Electronic Journal of Linear Algebra

Volume 23 *ELA Volume 23 (2012)* Article 67

2012

Eventual properties of matrices

Leslie Hogben
hogben@aimath.org

Ulrica Wilson

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

EVENTUAL PROPERTIES OF MATRICES*

LESLIE HOGBEN† AND ULRICA WILSON‡

Abstract. An eventual property of a matrix $M \in \mathbb{C}^{n\times n}$ is a property that holds for all powers $M^k, k \geq k_0$, for some positive integer k_0. Eventually positive and eventually nonnegative matrices have been studied extensively, and some results are known for eventually r-cyclic matrices. This paper introduces and establishs properties of eventually reducible matrices, establishs properties of the eigenstructure of eventually r-cyclic matrices, and answers affirmatively a question of Zaslavsky and Tam about eventually nonnegative matrices.

Key words. Eventually reducible matrix, Eventually nonnegative matrix, Eventually r-cyclic matrix.

AMS subject classifications. 15A21, 15A18.

1. Introduction.

A matrix $M \in \mathbb{C}^{n\times n}$ is *eventually positive* (respectively, *eventually nonnegative*) if there exists a positive integer k_0 such that for all $k \geq k_0$, $M^k > 0$ (respectively, $M^k \geq 0$). Eventually positive matrices and eventually nonnegative matrices have applications to control theory and have been studied extensively since their introduction in [5] by Friedland; see [2, 3, 4, 6, 7, 8, 10, 11] and the references therein. A matrix M is *eventually irreducible* if there exists a positive integer k_0 such that for all $k \geq k_0$, M^k is irreducible. Eventually irreducible matrices were introduced by Zaslavsky and Tam in [11], where they showed a matrix that is eventually irreducible and eventually nonnegative is eventually positive. Eventually r-cyclic matrices were introduced in [6].

Following the usual protocol, a matrix is *eventually reducible* if there exists a positive integer k_0 such that for all $k \geq k_0$, M^k is reducible. Any irreducible nilpotent matrix of order at least 2, such as $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, is an example of an eventually reducible matrix that is not reducible. A more interesting example is given in Example 3.1 in Section 3. Clearly, an eventually nonnegative matrix will either have an irreducible nonnegative power so the analysis in [6] applies, or every nonnegative power is reducible, in which case the matrix is eventually reducible. Thus, results about eventually reducible matrices provide information for the study of eventually nonnegative matrices.

*Received by the editors on February 19, 2012. Accepted for publication on December 6, 2012. Handling Editor: Bit-Shun Tam.

†Department of Mathematics, Iowa State University, Ames, IA 50011, USA (lhogben@iastate.edu), and American Institute of Mathematics, 360 Portage Ave, Palo Alto, CA 94306 (hogben@aimath.org).

‡Department of Mathematics, Morehouse College, Atlanta, GA, USA 30314 (uwilson@morehouse.edu). Research supported by Woodrow Wilson National Fellowship Foundation.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

In [11], Zaslavsky and Tam raise a variety of open questions concerning eventually nonnegative matrices, and in Section 2, we answer one of these questions affirmatively for several eventual properties that include eventually nonnegative matrices, eventually positive matrices, eventually reducible matrices (see Theorem 2.9 below); the analogous result for eventually r-cyclic matrices is established in Section 4 (see Theorem 4.3 below). In Section 2, we also show that if M is eventually reducible, then it is eventually reducible with a fixed partition (see Theorem 2.6 below); the analogous result for eventually r-cyclic matrices is established in Section 4 (see Theorem 4.1 below). In Section 3, we establish various properties of eventually reducible matrices. It is shown in [6] that an eventually r-cyclic matrix M having $\operatorname{rank} M^2 = \operatorname{rank} M$ is r-cyclic. We establish the analogous result for eventually reducible matrices, namely that an eventually reducible matrix M having $\operatorname{rank} M^2 = \operatorname{rank} M$ is reducible. The implication of these results is that to determine whether an $n \times n$ matrix M is eventually reducible or eventually r-cyclic, only M^n and M^{n+1} need be checked for the desired property (see Corollary 3.7 and Theorem 4.6 below). In Section 4, we also show that the spectrum of an eventually r-cyclic matrix is invariant under multiplication by a primitive rth root of unity, extending the well known result that the spectrum of an r-cyclic matrix is invariant under such multiplication, and establish relationships between eigenvectors.

In the remainder of this section, we introduce additional notation and terminology. If $M \in \mathbb{C}^{n\times n}$ and $R, C \subseteq \{1, 2, \ldots, n\}$, then $M[R|C]$ denotes the *submatrix* of M whose rows and columns are indexed by R and C, respectively. For an ordered partition $\Pi = (V_1, \ldots, V_s)$ of $\{1, \ldots, n\}$ into $s \geq 2$ nonempty sets, a square matrix M is Π-*reducible* if $M[V_i|V_j] = 0$ for $i > j$. We say M is *reducible* if M is Π-reducible for some such partition Π; M is *irreducible* if M is not reducible.

An ordered partition $\Pi = (V_1, \ldots, V_s)$ of $\{1, \ldots, n\}$ into $s \geq 2$ nonempty sets is *consecutive* if $V_1 = \{1, \ldots, i_1\}, V_2 = \{i_1 + 1, \ldots, i_2\}, \ldots, V_s = \{i_{s-1} + 1, \ldots, n\}$. Observe that if $\Pi = (V_1, \ldots, V_s)$ is a consecutive ordered partition into $s \geq 2$ nonempty sets and $M \in \mathbb{C}^{n\times n}$ is Π-reducible, then

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1,s-1} & M_{1s} \\ 0 & M_{22} & \cdots & M_{2,s-1} & M_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{s-1,s-1} & M_{s-1,s} \\ 0 & 0 & \cdots & 0 & M_{ss} \end{bmatrix},$$

where $M_{ij} = M[V_i|V_j]$.

