next we shall introduce some bounds for $\lambda(G)$ and $\mu(G)$, which will play prominent roles in the proof of our main results.

LEMMA 2.3. [12, 13] *If G is a connected graph with n vertices, then $\mu(G) \geq \lambda(G) \geq d_1 + 1$, where the first equality holds if and only if G is bipartite, and the second equality holds if and only if $d_1 = n-1$.*

LEMMA 2.4. [6, 13] *If G is connected, then*

$$\mu(G) \leq 2 + \sqrt{(d_1 + d_2 - 2)(d_1 + d_3 - 2)},$$

where the equality holds if and only if G is regular or a star or a path with four vertices.

LEMMA 2.5. [7, 13] *If G is connected, then*

$$\mu(G) \leq \max\left\{\frac{d(u)(d(u)+m(u)) + d(v)(d(v)+m(v))}{d(u)+d(v)} : uv \in E(G)\right\},$$

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where the equality holds if and only if G is regular or a bipartite semiregular graph.

LEMMA 2.6. [2] *Let v be a vertex of a connected graph G. Suppose that $v_1, \dots, v_s$ are pendant vertices of G which are adjacent to v. Let G^* be the graph obtained from G by adding any b $\left(1 \le b \le \frac{s(s-1)}{2}\right)$ edges among $v_1, \dots, v_s$. Then, $\lambda(G) = \lambda(G^*)$.*

3. The proofs of Theorems 1.1 and 1.2. The following simple necessary condition turns out to be surprisingly useful in the proof of our main results.

LEMMA 3.1. *Suppose G is a graph of $\mathbb{C}(n,k,c)$, where $c \ge 2$ and $k \ge 1$. If either $\lambda(G) \ge k+2c+1$ or $\mu(G) \ge k+2c+1$, then G is obtained by attaching k paths and c cycles, respectively, to a common vertex.*

Proof. Suppose the degree sequence of G is $(d_1, d_2, \dots, d_n)$. Since $G \in \mathbb{C}(n,k,c)$, we have $2(n+c-1) = \sum_{i=1}^n d_i$.

If $d_1 + d_2 \ge k+2c+3$, then $2(n+c-1) = \sum_{i=1}^n d_i \ge k+2c+3+2(n-2-k)+k = 2n+2c-1$, a contradiction. By Lemmas 2.3–2.4 and the facts that $c \ge 2$ and $k \ge 1$, if $d_1+d_2 \le k+2c+1$, then

$$\begin{aligned}\lambda(G) \le \mu(G) &< 2+\sqrt{(d_1+d_2-2)(d_1+d_3-2)}\\ &\le 2+\sqrt{(d_1+d_2-2)^2} = d_1+d_2 \le k+2c+1,\end{aligned}$$

a contradiction.

Thus, $\lambda(G) \ge k+2c+1$ or $\mu(G) \ge k+2c+1$ implies that $d_1+d_2 = k+2c+2$. Note that $2(n+c-1) = \sum_{i=1}^n d_i$ and G contains exactly k pendant vertices. So, $d_3 = \cdots = d_{n-k} = 2$ and $d_{n-k+1} = d_{n-k+2} = \cdots = d_n = 1$. Since $d_1 = k+2c+2-d_2 \le k+2c$, we divide the proof into the following three cases.

Case 1. $d_1 \le k+2c-2$.

By Lemmas 2.3–2.4,

$$\begin{aligned}\lambda(G) \le \mu(G) &\le 2+\sqrt{(d_1+d_2-2)(d_1+d_3-2)} = 2+\sqrt{(k+2c)d_1}\\ &\le 2+\sqrt{(k+2c)(k+2c-2)} < k+2c+1,\end{aligned}$$

a contradiction.

Case 2. $d_1 = k+2c-1$.

