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# Decomposition of a ring induced by minus partial order

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# DECOMPOSITION OF A RING INDUCED BY MINUS PARTIAL ORDER*

DRAGAN S. RAKIĆ†

**Abstract.** The minus partial order linear algebraic methods have proven to be useful in the study of complex matrices. This paper extends study of minus partial orders to general rings. It is shown that the condition $a <^- b$, where $<^-$ is the minus partial order, defines two triples of orthogonal idempotents, and thus, a decomposition of a ring into a direct sum of abelian groups. Hence, several well-known results concerning minus partial order on real and complex matrices are generalized. Analogous decompositions for rectangular matrices over a ring and for Banach space operators are obtained. The equivalent conditions for the invariance of $ab^{(1)}a$ under the choices of $b^{(1)}$ are also obtained. An original inspiration for this work came from the study of minus partial order on complex matrices and from linear algebra methods.

**Key words.** Minus partial order, Generalized inverse, Idempotent, Von Neumann regular ring, Operator matrices, Peirce decomposition.

**AMS subject classifications.** 15A09, 06A06, 16U99.

## 1. Introduction.

Throughout the paper, $R$ denotes a ring with identity 1. The minus partial order on a ring $R$, introduced by Hartwig [7], is defined by $a <^- b$ if there exists an $x \in R$ such that

$$ax = bx, \quad xa = xb \quad \text{and} \quad axa = a. \tag{1.1}$$

Our aim is to find an equivalent condition for $a <^- b$ which will enable us to obtain some properties of minus partial order. For this purpose, we look how this problem is solved for real and complex matrices.

It is known that for real or complex matrices $A$ and $B$ the condition $A <^- B$ is equivalent to the existence of non-singular matrices $S$ and $T$ such that

$$A = S \begin{bmatrix} I_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} T \quad \text{and} \quad B = S \begin{bmatrix} I_A & 0 & 0 \\ 0 & I_{B-A} & 0 \\ 0 & 0 & 0 \end{bmatrix} T, \tag{1.2}$$

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where $I_A$ and $I_{B-A}$ are identity matrices. The above simultaneous diagonalization proves very useful in obtaining many of the properties of minus partial order. For more details concerning minus partial order on matrices see the monograph [13] and the references given there. The proofs in [13] are mostly based on linear algebra and finite dimensional methods. Of course, we cannot use these techniques when $a, b$ are elements of arbitrary ring.

We are interested in finding the matrix forms of $a, b \in R$, which are analogous to those given in (1.2), when $a <^- b$. We will show in our main Theorem 3.5 that the condition $a <^- b$, where $a$, $b$ are regular elements, naturally defines two triples of orthogonal idempotents $(e_i)$ and $(f_i)$, $i = 1, 2, 3$ with sums equal to 1. This yields the decomposition of the ring $R$ into a direct sum of abelian groups

$$R = \bigoplus_{i,j=1}^{3} e_i R f_j, \tag{1.3}$$

such that $a \in e_1Rf_1$ and $b - a \in e_2Rf_2$.

Using this result we will generalize a considerable number of well-known results for real and complex matrices. For example, when $a <^- b$, we can easily characterize all elements $x$ appearing in (1.1) and the class of all idempotents $p$ and $q$ such that $a = pb = bq$. The same notion of decomposition enable us to find equivalent conditions for the invariance of $ab^{(1)}a$ under the choices of $b^{(1)} \in b\{1\}$. Also, it is proven that the conditions $a <^- b$, $b\{1\} \subseteq a\{1\}$ and $b\{1, 2\} \subseteq a\{1\}$ are equivalent, which is known for matrices, (see [12, 17]).

Another advantage of using the decomposition (1.3) lies in the fact that the same idea can be applied to rectangular matrices over a ring and even to the class of bounded operators on Banach spaces. This concepts will be discussed in the last section.