2. Eventual properties. To discuss matrices that eventually have a property from a general perspective that is valid for several properties, we introduce additional terminology for properties. Defining a class $\mathcal{P}$ of $n \times n$ matrices is equivalent to defining the property of being a member of the class, and we will use both the terms "class" and "property". We use the term *nonempty property* to mean there is a matrix

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

that has property $\mathcal{P}$.

DEFINITION 2.1. Let $\mathcal{P}$ be a nonempty property. A matrix $M \in \mathbb{C}^{n\times n}$ is *eventually* $\mathcal{P}$ if there exists a positive integer k_0 such that for all $k \geq k_0$, $M^k \in \mathcal{P}$.

Definition 2.1 coincides with the usual definition of eventually nonnegative matrices and eventually positive matrices, and with the definition of eventually reducible matrices given in the introduction.

DEFINITION 2.2. A nonempty property $\mathcal{P}$ is *power hereditary* if $M^{k+\ell} \in \mathcal{P}$ whenever $M^k \in \mathcal{P}$ and $M^\ell \in \mathcal{P}$.

Clearly, the properties of nonnegativity and positivity are power hereditary. However, it is not immediately clear that reducibility is a power hereditary property because the partition used to reduce M^k might vary with k. Similarly, in the definition of an eventually reducible matrix M given above, the partition used to reduce M^k might vary with k. The next definition fixes the partition to avoid this potential difficulty.

DEFINITION 2.3. For an ordered partition $\Pi = (V_1, \ldots, V_s)$ of $\{1, \ldots, n\}$ into $s \geq 2$ nonempty sets, a matrix M is *eventually* Π*-reducible* if there exists a positive integer k_0 such that for all $k \geq k_0$, M^k is Π-reducible.

Clearly, the property of Π-reducibility is power hereditary. To establish the equivalence of eventual reducibility and eventual Π-reducibility, we need the well known "postage stamp" lemma. This follows from the fact that if S is a nonempty set of positive integers that is closed under addition and $d = \gcd(S)$, then there exists a positive integer t_0 such that for all $t \geq t_0$, $td \in S$ [1, Lemma 3.4.2].

LEMMA 2.4. *Let $k_1, k_2, \ldots, k_r \in \mathbb{Z}^+$ with $\gcd(k_1, k_2, \ldots, k_r) = d$. Then there exists $K = K(k_1, k_2, \ldots, k_r)$ such that for all $k \geq K$, $kd = \sum_{i=1}^{r} x_i k_i$ for some nonnegative integers $x_1, x_2, \ldots, x_r$.*

The next corollary has been observed by various authors for the properties eventually positive and eventually nonnegative in the case $r = 2$.

COROLLARY 2.5. *Let $k_1, k_2, \ldots, k_r \in \mathbb{Z}^+$ with $\gcd(k_1, k_2, \ldots, k_r) = 1$. If $\mathcal{P}$ is a power hereditary property and $M^{k_i} \in \mathcal{P}$ for all $i = 1, \ldots, r$ then M is eventually $\mathcal{P}$.*

Proof. Since $\gcd(k_1, k_2, \ldots, k_r) = 1$, by Lemma 2.4 there exists an integer k_0 such that every integer $k \geq k_0$ is a nonnegative linear combination of $k_1, k_2, \ldots, k_r$. Since $M^{k_i} \in \mathcal{P}$ for all $i = 1, \ldots, r$ and $\mathcal{P}$ is power hereditary, $M^k \in \mathcal{P}$ for every $k \geq k_0$. ☐

Corollary 2.5 applies to Π-reducible matrices, and we use this to show that the two definitions related to eventual reducibility are equivalent.

THEOREM 2.6. *Every eventually reducible matrix $M \in \mathbb{C}^{n\times n}$ is eventually* Π-*reducible for some ordered partition* Π.

Proof. Suppose M is not eventually Π-reducible for any ordered partition Π of

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

$\{1,\dots,n\}$ into $s \geq 2$ nonempty sets. Let $P = \{\Pi_1,\dots,\Pi_p\}$ be the set of ordered partitions of $\{1,\dots,n\}$ into $s \geq 2$ nonempty sets (note that there are only finitely many such partitions). For each ordered partition $\Pi_i \in P$, define

$$\mathcal{S}_i = \{k \in \mathbb{Z}^+ : M^k \text{ is } \Pi_i\text{-reducible}\}.$$

If for some $\Pi_j \in P$, $\gcd(\mathcal{S}_j) = 1$, then by Corollary 2.5 M is eventually Π_j-reducible. So $\gcd(\mathcal{S}_i) = d_i > 1$ for $i = 1,\dots,p$. For $\ell \in \mathbb{Z}^+$, define $u = \ell d_1 \cdots d_p + 1$. Then $u \notin \mathcal{S}_i$ for all $i = 1,\dots,p$. Since M^u is not reducible with any partition, M^u is irreducible. Since there exists $u > \ell$ with M^u irreducible for every $\ell \in \mathbb{Z}^+$, M is not eventually reducible. □

Notice that Corollary 2.5 requires at least two relatively prime powers. It is known that having a single power that is positive (or nonnegative) is not sufficient to ensure a matrix is eventually positive (or eventually nonnegative). Similarly, a reducible power of a matrix does not guarantee the matrix is eventually reducible, as seen in the next example.

Example 2.7. Let $M = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$. Then $M^2 = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$ is reducible, and clearly $M^{2k+1} = 7^k M$ is not reducible, so M is not eventually reducible.

In their study of eventually nonnegative matrices, Zaslavsky and Tam raised the following question.

Question 2.8. [11, Question 6.5] Let $M \in \mathbb{C}^{n\times n}$ and

$$S = \{k : k \text{ is a nonnegative integer and } M^{k+1} \geq 0\}.$$

If for every positive integer p, there exists $k \in S$ such that p divides k, does it follow that M is eventually nonnegative?

We can now answer this question affirmatively not only for nonnegative matrices but also for any property of square matrices that is power hereditary, including positive and reducible; a version appropriate for r-cyclic appears in Section 4.