Then, $d_2 = 3$, which implies that $d(w) \in \{k+2c-1, 3, 2, 1\}$ holds for any $w \in V(G)$. By $c \ge 2$ and $k \ge 1$, G is neither a regular nor a bipartite semiregular graph. According to Lemmas 2.3 and 2.5, we have

$$\lambda(G) \le \mu(G) < \max\left\{\frac{d(u)(d(u)+m(u))+d(v)(d(v)+m(v))}{d(u)+d(v)}, uv \in E(G)\right\}. \quad (3.1)$$

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Suppose that $f(u_0v_0) = \max\left\{\frac{d(u)(d(u)+m(u))+d(v)(d(v)+m(v))}{d(u)+d(v)}, uv \in E(G)\right\}$ occurs at the edge u_0v_0, where $d(u_0) \geq d(v_0)$. Then, $d(u_0) \in \{k+2c-1, 3, 2\}$, as G is connected and $c \geq 2$.

Subcase 2.1. $d(u_0) = k+2c-1$.

Then, $d(u_0) = d_1 \geq 4 > d(v_0)$.

If $d(v_0) = 3$, then $d(u_0) = d_1$ and $d(v_0) = d_2$. By inequality (3.1),

$$\begin{aligned}\mu(G) < f(u_0v_0) &\leq \frac{d_1^2 + d_2 + 2(d_1-1) + d_2^2 + d_1 + 2(d_2-1)}{d_1+d_2} \\ &= k+2c-1+\frac{14}{k+2+2c} \leq k+2c+1,\end{aligned}$$

a contradiction.

If $d(v_0) = 2$, then by inequality (3.1),

$$\mu(G) < f(u_0v_0) \leq \frac{d_1^2 + 3 + 2(d_1-1) + 4 + d_1 + 3}{d_1+2} = k+2c+\frac{6}{k+2c+1} \leq k+2c+1,$$

a contradiction.

If $d(v_0) = 1$, then by inequality (3.1),

$$\mu(G) < f(u_0v_0) \leq \frac{d_1^2 + 1 + 3 + 2(d_1-2) + 1 + d_1}{d_1+1} = k+2c+1-\frac{1}{k+2c} < k+2c+1,$$

a contradiction.

Subcase 2.2. $d(u_0) = 3$.

Since $d_1 \geq 4$ and $d(u_0) = 3 = d_2 > d_3$, we have $1 \leq d(v_0) \leq 2$.

If $d(v_0) = 2$, then by inequality (3.1),

$$\mu(G) < f(u_0v_0) \leq \frac{d_2^2 + d_1 + 2(d_2-1) + 4 + d_1 + d_2}{d_2+2} = \frac{2(k+2c)+18}{5} < k+2c+1,$$

a contradiction.

If $d(v_0) = 1$, then by inequality (3.1),

$$\mu(G) < f(u_0v_0) \leq \frac{d_2^2 + d_1 + 2 + 1 + 1 + d_2}{d_2+1} = \frac{k+2c+15}{4} < k+2c+1,$$

a contradiction.

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Subcase 2.3. $d(u_0) = 2$.

Then, $1 \leq d(v_0) \leq 2$.

If $d(v_0) = 2$, then by inequality (3.1),

$$\lambda(G) \leq \mu(G) < f(u_0v_0) \leq \frac{2(4 + d_1 + 2)}{2 + 2} = \frac{k + 2c + 5}{2} \leq k + 2c + 1,$$

a contradiction.

If $d(v_0) = 1$, then by inequality (3.1),

$$\lambda(G) \leq \mu(G) < f(u_0v_0) \leq \frac{4 + d_1 + 1 + 1 + 2}{2 + 1} = \frac{k + 2c + 7}{3} < k + 2c + 1,$$

a contradiction.

Case 3. $d_1 = k + 2c$.

Then, $d_2 = 2$, and hence $d_2 = \cdots = d_{n-k} = 2$ and $d_{n-k+1} = \cdots = d_n = 1$, which implies that G is obtained by attaching k paths and c cycles to a common vertex. □

Proof of Theorem 1.1. When $0 \leq c \leq 1$, the result had been proved in [8, 10, 14, 15]. So, we may suppose that $c \geq 2$ and G is an extremal graph in $\mathbb{C}(n, k, c)$ of first type in the sequel.