**2. Preliminaries.** An element $a \in R$ is called *von Neumann regular* (regular for short) if there exists $x \in R$ satisfying $axa = a$. The element $x$ is called a *generalized inverse* of $a$. If $axa = a$ and $xax = x$, then $x$ is called a *reflexive generalized inverse* of $a$. If $x_1$ and $x_2$ are generalized inverses of $a$, then $x_1ax_2$ is a reflexive generalized inverse of $a$. We denote by $a\{1\}$ the set of all generalized inverses of $a$ and by $a\{1, 2\}$ the set of all reflexive generalized inverses of $a$. The set of all regular elements of $R$ is denoted by $R^{(1)}$. Some properties of generalized inverses in a ring was studied in [2, 4, 15].

A ring $R$ is *von Neumann regular* (regular for short) if every element of $R$ is regular. For details concerning regular rings, we refer the reader to [5]. Note that a

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ELA
http://math.technion.ac.il/iic/ela



ring of all $n \times n$ complex matrices is regular.

Definition 2.1. Let $a, b \in R$. Then *a is below b under the minus partial order*, denoted by $a <^- b$, if $a \in R^{(1)}$ and $ax = bx$, $xa = xb$ for some $x \in a\{1\}$.

The idempotents $e, f \in R$ are *orthogonal* if $ef = fe = 0$. The idempotents $e_1, e_2, \ldots, e_n \in R$ are called *orthogonal* if they are mutually orthogonal. An equality $1 = e_1 + e_2 + \cdots + e_n$, where $e_1, e_2, \ldots, e_n \in R$ are orthogonal idempotents, is called a *decomposition of the identity of the ring R*.

Remark 2.2. Let $1 = e_1 + \cdots + e_m$ and $1 = f_1 + \cdots + f_n$ be two decompositions of the identity of a ring $R$. For any $x \in R$ we have

$$x = 1 \cdot x \cdot 1 = (e_1 + \cdots + e_m)x(f_1 + \cdots + f_n) = \sum_{i=1}^{m} \sum_{j=1}^{n} e_i x f_j.$$

It is not difficult to verify that above sum defines a decomposition of $R$ into a direct sum of abelian groups $e_i R f_j := \{e_i x f_j : x \in R\}$:

$$R = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} e_i R f_j. \tag{2.1}$$

It is convenient to write $x$ as a matrix

$$x = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}_{e \times f},$$

where $x_{ij} = e_i x f_j \in e_i R f_j$. Similarly,

$$R = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} f_i R e_j,$$

and any $y \in R$ can be written in a matrix form

$$y = [y_{ij}]_{f \times e},$$

where $y_{ij} = f_i y e_j \in f_i R e_j$, $i = \overline{1,n}$, $j = \overline{1,m}$. By the orthogonality of idempotents involved, one can use usual matrix rules in order to add and multiply $x$ and $y$.

When $m = n$ and $e_i = f_i$, $i = \overline{1,n}$, the decomposition (2.1) is known as the two-sided Peirce decomposition of the ring $R$, [8].

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**3. The minus partial order.** Before moving to the minus partial order, we consider another order which is associated to the minus partial order.

If $a, b \in R$, then we say that *$a$ is below $b$ under the space pre-order*, denoted by $a <^s b$, if

$$aR \subseteq bR \quad \text{and} \quad Ra \subseteq Rb.$$

This definition is analogous to the definition of space pre-order on complex matrices (see [11]) in which case $A <^s B$ if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}(A^*)$, where $\mathcal{R}(A)$ denotes the column space of matrix $A$, and $A^*$ is the conjugate transpose of $A$. It is easily seen that $<^s$ is pre-order and that $a <^- b$ implies $a <^s b$.

The following result is well-known in the matrix case, [1].

THEOREM 3.1. *Let $R$ be a regular ring and $a, b \in R$. Then the following conditions are equivalent:*

(i) $a <^s b$;
(ii) $a = bb^{(1)}a = ab^{(1)}b$ *for all* $b^{(1)} \in b\{1\}$;
(iii) $a = bb^{(1)}ab^{(1)}b$ *for all* $b^{(1)} \in b\{1\}$; *and*
(iv) $ab^{(1)}a$ *is invariant under the all choices of* $b^{(1)} \in b\{1\}$.