Theorem 2.9. *Let $\mathcal{P}$ be a power hereditary property. For $M \in \mathbb{C}^{n\times n}$, let*

$$S = \{k : k \text{ is a nonnegative integer and } M^{k+1} \in \mathcal{P}\}.$$

If for every positive integer p there exists $k \in S$ such that p divides k, then M is eventually $\mathcal{P}$.

Proof. Assume the hypothesis. If $S = \emptyset$, then there would be no $k \in S$ such that p divides k, so $S \neq \emptyset$. Define $T = \{k \in \mathbb{Z}^+ : M^k \in \mathcal{P}\}$. Note that $k \in S$ if and only if $k+1 \in T$. Let $\gcd(T) = d$. By hypothesis, there is an $\ell \in S$ such that d divides ℓ. Since d divides $k+1$ for every $k \in S$, d divides $\ell + 1$. Thus $\gcd(T) = 1$, and there is some finite subset $\{k_1, k_2, \dots, k_r\} \subseteq T$ with $\gcd(k_1, k_2, \dots, k_r) = 1$. So M is eventually $\mathcal{P}$ by Corollary 2.5. □

Corollary 2.10. *Let $M \in \mathbb{C}^{n\times n}$ and $S = \{k \in \mathbb{Z} : k \geq 0$ and $M^{k+1} \geq 0\}$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

If for every $p \in \mathbb{Z}^+$, there exists $k \in S$ such that p divides k, then M is eventually nonnegative.

The hypothesis of Theorem 2.9 and Corollary 2.10, that for every positive integer p there exists $k \in S$ such that p divides k, can be weakened to: for every prime p there exists $k \in S$ such that p divides k.

Note that eventually irreducible matrices are fundamentally different from most of the other classes we discuss, where M is $\mathcal{P}$ trivially implies M is eventually $\mathcal{P}$, and the converse is false. The opposite is true for irreducibility: if M is eventually irreducible, then M is necessarily irreducible, but not conversely. Irreducibility is not power hereditary, as any irreducible nilpotent matrix shows.

3. Eventually reducible matrices. We begin this section with an example of an irreducible, eventually reducible matrix M that we decompose into a sum of a reducible matrix M_1 having $\operatorname{rank} {M_1}^2 = \operatorname{rank} M_1$ and a nilpotent matrix M_0 with $M_0M_1 = M_1M_0 = 0$. Theorem 3.5 below shows that every eventually reducible matrix has this type of decomposition.

Example 3.1. The matrix

$$M = \begin{bmatrix} 1 & 1 & -2 & 1 & 0 & -1 \\ 3 & 0 & -3 & 0 & 1 & -1 \\ -4 & -1 & 5 & -1 & -1 & 2 \\ 1 & 1 & 1 & 1 & 1 & -2 \\ 1 & 1 & 1 & 3 & 0 & -3 \\ 1 & 1 & 1 & -4 & -1 & 5 \end{bmatrix}$$

is eventually reducible, because M^2 and M^3 are reducible with ordered partition $\Pi = (\{1, 2, 3\}, \{4, 5, 6\})$. Clearly, M is irreducible. Furthermore, if we define

$$M_1 = \begin{bmatrix} 1 & 1 & -2 & 1 & 0 & -1 \\ 3 & 0 & -3 & 0 & 1 & -1 \\ -4 & -1 & 5 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 3 & 0 & -3 \\ 0 & 0 & 0 & -4 & -1 & 5 \end{bmatrix} \quad \text{and} \quad M_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

we see that $M = M_1 + M_0$, M_1 is reducible, $\operatorname{rank} {M_1}^2 = \operatorname{rank} M_1$, $M_1M_0 = M_0M_1 = 0$, ${M_0}^2 = 0$, and for all $k \geq 2$, $M^k = {M_1}^k$.

For $M \in \mathbb{C}^{n\times n}$ and a consecutive partition $\Pi = (V_1, V_2)$ of $\{1, \ldots, n\}$ with $V_1 = \{1, \ldots, t\}$ and $V_2 = \{t + 1, \ldots, n\}$, we denote the 2×2 block matrix partition of M^k defined by Π as

$$M^k = \begin{bmatrix} (M^k)_{11} & (M^k)_{12} \\ (M^k)_{21} & (M^k)_{22} \end{bmatrix},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

where $(M^k)_{11}$ is $t \times t$. The next lemma gives the key step in the proof that an eventually Π-reducible matrix M with $\operatorname{rank} M = \operatorname{rank} M^2$ is Π-reducible.

LEMMA 3.2. *Suppose* $M \in \mathbb{C}^{n\times n}$, $\operatorname{rank} M^2 = \operatorname{rank} M$ *and there is a consecutive partition* $\Pi = (V_1, V_2)$ *such that* $(M^2)_{21} = (M^3)_{21} = 0$ *in the* 2×2 *block matrix partition defined by* Π*. Then* $M_{21} = 0$*, i.e.,* M *itself is* Π*-reducible.*

Proof. Denote the partitioned form of M as $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A a $t \times t$ matrix. Since $(M^2)_{21} = 0$, we have $M^2 = \begin{bmatrix} A^2 + BC & AB + BD \\ 0 & CB + D^2 \end{bmatrix}$. Since $\operatorname{rank} M^2 = \operatorname{rank} M$, $\ker M^2 = \ker M$. If $\mathbf{v} \in \ker(A^2 + BC)$, then $[\mathbf{v}^T, 0^T]^T \in \ker M^2$, so $\mathbf{v} \in \ker A \cap \ker C$. Thus,

$$\ker(A^2 + BC) = \ker A \cap \ker C. \tag{3.1}$$

Since $(M^3)_{21} = 0$,

$$C(A^2 + BC) = 0.$$

Thus,

$$\operatorname{range}(A^2 + BC) \subseteq \ker C. \tag{3.2}$$

Since $\ker M^4 = \ker M^2$,

$$\ker(A^2 + BC)^2 = \ker(A^2 + BC).$$

Thus,

$$\operatorname{range}(A^2 + BC) \cap \ker(A^2 + BC) = \{0\}, \tag{3.3}$$

From (3.1) and (3.2),

$$\operatorname{range}(A^2 + BC), \ker(A^2 + BC) \subseteq \ker C,$$

and from (3.3), we have

$$\dim(\operatorname{range}(A^2 + BC) + \ker(A^2 + BC)) = \operatorname{rank}(A^2 + BC) + \dim(\ker(A^2 + BC))$$

i.e., $t \le \dim \ker C$. Since C is an $(n - t) \times t$ matrix, we conclude $C = 0$. ◻

Observe that Example 3.1 does not contradict Lemma 3.2 because $\operatorname{rank} M = 5$ but $\operatorname{rank} M^2 = 4$.