Since $F_n(k, C_3^{(c)}) \in \mathbb{C}(n, k, c)$ and $F_n(k, C_3^{(c)})$ is non-bipartite, by the choice of G and Lemma 2.3, $\mu(G) \geq \mu(F_n(k, C_3^{(c)})) > k + 2c + 1$. Thus, by Lemma 3.1, G is obtained by attaching k paths and c cycles, respectively, to a common vertex, say u_0.

Suppose that G contains a cycle, say C, of length at least four. Let u, v and w be three vertices of C such that $uv \in E(C)$, $vw \in E(C)$ and $u_0 \notin \{u, v, w\}$. Suppose x is a pendant vertex of G. Let $G_1 = G + uw - uv - vw$, $G_2 = G_1 - v$ and let $G_3 = G_1 + xv$. Then, $G_3 \in \mathbb{C}(n, k, c)$. Since $c \geq 2$, uw lies on an internal path of G_2 and $G = W_{G_2}(uw)$. By Lemma 2.1, $\mu(G) < \mu(G_2)$. Furthermore, since $G_1 \subset G_3$, we have $\mu(G_1) < \mu(G_3)$. Thus, $\mu(G) < \mu(G_2) = \mu(G_1) < \mu(G_3)$, contrary to the choice of G. So, every cycle of G is a triangle, and hence G is obtained by attaching k paths (say $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$, where $l_i \geq 2$ for $1 \leq i \leq k$) and c triangles, respectively, to u_0.

If there exists two paths, the length of which differ at least two, without loss of generality, we suppose that $l_1 - l_2 \geq 2$. Suppose $P_{l_1} = u_0w_1w_2 \cdots w_{l_1-1}$ and $P_{l_2} = u_0z_1z_2 \cdots z_{l_2-1}$. Let $G_5 = G - w_{l_1-2}w_{l_1-1} + z_{l_2-1}w_{l_1-1}$. Then, $G_5 \in \mathbb{C}(n, k, c)$. By Lemma 2.2, we have $\mu(G) < \mu(G_5)$, which contradicts the choice of G. Thus, the k paths have almost equal lengths and hence the result follows. □

Suppose $P_t = u_1u_2 \cdots u_t$ is a path. If $d(u_2) = d(u_3) = \cdots = d(u_{t-1}) = 2$ and $d(u_t) = 1$, then P_t is called a *pendant path*. Denote by $g(G)$ the *girth*, i.e., the length

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of a shortest cycle of G. Let $F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$ be the connected $(c-s)$-cyclic graph obtained from $F_n(k, C_4^{(c-s)}, C_3^{(s)})$ by deleting s edges, the degrees of whose end vertices are two, in the s triangles of $F_n(k, C_4^{(c-s)}, C_3^{(s)})$. In other words, $F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$ is obtained from $F_{n-2s}(k, C_4^{(c-s)})$ by attaching $2s$ pendant edges to the vertex of degree $k+2(c-s)$ of $F_{n-2s}(k, C_4^{(c-s)})$.

Lemma 3.2. *Suppose G is a connected c-cyclic graph on n vertices obtained by attaching k paths, s triangles, and $c-s$ cycles of order at least four, respectively, to a common vertex, where $1 \le s \le c$ and $k \ge 1$. Then,*

$$\lambda(G) \le \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})),$$

where the first equality holds if and only if $G = F_n(k, C_4^{(c-s)}, C_3^{(s)})$.

Proof. Let G_1 be the connected $(c-s)$-cyclic graph obtained from G by deleting s edges, the degrees of whose end vertices are two, in the s triangles of G. Then, $g(G_1) \ge 4$. Suppose u_0 has the maximum degree of G. Then, $d(u_0) = k+2c$. By Lemma 2.6, $\lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)}))$ and $\lambda(G) = \lambda(G_1)$. We consider the following two cases.

Case 1. The length of every cycle of G_1 is four.