*Proof.* If $a = 0$ or $b = 0$, then the theorem holds. Suppose that $a \neq 0$ and $b \neq 0$.

(i) $\Longrightarrow$ (ii): Since $a <^s b$ we have $aR \subseteq bR$, so there exists an $x \in R$ such that $a = bx$. Hence, $a = bb^{(1)}bx = bb^{(1)}a$ for all $b^{(1)} \in b\{1\}$. Similarly, $a = ab^{(1)}b$ for all $b^{(1)} \in b\{1\}$.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (iv): Fix $h \in b\{1\}$. For every $b^{(1)} \in b\{1\}$ we have

$$ab^{(1)}a = (bhahb)b^{(1)}(bhahb) = bhahbhahb,$$

which does not depend on $b^{(1)}$.

(iv) $\Longrightarrow$ (i): Fix $h \in b\{1\}$ and set

$$\begin{aligned} e_1 &= bh, \quad e_2 = 1 - bh, \\ f_1 &= hb, \quad f_2 = 1 - hb. \end{aligned}$$

Then

$$1 = e_1 + e_2 \quad \text{and} \quad 1 = f_1 + f_2$$

are two decompositions of the identity of the ring $R$. If $b^{(1)} \in b\{1\}$, then $f_1 b^{(1)} e_1 = hbb^{(1)}bh = hbh \; (= f_1 h e_1)$. If $f_1 b^{(1)} e_1 = hbh$, then $b(f_1 b^{(1)} e_1)b = b(hbh)b$. Thus,

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ELA
http://math.technion.ac.il/iic/ela



$bb^{(1)}b = b$. Therefore, $b^{(1)} \in b\{1\}$ if and only if

$$b^{(1)} = \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{f\times e}, \tag{3.1}$$

where $x_{ij} \in f_iRe_j$ are arbitrary. Now,

$$\begin{aligned} ab^{(1)}a &= [af_1 \quad af_2]_{1\times f} \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{f\times e} \begin{bmatrix} e_1a \\ e_2a \end{bmatrix}_{e\times 1} \\ &= ahbha + ax_{21}a + ax_{12}a + ax_{22}a \end{aligned}$$

does not depend on $x_{21}, x_{12}, x_{22}$. Setting $x_{12} = x_{22} = 0$, it follows that $ax_{21}a = 0$ for all $x_{21} \in f_2Re_1$, i.e., $af_2xe_1a = 0$ for all $x \in R$. Multiplying this equation by $e_1$ from the left and by $f_2$ from the right, we obtain

$$(e_1af_2)x(e_1af_2) = 0.$$

Since $R$ is regular, we can choose $x = (e_1af_2)^{(1)} \in (e_1af_2)\{1\}$. Hence, $e_1af_2 = 0$. Similarly, $e_2af_1 = 0$ and $e_2af_2 = 0$, so we conclude that $a = e_1af_1 = bhahb$. This implies $a <^s b$. □

Note that the regularity of $R$ is used only in the part (iv) ⇒ (i), and the set $b\{1\}$ is characterized by (3.1).

Lemma 3.2. *Let $b \in R^{(1)}$ ($R$ is not necessarily regular). Under the notation of the proof of Theorem 3.1, the set of all reflexive generalized inverses of $b$ is given by*

$$b^{(1,2)} = \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{21}bx_{12} \end{bmatrix}_{f\times e}, \tag{3.2}$$

*where $x_{12} \in f_1Re_2$ and $x_{21} \in f_2Re_1$ are arbitrary.*

*Proof.* We already know that $b^{(1,2)}$ must be of the form (3.1). We have $x_{22} = x_{21}bx_{12}$, because

$$\begin{aligned} \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{f\times e} &= b^{(1,2)} = b^{(1,2)}bb^{(1,2)} = \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{f\times e} \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{e\times f} b^{(1,2)} \\ &= \begin{bmatrix} f_1 & 0 \\ x_{21}b & 0 \end{bmatrix}_{f\times f} \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{f\times e} = \begin{bmatrix} hbh & x_{12} \\ x_{21} & x_{21}bx_{12} \end{bmatrix}_{f\times e}. \end{aligned}$$