THEOREM 3.3. *Let* $M \in \mathbb{C}^{n\times n}$ *and* $\Pi = (V_1, V_2, \ldots, V_s)$ *be an ordered partition of* $\{1, \ldots, n\}$ *into* $s \ge 2$ *nonempty sets. If* $\operatorname{rank} M^2 = \operatorname{rank} M$*, then* M *is eventually* Π*-reducible if and only if* M *is* Π*-reducible.*

Proof. Without loss of generality (by use of a permutation similarity if necessary), we assume $M \in \mathbb{C}^{n\times n}$ is eventually Π-reducible with a consecutive ordered partition

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

Π. First we assume $\Pi = (V_1, V_2)$ and we write M as a 2×2 block matrix $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ with square diagonal blocks. We show by induction on k_0 that $(M^k)_{21} = 0$ for all $k \geq k_0$ implies $M_{21} = 0$. Lemma 3.2 gives us the case $k_0 = 2$. Assume true for $k_0 < \ell$. Assume M is a matrix such that for all integers $k \geq \ell$, $(M^k)_{21} = 0$. Let $u = \ell - 1$ and set $H = M^u$. Then $\operatorname{rank} H = \operatorname{rank} M^u = \operatorname{rank} M^{2u} = \operatorname{rank} H^2$. Since $\ell > 2$, $2u = 2(\ell - 1) > \ell$ and thus $(H^2)_{21} = 0 = (H^3)_{21}$. So, by applying Lemma 3.2 to $H = M^u$, we have $(M^u)_{21} = H_{21} = 0$. Thus, $(M^k)_{21} = 0$ for all $k \geq \ell - 1$, and so by the induction hypothesis, $M_{21} = 0$.

Now suppose that M is eventually reducible with an ordered partition $\Pi = (V_1, V_2, \ldots, V_s)$ into $s > 2$ nonempty sets. For $j = 1, 2, \ldots, s-1$, define an ordered partition into two sets $\Pi_j = (\cup_{i=1}^{j} V_i, \cup_{j+1}^{s} V_i)$. Since M is eventually Π-reducible, M is eventually Π_j-reducible for $j = 1, \ldots, s-1$. Since Π_j is a partition into two sets, M is Π_j-reducible. Since M is Π_j-reducible for $j = 1, \ldots, s-1$, M is Π-reducible. $\square$

The next corollary is a consequence of Theorems 2.6 and 3.3.

Corollary 3.4. *Let $M \in \mathbb{C}^{n\times n}$ have $\operatorname{rank} M^2 = \operatorname{rank} M$. Then M is eventually reducible if and only if M is reducible.*

The *index* of an eigenvalue λ of $M \in \mathbb{C}^{n\times n}$ is the multiplicity of λ as a root of the minimal polynomial of M, or equivalently the maximum size of a Jordan block for λ; if λ is not an eigenvalue, the index of λ for M is zero. For $M \in \mathbb{C}^{n\times n}$, $\operatorname{rank} M^2 = \operatorname{rank} M$ if and only if the index of the number zero for M is at most one. It is shown in [11, Theorem 3.6] that any $M \in \mathbb{C}^{n\times n}$ has a unique decomposition $M = M_1 + M_0$ where the index of the number zero for M_1 is at most one, M_0 is nilpotent, and $M_1 M_0 = M_0 M_1 = 0$. The next theorem follows from this result and Theorem 3.3.

Theorem 3.5. *If Π is an ordered partition of $\{1, \ldots, n\}$ into $s \geq 2$ nonempty sets and $M \in \mathbb{C}^{n\times n}$ is eventually Π-reducible, then for the unique decomposition $M = M_1 + M_0$ such that $\operatorname{rank} {M_1}^2 = \operatorname{rank} M_1$, $M_1 M_0 = M_0 M_1 = 0$, and ${M_0}^n = 0$, M_1 is Π-reducible.*

It is well known from results in [5] that an eventually positive (or nonnegative) $n \times n$ matrix may need to be raised to an arbitrarily large power before it becomes positive. This is not the case for an eventually reducible matrix, as the next two corollaries show.

Corollary 3.6. *Let $\Pi = (V_1, V_2, \ldots, V_s)$ be an ordered partition of $\{1, \ldots, n\}$ into $s \geq 2$ nonempty sets and let $M \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*

1. *M is eventually Π-reducible.*
2. *M^n and M^{n+1} are Π-reducible.*
3. *For every integer $k \geq n$, M^k is Π-reducible.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

Corollary 3.7. *Let $M \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*

1. *M is eventually reducible.*
2. *M^n and M^{n+1} are reducible.*
3. *For every integer $k \geq n$, M^k is reducible.*

Remark 3.8. The extension of Corollary 3.4 to Corollary 3.7 applies to any power hereditary property $\mathcal{P}$ for which M is $\mathcal{P}$ whenever M is eventually $\mathcal{P}$ and $\operatorname{rank} M^2 = \operatorname{rank} M$.