Then, G_1 is a connected $(c-s)$-cyclic graph obtained by attaching $k+2s$ paths (among which at least $2s$ paths have lengths 1), and $c-s$ quadrilaterals, respectively, to u_0. Suppose that $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ are k pendant paths of G_1 with the first k largest lengths among all the pendant paths of G_1. If there exists two pendant paths of $\{P_{l_1}, P_{l_2}, \ldots, P_{l_k}\}$, the length of which differ at least two, without loss of generality, we may suppose that $l_1 - l_2 \ge 2$. Suppose $P_{l_1} = u_0 w_1 w_2 \cdots w_{l_1-1}$ and $P_{l_2} = u_0 z_1 z_2 \cdots z_{l_2-1}$. Let $G_2 = G_1 - w_{l_1-2}w_{l_1-1} + z_{l_2-1}w_{l_1-1}$. By Lemma 2.2, $\mu(G_1) < \mu(G_2)$. Repeating the above process, we see by Lemma 2.2 that $\mu(G_1) \le \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$, where the equality holds if and only if $G_1 = F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$ namely, $G = F_n(k, C_4^{(c-s)}, C_3^{(s)})$.

Since $\lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$ is bipartite, $\mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$ by Lemma 2.3. Thus, by Lemma 2.3, we have

$$\lambda(G) = \lambda(G_1) \le \mu(G_1) \le \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})).$$

If $\lambda(G) = \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$, then $\mu(G_1) = \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$, and hence $G = F_n(k, C_4^{(c-s)}, C_3^{(s)})$. Conversely, if $G = F_n(k, C_4^{(c-s)}, C_3^{(s)})$, then $\lambda(G) = \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$.

Case 2. G_1 contains a cycle, say C, of length at least five.

Let u, v and w be three vertices of C such that $uv \in E(C)$, $vw \in E(C)$ and $u_0 \notin \{u, v, w\}$. Let P be a longest pendant path of G_1 with initial vertex u_0, and

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let x be the pendant vertex of P. Let $G_2 = G_1 + uw - uv - vw$, $G_3 = G_2 - v$ and let $G_4 = G_2 + xv$. Since $d(u_0) = k + 2c \geq 3$, uw lies on an internal path of G_3 and $G_1 = W_{G_3}(uw)$. By Lemma 2.1, $\mu(G_1) < \mu(G_3)$. Furthermore, since $G_2 \subset G_4$, we have $\mu(G_2) < \mu(G_4)$. Thus, $\mu(G_1) < \mu(G_3) = \mu(G_2) < \mu(G_4)$.

Note that G_4 contains exactly $k + 2s$ pendant vertices, and there are at least $2s$ pendant vertices being adjacent to u_0 in G_4. By repeating the above process, we see that there exists some $(c - s)$-cyclic graph, say G_5, such that $\mu(G_4) \leq \mu(G_5)$, where G_5 is obtained by attaching $k + 2s$ paths (among which at least $2s$ paths have lengths 1) and $c - s$ quadrilaterals, respectively, to u_0. By the proof of Case 1, we have $\mu(G_5) \leq \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$.

Since $F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$ is bipartite, Lemma 2.3 implies that $\mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$. Now, by Lemma 2.3, we can conclude that

$$\lambda(G) = \lambda(G_1) \leq \mu(G_1) < \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})).$$

This completes the proof of this result. $\square$

LEMMA 3.3. *Suppose* $1 \leq s \leq c$, $k \geq 1$ *and* $n \geq k + 3c + 2 - s$. *Then,* $\lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) < \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$.

Proof. Suppose u_0 has the maximum degree of $F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$. Then, $d(u_0) = k + 2c$. Let P be a longest pendant path of $F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$ with initial vertex u_0, and let y be the pendant vertex of P. If $|V(P)| = 2$, then $n \leq 2s + 3(c - s) + 1 + k = k + 3c + 1 - s$, a contradiction. Thus, $|V(P)| \geq 3$. Let x be a pendant vertex, which is adjacent to u_0.