If $b^{(1,2)}$ is given by (3.2), then it is easy to check that $b^{(1,2)} \in b\{1,2\}$. □

It is stated in [12] that if $R$ is a regular ring, then $a <^- b$ is equivalent to $b\{1\} \subseteq a\{1\}$ with the additional hypothesis, $a \in bRb$. Using the direct sum partial

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ELA
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order as an intermediate step, it is proved in [3] that the equivalence still holds without the additional hypothesis. We show that a stronger result is in fact true.

THEOREM 3.3. *Let $R$ be a regular ring and $a, b \in R$. Then the following conditions are equivalent:*

(i) $a <^- b$;
(ii) $b\{1\} \subseteq a\{1\}$; *and*
(iii) $b\{1,2\} \subseteq a\{1\}$.

*Proof.* (i) $\Longrightarrow$ (ii): This is well-known. However, we prove it here for completeness. As $a <^- b$, there exists $a^{(1)} \in a\{1\}$ such that $aa^{(1)} = ba^{(1)}$ and $a^{(1)}a = a^{(1)}b$. For any $b^{(1)} \in b\{1\}$, we have

$$ab^{(1)}a = aa^{(1)}ab^{(1)}aa^{(1)}a = aa^{(1)}bb^{(1)}ba^{(1)}a = aa^{(1)}ba^{(1)}a = a.$$

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i): Lemma 3.2 shows that

$$a = ab^{(1,2)}a = ahbha + ax_{21}a + ax_{12}a + ax_{21}bx_{12}a,$$

for all $x_{12} \in f_1Re_2$ and $x_{21} \in f_2Re_1$. As in the proof of Theorem 3.1, we get

$$e_1af_2 = e_2af_1 = 0 \quad \text{and} \quad a = ahbha = af_1he_1a. \tag{3.3}$$

By (3.3), we obtain

$$a = (e_1 + e_2)(af_1he_1a)(f_1 + f_2) = e_1(af_1he_1a)f_1 = e_1af_1 = e_1a = af_1.$$

Let $x = f_1hahe_1$. Then

$$x = he_1af_1h = he_1ah = haf_1h = hah$$

and

$$he_1 = f_1h = hbh \in b\{1,2\} \subseteq a\{1\}.$$

Thus,

$$axa = a$$

and

$$ax = ahe_1ah = ah = e_1ah = bhah = bx,$$
$$xa = haf_1ha = ha = haf_1 = hahb = xb.$$

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ELA
http://math.technion.ac.il/iic/ela



By definition, $a <^- b$ holds. □

It is known that minus partial order is a partial order on $R$, when $R$ is regular. Reflexivity and transitivity follows from Theorem 3.3. If $a <^- b$ and $b <^- a$, then $a\{1\} \subseteq b\{1\}$ and there exists $a^{(1)} \in a\{1\}$ such that $aa^{(1)} = ba^{(1)}$ and $a^{(1)}a = a^{(1)}b$. So $a = aa^{(1)}a = ba^{(1)}b = b$, and $<^-$ is a partial order.

Note that $0 <^- b$, for all $b$ in $R$ and that $a <^- 0 \Leftrightarrow a = 0$.

Lemma 3.4. *Let $a, b \in R^{(1)}$. The following conditions are equivalent:*

(i) $a <^- b$;
(ii) $a = ab^{(1)}a = ab^{(1)}b = bb^{(1)}a$ *for all* $b^{(1)} \in b\{1\}$;
(iii) $b - a = (b-a)b^{(1)}(b-a) = (b-a)b^{(1)}b = bb^{(1)}(b-a)$ *for all* $b^{(1)} \in b\{1\}$; *and*
(iv) $b - a <^- b$.