4. Eventually r-cyclic matrices. In this section, we establish properties of an eventually r-cyclic matrix. For $r \geq 2$, a square matrix $B \in \mathbb{C}^{n\times n}$ is called *r-cyclic* if there exists an ordered partition $\Pi = (V_1, \dots, V_r)$ of $\{1, \dots, n\}$ into r nonempty sets such that $B[V_i|V_j] = 0$ unless $j \equiv i + 1 \mod r$. Such a matrix is also called r-cyclic with partition Π. For an r-cyclic matrix B, there exists a permutation matrix P such that PBP^T has the block form

$$\begin{bmatrix} 0 & B_{12} & 0 & \cdots & 0 \\ 0 & 0 & B_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & B_{r-1,r} \\ B_{r1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{4.1}$$

where the diagonal 0 blocks are square and the blocks $B_{j,j+1}, j = 1, \dots, r$ may be rectangular (as an index $r + 1$ is interpreted as 1). If B is r-cyclic with consecutive ordered partition Π, then B itself has the block form (4.1).

Any power of a nonnegative (respectively, reducible) matrix remains nonnegative (respectively, reducible), whereas it is not true that if B is r-cyclic then B^k is r-cyclic for all $k \in \mathbb{Z}^+$. To discuss eventually r-cyclic matrices and their powers, we need some additional terminology, much of which is taken from [6]. For an ordered partition $\Pi = (V_1, \dots, V_r)$ of $\{1, \dots, n\}$ into r nonempty sets, the *characteristic matrix* $C_\Pi = [c_{ij}]$ of Π is the $n \times n$ matrix such that $c_{ij} = 1$ if there exists $\ell \in \{1, \dots, r\}$ such that $i \in V_\ell$ and $j \in V_{\ell+1}$ and $c_{ij} = 0$ otherwise. Note that for any ordered partition $\Pi = (V_1, \dots, V_r)$ of $\{1, \dots, n\}$ into r nonempty sets, C_Π is r-cyclic. For matrices $M = [m_{ij}], C = [c_{ij}] \in \mathbb{C}^{n\times n}$, matrix M *conforms* to C if for all $i, j = 1, \dots, n$, $c_{ij} = 0$ implies $m_{ij} = 0$. For an ordered partition Π of $\{1, \dots, n\}$ into r nonempty sets, B is r-cyclic with partition Π if and only if B conforms to C_Π. It should be noted that what we here call "conforms to" was referred to as "conformal with" in [6]. The latter terminology is unfortunate, because "conformal with" usually means having the same block structure or partitioning, not having zeros in a certain pattern.

For a positive integer $r \geq 2$, a matrix $M \in \mathbb{C}^{n\times n}$ is *eventually r-cyclic* if there exists a positive integer k_0 such that $k \geq k_0$ and $k \equiv 1 \mod r$ implies M^k is r-cyclic (this definition allows the partition into r nonempty sets to depend on k). For an ordered partition $\Pi = (V_1, \dots, V_s)$ of $\{1, \dots, n\}$ into $s \geq 2$ nonempty sets, a

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

matrix M is *eventually r-cyclic with partition* Π if there exists an ordered partition Π of $\{1,\ldots,n\}$ into $r \geq 2$ nonempty sets, and a positive integer k_0 such that for all $k \geq k_0$, M^k conforms to ${C_\Pi}^k$ (this is the way that 'eventually r-cyclic' is defined in [6]). We will show that M is eventually r-cyclic (as defined here) implies that there is a partition Π such that M is eventually r-cyclic with partition Π. Of course, if M is an r-cyclic matrix, then there is a partition Π such that M is r-cyclic with partition Π, so M^k conforms to ${C_\Pi}^k$, and M is eventually r-cyclic with partition Π.

THEOREM 4.1. *If $M \in \mathbb{C}^{n\times n}$ is eventually r-cyclic, then there is some ordered partition Π such that M is eventually r-cyclic with partition Π.*

Proof. Suppose M is eventually r-cyclic for $r \geq 2$, i.e., there is a positive integer k_0 such that $k \geq k_0$ and $k \equiv 1 \mod r$ implies M^k is r-cyclic. Let $P = \{\Pi_1,\ldots,\Pi_p\}$ be the set of all ordered partitions of $\{1,\ldots,n\}$ into r nonempty sets. For each ordered partition $\Pi_i \in P$, define

$$\mathcal{S}_i = \{k \in \mathbb{Z}^+ : k \equiv 1 \mod r \text{ and } M^k \text{ is } r\text{-cyclic with partition } \Pi_i\}$$

and $d_i = \gcd(\mathcal{S}_i)$. Define $u = rk_0d_1\cdots d_p + 1$. Then M^u is r-cyclic with some ordered partition Π_j, so $u \in \mathcal{S}_j$ and d_j divides u. Since d_j also divides $u-1 = rk_0d_1\cdots d_p$, $d_j = 1$. Then $1 = \gcd(k_1,\ldots,k_s)$ for some $k_1,\ldots,k_s \in \mathcal{S}_j$, and by [1, Lemma 3.4.2], there is a positive integer t_0 so that for all $k \geq t_0$, k is a nonnegative linear combination of $k_1,\ldots,k_s$. Then M^k conforms to ${C_{\Pi_j}}^k$ and so M is r-cyclic with partition Π. ☐

Recall that in Section 2 we proved a generalized answer to a question of Zaslavsky and Tam. An analogous result is true for eventually r-cyclic matrices, but we need to refine our terminology for power hereditary properties, because the situation for powers of r-cyclic matrices is more delicate. We extend the definition of a property to include a family of classes indexed by the positive integers: For each $k \in \mathbb{Z}^+$, let $\mathcal{P}(k)$ denote a class of $n \times n$ matrices. The matrix A has *indexed property* $\mathcal{P}$ if $A^k \in \mathcal{P}(k)$. A nonempty indexed property $\mathcal{P}$ is *power hereditary* if $M^{k+\ell} \in \mathcal{P}(k+\ell)$ whenever $M^k \in \mathcal{P}(k)$ and $M^\ell \in \mathcal{P}(\ell)$. Of course, an (ordinary) property $\mathcal{P}$ that is power hereditary can be viewed as a power hereditary indexed property by defining $\mathcal{P}(k) = \mathcal{P}$ for all $k \in \mathbb{Z}^+$. We use the notion of an indexed property to describe the behavior of the powers of r-cyclic matrices. For an ordered partition Π of $\{1,\ldots,n\}$ into $r \geq 2$ nonempty sets, define the indexed property $\mathcal{C}_\Pi$ by

$$\mathcal{C}_\Pi(k) = \{M \in \mathbb{C}^{n\times n} :\ M \text{ conforms to } {C_\Pi}^k\}.$$