Let $G_1 = F_n^*(k, C_4^{(c-s)}, C_3^{(s)}) + xy$. Then, $g(G_1) \geq 4$ and G_1 is obtained by attaching $k + 2(s - 1)$ paths (among which at least $2(s - 1)$ paths have lengths 1), $c - s$ quadrilaterals and a cycle (say C_q, where $q \geq 4$), respectively, to u_0.

If $q = 4$, then by Lemma 2.2, $\mu(G_1) \leq \mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$. If $q \geq 5$, then since $d(u_0) = k + 2c \geq 3$, by Lemma 2.1 there exists some graph, say G_2, such that $\mu(G_1) < \mu(G_2)$, where G_2 is obtained by attaching $k + 2(s - 1)$ paths (among which at least $2(s-1)$ paths have lengths 1) and $c-s+1$ quadrilaterals, respectively, to u_0. Now, Lemma 2.2 implies that $\mu(G_2) \leq \mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$. Thus, $\mu(G_1) \leq \mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$.

Note that $F_n^*(k, C_4^{(c-s)}, C_3^{(s)}) \subset G_1$. Then, $\mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) < \mu(G_1)$. By Lemma 2.6, we have $\lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) = \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$ and $\lambda(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})) = \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$. Since $F_n^*(k, C_4^{(c-s)}, C_3^{(s)})$ and $F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})$ are two bipartite graphs, by Lemma 2.3, $\lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)}))$ and $\mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})) = \lambda(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$.

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Now, we can conclude that

$$\begin{aligned}\lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) &= \lambda(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) = \mu(F_n^*(k, C_4^{(c-s)}, C_3^{(s)})) \\ &< \mu(G_1) \leq \mu(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})) \\ &= \lambda(F_n^*(k, C_4^{(c-s+1)}, C_3^{(s-1)})) = \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)})).\end{aligned}$$

Thus, $\lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) < \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)}))$. □

Proof of Theorem 1.2 (*i*). When $0 \leq c \leq 1$, the result had been proved in [3, 8, 10, 14]. So, we may suppose that $c \geq 2$ and G is an extremal graph in $\mathbb{C}(n, k, c)$ of second type in the sequel.

Note that $F_n(k, C_4^{(c)}) \in \mathbb{C}(n, k, c)$. By Lemma 2.3 and the choice of G, $\lambda(G) \geq \lambda(F_n(k, C_4^{(c)})) > k + 2c + 1$. Thus, G is obtained by attaching k paths and c cycles, respectively, to a common vertex by Lemma 3.1. We consider the following two cases.

Case 1. $g(G) \geq 4$.

If every cycle of G is a quadrilateral, by Lemma 2.2 it follows that $\mu(G) \leq \mu(F_n(k, C_4^{(c)}))$, where the equality holds if and only if $G = F_n(k, C_4^{(c)})$. If G contains at least one cycle of length at least five, since $c \geq 2$, by Lemma 2.1 there exists some c-cyclic graph, say G_1, such that $\mu(G) < \mu(G_1)$, where G_1 is obtained by attaching k paths and c quadrilaterals, respectively, to a common vertex. Furthermore, Lemma 2.2 implies that $\mu(G_1) \leq \mu(F_n(k, C_4^{(c)}))$, and hence, $\mu(G) < \mu(F_n(k, C_4^{(c)}))$.

So, we can conclude that $\mu(G) \leq \mu(F_n(k, C_4^{(c)}))$, where the equality holds if and only if $G = F_n(k, C_4^{(c)})$. Since $F_n(k, C_4^{(c)})$ is bipartite, by Lemma 2.3 we have $\lambda(G) \leq \mu(G) \leq \mu(F_n(k, C_4^{(c)})) = \lambda(F_n(k, C_4^{(c)}))$. Now, if $\lambda(G) = \lambda(F_n(k, C_4^{(c)}))$, then $\mu(G) = \mu(F_n(k, C_4^{(c)}))$, and hence, $G = F_n(k, C_4^{(c)})$.

Case 2. $g(G) = 3$.