*Proof.* (i) $\Longrightarrow$ (ii): The proof is similar to that of Theorem 3.3 (i) $\Rightarrow$ (ii).

(ii) $\Longrightarrow$ (i): Let $x = b^{(1)}ab^{(1)}$. Then $x \in a\{1\}$ and

$$ax = ab^{(1)}ab^{(1)} = ab^{(1)} = bb^{(1)}ab^{(1)} = bx.$$

Likewise, $xa = xb$, and hence, $a <^- b$.

(ii) $\Longleftrightarrow$ (iii) is a matter of direct computation.

(iii) $\Longleftrightarrow$ (iv) follows from the equivalence of (i) and (ii). □

Note that the word "all" can be replaced by "some" in the conditions (ii) and (iii) of Lemma 3.4. From Lemma 3.4 we see that the condition $a <^- b$ is symmetric in $a$ and $b - a$.

Next, if $A$ and $B$ are complex matrices, then [7]

$$A <^- B \Longleftrightarrow \operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(B - A), \tag{3.4}$$

and the latter condition is symmetric in $A$ and $B - A$. The element $x \in a\{1\}$ that appears in Definition 2.1 does not imply the symmetry, since $(b-a)x = 0 \neq bx$. But from Lemma 3.4 we see that elements of $b\{1\}$ somehow point out that symmetry. For this reason, we start from arbitrary but fixed $h \in b\{1\}$ in order to obtain the decomposition of $R$ induced by the condition $a <^- b$.

Remark 2.2 is crucial for our main result which follows.

Theorem 3.5. *Let $a, b \in R^{(1)}$. Then the following conditions are equivalent:*

(i) $a <^- b$;
(ii) *There exist decompositions of the identity of the ring* $R$

$$1 = e_1 + e_2 + e_3, \quad 1 = f_1 + f_2 + f_3 \tag{3.5}$$

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ELA
http://math.technion.ac.il/iic/ela



*with respect to which $a$ and $b$ have the following matrix forms:*

$$a = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f}, \quad b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f}. \tag{3.6}$$

*Proof.* The cases $a = 0$ or $b = 0$ are trivial.

(i) $\Longrightarrow$ (ii): Fix $h \in b\{1\}$ and set

$$\begin{aligned} e_1 &= ah, \quad e_2 = (b-a)h, \quad e_3 = 1 - bh, \\ f_1 &= ha, \quad f_2 = h(b-a), \quad f_3 = 1 - hb. \end{aligned} \tag{3.7}$$

Since $a <^- b$, Lemma 3.4 shows that

$$(ah)(ah) = (ah)(bh) = (bh)(ah) = ah, \quad (bh)(bh) = bh$$

and

$$(ha)(ha) = (ha)(hb) = (hb)(ha) = ha, \quad (hb)(hb) = hb.$$

It follows that

$$1 = e_1 + e_2 + e_3, \quad 1 = f_1 + f_2 + f_3$$

are two decompositions of the identity of the ring $R$.

From

$$e_1 a f_1 = ahaha = a$$

and

$$e_2(b-a)f_2 = (b-a)h(b-a)h(b-a) = b - a,$$

we conclude that $a$ and $b$ have the matrix forms given by (3.6).

(ii) $\Longrightarrow$ (i): Fix $a^{(1)} \in a\{1\}$ and set $x = f_1 a^{(1)} e_1$. It is easily seen that $x \in a\{1\}$, $ax = bx$ and $xa = xb$. □

When it is the case as in Theorem 3.5, we say that the decompositions (3.5), where idempotents are defined by (3.7), are *standard decompositions.*