Observe that $\mathcal{C}_\Pi(k+r) = \mathcal{C}_\Pi(k)$. This definition has the desirable result that if $M \in \mathcal{C}_\Pi(1)$, then $M^k \in \mathcal{C}_\Pi(k)$. For an ordered partition Π into r nonempty sets, M is r-cyclic with partition Π if and only if M has the indexed property $\mathcal{C}_\Pi$.

We now restate Corollary 2.5 and Theorem 2.9 for indexed properties, omitting the proofs, which adapt naturally to an indexed property.

PROPOSITION 4.2. *Let $k_1,k_2,\ldots,k_r \in \mathbb{Z}^+$ with $\gcd(k_1,k_2,\ldots,k_r) = 1$. If $\mathcal{P}$ is a power hereditary indexed property and $M^{k_i} \in \mathcal{P}(k_i)$ for all $i = 1,\ldots,r$, then M is*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

eventually $\mathcal{P}$.

THEOREM 4.3. *Let* $\mathcal{P}$ *be a power hereditary indexed property. For* $M \in \mathbb{C}^{n\times n}$, *let* $S = \{k : k$ *is a nonnegative integer and* $M^{k+1} \in \mathcal{P}(k+1)\}$. *If for every positive integer* p *there exists* $k \in S$ *such that* p *divides* k, *then* M *is eventually* $\mathcal{P}$.

As is the case for properties discussed earlier, it is possible to have a matrix that is not eventually r-cyclic even though it has an r-cyclic power.

EXAMPLE 4.4. Let $M = \begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$. Then $M^3 = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}$ is 2-cyclic, and clearly $M^{6k+1} = (-64)^k M$ does not conform to any power of C_Π for either of the two ordered partitions Π of $\{1,2\}$ into two nonempty sets, so M is not eventually 2-cyclic.

We will use the following result (as originally stated it was for real matrices but it is clear the proof is valid for complex matrices).

THEOREM 4.5. [6, Corollary 2.8] *Let* Π *be an ordered partition of* $\{1,\ldots,n\}$ *into* $r \geq 2$ *nonempty sets and let* $M \in \mathbb{C}^{n\times n}$ *have* $\operatorname{rank} M^2 = \operatorname{rank} M$. *Then* M *is eventually* r*-cyclic with partition* Π *if and only if* M *is* r*-cyclic with partition* Π.

THEOREM 4.6. *Let* Π *be an ordered partition of* $\{1,\ldots,n\}$ *into* $r \geq 2$ *nonempty sets and let* $M \in \mathbb{C}^{n\times n}$. *Then the following are equivalent:*

1. M *is eventually* r*-cyclic with partition* Π.
2. *For the unique decomposition* $M = B + N$ *that satisfies* $\operatorname{rank} B^2 = \operatorname{rank} B$, $BN = NB = 0$, *and* $N^n = 0$ [11, Theorem 3.6], B *is* r*-cyclic with partition* Π.
3. *For all* $k \geq n$, M^k *conforms to* ${C_\Pi}^k$.
4. M^{qr} *conforms to* ${C_\Pi}^r$ *and* M^{qr+1} *conforms to* C_Π *where* $q = \lceil \frac{n}{r} \rceil$.

Proof. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are immediate. (4) $\Rightarrow$ (1) is a consequence of Proposition 4.2. So assume M is eventually r-cyclic with partition Π. Decompose $M \in \mathbb{C}^{n\times n}$ as $M = B + N$ as specified. Since M is eventually r-cyclic with partition Π, B is eventually r-cyclic with partition Π. Since $\operatorname{rank} B^2 = \operatorname{rank} B$, B is r-cyclic with partition Π. $\square$

REMARK 4.7. A result analogous to Theorem 4.6 also holds when 'r-cyclic' is replaced by a power hereditary indexed property $\mathcal{P}$ for which M is $\mathcal{P}$ whenever $\operatorname{rank} M = \operatorname{rank} M^2$ and M is eventually $\mathcal{P}$.

We now turn our attention to the eigenstructure of an eventually r-cyclic matrix. If $\sigma = \{\lambda_1,\ldots,\lambda_t\}$ is a finite multiset of complex numbers, $\omega \in \mathbb{C}$, and $k \in \mathbb{Z}^+$, then $\omega\sigma = \{\omega\lambda_1,\ldots,\omega\lambda_t\}$ and $\sigma^k = \{{\lambda_1}^k,\ldots,{\lambda_t}^k\}$. If $\sigma = \omega\sigma$, then we say that σ is *ω-invariant.*

For an r-cyclic matrix B of the form (4.1), define

$$A_j = (B_{j,j+1}\cdots B_{r-1,r}B_{r1})(B_{12}\cdots B_{j-1,j}), \qquad j = 1,\ldots,r.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

Then by [1, Theorem 3.4.7] there exists a polynomial $f(x)$ and nonnegative integers $p_j, j = 1, \dots, r$ such that the following hold.

1. $f(0) \neq 0$.
2. The characteristic polynomial of A_j is $f(x)x^{p_j}$ for $j = 1, \dots, r$.
3. The characteristic polynomial of B is $f(x^r)x^{p_1+\cdots+p_r}$.