We may suppose that G contains exactly $s \geq 1$ triangles. By Lemma 3.2, $\lambda(G) \leq \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)}))$. Since $s \geq 1$, Lemma 3.3 implies that

$$\lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) < \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)})) \leq \cdots \leq \lambda(F_n(k, C_4^{(c)}, C_3^{(0)})).$$

Thus, $\lambda(G) < \lambda(F_n(k, C_4^{(c)}, C_3^{(0)})) = \lambda(F_n(k, C_4^{(c)}))$. □

Proof of Theorem 1.2 (*ii*). When $c = 1$, then $n = k + 3$, and the result clearly follows. So, we may suppose that $c \geq 2$ and G is an extremal graph in $\mathbb{C}(n, k, c)$ of second type in the sequel. Since $F_n(k, C_4^{(t)}, C_3^{(c-t)}) \in \mathbb{C}(n, k, c)$, by Lemma 2.3 and the choice of G, $\lambda(G) \geq \lambda(F_n(k, C_4^{(t)}, C_3^{(c-t)})) \geq k + 2c + 1$. Thus, G is obtained by attaching k paths and c cycles, respectively, to a common vertex by Lemma 3.1.

Note that $n \leq 3c+k$. So, we may suppose that G contains exactly $s \geq 1$ triangles.

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If $s \le c-t-1$, then

$$\begin{aligned} n &\ge 2s + 3(c-s) + k + 1 = 3c + k + 1 - s \\ &\ge 3c + k + 1 - (c-t-1) = k + 2c + t + 2, \end{aligned}$$

a contradiction. Thus, $s \ge c - t$.

If $s = c-t$, then by Lemma 3.2, $\lambda(G) \le \lambda(F_n(k, C_4^{(t)}, C_3^{(c-t)}))$, where the equality holds if and only if $G = F_n(k, C_4^{(t)}, C_3^{(c-t)})$. If $s \ge c-t+1$, then since $n = 2c+k+1+t \ge 3c+k+2-s$, by Lemmas 3.2–3.3, we can conclude that

$$\begin{aligned} \lambda(G) &\le \lambda(F_n(k, C_4^{(c-s)}, C_3^{(s)})) < \lambda(F_n(k, C_4^{(c-s+1)}, C_3^{(s-1)})) \\ &\le \cdots \le \lambda(F_n(k, C_4^{(t)}, C_3^{(c-t)})). \quad \square \end{aligned}$$

4. Further discussion. By Theorems 1.1 and 1.2, the unique extremal graph in $\mathbb{C}(n,k,c)$ of first type and the unique extremal graph in $\mathbb{C}(n,k,c)$ of second type are, respectively, determined for $c \ge 0$, $k \ge 1$ and $n \ge 2c+k+1$.

When $c \ge 0$, $k \ge 1$ and $n \le 2c+k$, for any $G \in \mathbb{C}(n,k,c)$, by Lemma 2.3 we have $\lambda(G) \le n$, where the equality holds if and only if $d_1 = n-1$. Furthermore, when $c \ge 3$, the extremal graphs in $\mathbb{C}(n,k,c)$ of second type are always not unique. For instance, let W_1 and W_2 be the two tricyclic graphs on n vertices as shown in Fig. 4.1. Then, $\{W_1, W_2\} \subseteq \mathbb{C}(n,k,3)$. Since $d_1(W_1) = n-1 = d_1(W_2)$, by Lemma 2.3 we have $\lambda(W_1) = n = \lambda(W_2)$.

When $c \ge 0$, $k \ge 1$ and $n \le 2c+k$, it is still an open problem to characterize the extremal graphs in $\mathbb{C}(n,k,c)$ of first type.

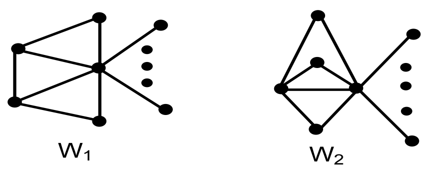

FIG. 4.1. *The tricyclic graphs W_1 and W_2.*

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