Consequently, if $B \in \mathbb{C}^{n\times n}$ is r-cyclic, then $\sigma(B)$ is ω-invariant, where $\omega = e^{2\pi i/r}$ and $i = \sqrt{-1}$. We adopt much of the terminology of [11], introducing new terms as needed. For a finite multiset σ of complex numbers, the *radius* of σ is $\rho(\sigma) = \max\{|\lambda| : \lambda \in \sigma\}$ and the *periphery* or *boundary* of σ is $\partial(\sigma) = \sigma \cap \{z \in \mathbb{C} : |z| = \rho(\sigma)\}$. We say σ is a *Frobenius multiset* if for $r = |\partial(\sigma)|$, $\omega = e^{2\pi i/r}$, and $Z_r = \{1, \omega, \omega^2, \dots, \omega^{r-1}\}$ we have

1. $\rho(\sigma) > 0$,
2. $\partial(\sigma) = \rho(\sigma)Z_r$, and
3. σ is ω-*invariant*.

Although not explicitly stated in [11], the next result is implicitly established within Zaslavsky and Tam's proof of Friedland's lemma that the spectrum of an eventually nonnegative matrix is the union of self-conjugate Frobenius multisets. Because the statement is quite different, we include the brief proof here.

Theorem 4.8. [11, Theorem 3.1] *Let σ be a multiset of complex numbers and let $r = |\partial(\sigma)|$. If there exists a positive integer ℓ_0 such that for all $\ell \geq \ell_0$, $\sigma^{\ell r+1}$ is a single Frobenius multiset, then σ is a single Frobenius multiset.*

Proof. Assume that $\sigma^{\ell r+1}$ is a single Frobenius multiset for all $\ell \geq \ell_0$. For any $x, y \in \sigma$ for which $\frac{x}{y}$ is a root of unity, let $m(x, y)$ denote the smallest positive integer such that $\left(\frac{x}{y}\right)^{m(x,y)} = 1$. For any nonzero number $z \in \sigma$ for which $\frac{z}{|z|}$ is a root of unity, let $m(z)$ denote the smallest positive integer such that $\left(\frac{z}{|z|}\right)^{m(z)} = 1$. Let m be the product of these $m(x, y)$'s and $m(z)$'s and set $k = m\ell_0 r + 1$.

By hypothesis, σ^k is a single Frobenius multiset, so σ^k is the multiset union of $\partial(\sigma^k) = \rho(\sigma^k)Z_r = \rho(\sigma)^k Z_r$, sets of the form $\lambda_i Z_r, i = 1, \dots, q$ with $0 < |\lambda_i| < \rho(\sigma^k)$, and a (possibly empty) multiset of zeros ζ. Let $\lambda_0 = \rho(\sigma^k)$. Decompose σ as a union of multisets $\sigma_i, i = 0, \dots, q$, and ζ, where $\sigma_i^k = \lambda_i Z_r, i = 0, \dots, q$. Let μ_i denote the element in σ_i such that ${\mu_i}^k = \lambda_i$. Then $\sigma_i = \{\mu_i, \mu_i\alpha_1, \dots, \mu_i\alpha_{r-1}\}$ where ${\alpha_j}^k = \omega^j, j = 1, \dots, r-1$. For $j \in \{1, \dots, r-1\}$, $\mu_i, \mu_i\alpha_j \in \sigma$ and $\alpha_j = \frac{\mu_i\alpha_j}{\mu_i}$, so by the choice of k, ${\alpha_j}^k = \alpha_j$ and thus $\sigma = \mu_i Z_r$. In the case $i = 0$, where ${\mu_0}^k = \rho(\sigma)^k > 0$, $\frac{\mu_0}{|\mu_0|}$ is a root of unity, so again by choice of k, ${\mu_0}^k = |\mu_0|^{k-1}\mu_0$ and thus $\mu_0 > 0$. Thus, σ is a Frobenius multiset. □

Theorem 4.9. *Let σ be a multiset of complex numbers and $\omega = e^{2\pi i/r}$ for some integer $r \geq 2$. If there exists a positive integer ℓ_0 such that $\sigma^{\ell r+1}$ is ω-invariant for all $\ell \geq \ell_0$, then σ is ω-invariant.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

Proof. Choose $\hat{\rho} > \rho(\sigma)$ and consider the multiset $\hat{\sigma} = \sigma \cup \hat{\rho} Z_r$. Then $\rho(\hat{\sigma}) = \hat{\rho} > \rho(\sigma) \geq 0$, $\partial(\hat{\sigma}) = \hat{\rho} Z_r$, and $r = |\partial(\hat{\sigma})|$. For $\ell \geq \ell_0$, $\sigma^{\ell r+1}$ is ω-invariant. And since $(\hat{\rho} Z_r)^{\ell r+1} = \hat{\rho}^{\ell r+1} Z_r = \hat{\rho}^{\ell r+1}\omega Z_r = \omega((\hat{\rho} Z_r)^{\ell r+1})$, we have that $\hat{\sigma}^{\ell r+1}$ is ω-invariant and hence a single Frobenius multiset. By Theorem 4.8, we can conclude that $\hat{\sigma}$ is a single Frobenius multiset, which implies that σ is ω-invariant. □

Corollary 4.10. *If $M \in \mathbb{C}^{n\times n}$ is eventually r-cyclic and $\omega = e^{2\pi i/r}$, then $\sigma(M)$ is ω-invariant.*

Proof. Assume M is eventually r-cyclic with partition Π. Let k_0 be a positive integer such that for all $k \geq k_0$, M^k conforms to ${C_\Pi}^k$. Then for $\ell \geq k_0$, $M^{\ell r+1}$ is r-cyclic with partition Π, hence $(\sigma(M))^{\ell r+1} = \sigma(M^{\ell r+1})$ is ω-invariant. Therefore, $\sigma(M)$ is ω-invariant. □

The converse to the previous corollary is not true as seen in the following example.

Example 4.11. The matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -\frac{1}{5} & 1 & -1 \\ -\frac{4}{5} & 1 & -2 \end{bmatrix}$$

has spectrum $\{0, \pm 1\}$, which is ω-invariant for $\omega = e^{2\pi i/2} = -1$. Since M is not 2-cyclic and $\operatorname{rank} M = \operatorname{rank} M^2$, M is not eventually 2-cyclic by Theorem 4.5.

Let B be an r-cyclic matrix of the form (4.1) with the ith zero diagonal block being $n_i \times n_i$ for $i = 1, \ldots, r$. Suppose B has eigenvalue λ and corresponding eigenvector $\mathbf{x}^T = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T & \cdots & \mathbf{x}_r^T \end{bmatrix}$ partitioned conformally with the r-cyclic structure (4.1). Let $\omega := e^{2\pi i/r}$ and $D := \operatorname{diag}(\omega I_{n_1}, \omega^2 I_{n_2}, \ldots, \omega^{r-1} I_{n_{r-1}}, I_{n_r})$. By [9, Theorem 4.1], $(D^s)^{-1} B D^s = \omega^s B$. Clearly $\mathbf{x}$ is an eigenvector for eigenvalue $\omega^s\lambda$ of $\omega^s B$. Thus, for $s = 1, \ldots, r-1$, $\mathbf{x}^{(s)} := D^s \mathbf{x} = \begin{bmatrix} \omega^s {\mathbf{x}_1}^T & \omega^{2s} {\mathbf{x}_2}^T & \cdots & \omega^{(r-1)s} {\mathbf{x}_{r-1}}^T & {\mathbf{x}_r}^T \end{bmatrix}^T$ is an eigenvector for eigenvalue $\omega^s \lambda$ of B. We can extend this result to an eventually r-cyclic matrix for an eigenvector of a nonzero eigenvalue.

Theorem 4.12. *Let M be an eventually r-cyclic matrix with consecutive ordered partition Π. Assume that μ is a nonzero eigenvalue of M with eigenvector $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T & \cdots & \mathbf{x}_r^T \end{bmatrix}$ partitioned conformally with the r-cyclic structure of Π (as given in (4.1)). Then for each $s = 1, \ldots, r-1$*

$$\mathbf{x}^{(s)} = \begin{bmatrix} \omega^s {\mathbf{x}_1}^T & \omega^{2s} {\mathbf{x}_2}^T & \cdots & \omega^{(r-1)s} {\mathbf{x}_{r-1}}^T & {\mathbf{x}_r}^T \end{bmatrix}^T$$

is an eigenvector for eigenvalue $\omega^s \mu$ of M.

Proof. Decompose $M = B + N$ with $\operatorname{rank} B^2 = \operatorname{rank} B$, $BN = NB = 0$, and $N^n = 0$, so B is r-cyclic with partition Π and has the block form (4.1). Since $\mu^n \mathbf{x} = M^n \mathbf{x} = B^n \mathbf{x}$ and $\mu \neq 0$, we have $\mathbf{x} = \frac{1}{\mu^n} B^n \mathbf{x}$. Thus, $N\mathbf{x} = \frac{1}{\mu^n} N B^n \mathbf{x} = 0$ as $NB = 0$. Therefore, $B\mathbf{x} = \mu \mathbf{x}$.

Partition $N = [N_{ij}]$ as a block matrix conformally with the block form (4.1) of B. Since $NB = 0$, $N_{ij} B_{j,j+1} = 0$ for $i, j \in \{1, \ldots, r\}$ (where index $r+1$ is interpreted

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 953-965, December 2012

ELA
http://math.technion.ac.il/iic/ela

as 1). Since $B\mathbf{x} = \mu\mathbf{x}$, $\mathbf{x}_j = \frac{1}{\mu} B_{j,j+1}\mathbf{x}_{j+1}$. Thus $N_{ij}\mathbf{x}_j = 0$, and therefore, $N\mathbf{x}^{(s)} = 0$. Thus, $M\mathbf{x}^{(s)} = B\mathbf{x}^{(s)} = \omega^s \mu \mathbf{x}^{(s)}$. ☐

For example, $N = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ is nilpotent and thus is eventually 2-cyclic, but has only one independent eigenvector.

Acknowledgment. The authors thank Bit-Shun Tam and the anonymous referee for many helpful suggestions that improved the paper.

REFERENCES

[1] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory.* Cambridge University Press, New York, 1991.

[2] S. Carnochan Naqvi and J.J. McDonald. The combinatorial structure of eventually nonnegative matrices. *Electronic Journal of Linear Algebra*, 9:255–269, 2002.

[3] S. Carnochan Naqvi and J.J. McDonald. Eventually nonnegative matrices are similar to seminonnegative matrices, *Linear Algebra and its Applications*, 381:245–258, 2004.

[4] A. Elhashhash and D. Szyld. On general matrices having the Perron-Frobenius property. *Electronic Journal of Linear Algebra*, 17:389–413, 2008.

[5] S. Friedland. On an inverse problem for nonnegative and eventually nonnegative matrices. *Israel Journal of Mathematics*, 29:43–60, 1978.

[6] L. Hogben. Eventually cyclic matrices and a test for strong eventual nonnegativity. *Electronic Journal of Linear Algebra*, 19:129–140, 2010.

[7] C.R. Johnson and P. Tarazaga. On matrices with Perron-Frobenius properties and some negative entries. *Positivity*, 8:327–338, 2004.

[8] D. Noutsos and M. Tsatsomeros. Reachability and holdability of nonnegative states. *SIAM Journal on Matrix Analysis and Applications*, 30:700–712, 2008.

[9] B.-S. Tam. On matrices with cyclic structure. *Linear Algebra and its Applications*, 302/303:377–410, 1999.

[10] B.G. Zaslavsky and J.J. McDonald. A characterization of Jordan canonical forms which are similar to eventually nonnegative matrices with the properties of nonnegative matrices. *Linear Algebra and its Applications*, 372:253–285, 2003.

[11] B.G. Zaslavsky and B.-S. Tam. On the Jordan form of an irreducible matrix with eventually non-negative powers. *Linear Algebra and its Applications*, 302/303:303–330, 1999.