Theorem 3.6. *Let $a, b \in R^{(1)}$ such that $a \neq 0$, $b \neq 0$ and $a <^- b$. Let $e_i$, $f_i$, $i = \overline{1,3}$, be defined by (3.7). Then there exist unique elements $x_a \in f_1Re_1$ and $x_{b-a} \in f_2Re_2$ such that*

$$\begin{aligned} ax_a &= e_1, & x_a a &= f_1, \\ (b-a)x_{b-a} &= e_2, & x_{b-a}(b-a) &= f_2. \end{aligned} \tag{3.8}$$

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ELA
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*Furthermore, with respect to the standard decompositions,* $a\{1\}$ *is given by*

$$a^{(1)} = [x_{ij}]_{f\times e}, \tag{3.9}$$

*where* $x_{11} = x_a$ *and* $x_{ij} \in f_iRe_j$, $(i,j) \neq (1,1)$ *are arbitrary, and the set* $b\{1\}$ *is given by*

$$b^{(1)} = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & x_{b-a} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}_{f\times e}, \tag{3.10}$$

*where* $x_{ij} \in f_iRe_j$ *are arbitrary.*

*Proof.* Let

$$\begin{aligned} x_a &:= f_1he_1 = hah, \\ x_{b-a} &:= f_2he_2 = h(b-a)h. \end{aligned} \tag{3.11}$$

One can check that these elements satisfy (3.8). The proof of the uniqueness is left to the reader.

Let $b^{(1)} = [f_ib^{(1)}e_j]_{f\times e} \in b\{1\}$ be arbitrary. From

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f} = b = bb^{(1)}b = \begin{bmatrix} ab^{(1)}a & ab^{(1)}(b-a) & 0 \\ (b-a)b^{(1)}a & (b-a)b^{(1)}(b-a) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times f},$$

it follows that $ab^{(1)}a = a$. Multiplying this equation by $x_a$ from the both sides yields $f_1b^{(1)}e_1 = x_a$. Also, multiplying $f_1b^{(1)}e_1 = x_a$ by $a$ from the both sides yields $ab^{(1)}a = a$. Similarly, $ab^{(1)}(b-a) = 0 \Leftrightarrow f_1b^{(1)}e_2 = 0$, $(b-a)b^{(1)}a = 0 \Leftrightarrow f_2b^{(1)}e_1 = 0$ and $(b-a)b^{(1)}(b-a) = b-a \Leftrightarrow f_2b^{(1)}e_2 = x_{b-a}$. Therefore, the set $b\{1\}$ is given by (3.10). The characterization of $a\{1\}$ can be proved analogously. $\square$

By the end of the section we will follow the notation of Theorems 3.5 and 3.6.

Remark 3.7. Let $a, b \in R^{(1)}$ such that $a <^- b$. Suppose that $h \in b\{1\}$ is regular. Fix $r_1 \in h\{1\}$ and set $r = b + e_3r_1f_3$. Then $r \in h\{1\}$. Let

$$\begin{aligned} &e_1' = e_1, \quad e_2' = e_2, \quad e_3' = (r-b)h, \quad e_4' = 1 - rh, \\ &f_1' = f_1, \quad f_2' = f_2, \quad f_3' = h(r-b), \quad f_4' = 1 - hr. \end{aligned}$$

It is not difficult to show that

$$1 = e_1' + \cdots + e_4', \quad 1 = f_1' + \cdots + f_4'$$

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are two decompositions of the identity of the ring $R$. Also $f_3'(h-hbh)e_3' = h-hbh$. It follows that $a$, $b$ and $h$ have the following matrix forms with respect to these decompositions:

$$a = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{e'\times f'}, \quad b = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b-a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{e'\times f'},$$

$$h = \begin{bmatrix} x_a & 0 & 0 & 0 \\ 0 & x_{b-a} & 0 & 0 \\ 0 & 0 & h-hbh & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{f'\times e'}.$$

Moreover, there exists unique $x \in e_3'Rf_3'$ (namely, $x = e_3'rf_3' = rhr - b$) such that $(h-hbh)x = f_3'$ and $x(h-hbh) = e_3'$.

Remark 3.8. We survey some known characterizations of minus partial order. Let $a, b \in R^{(1)}$ such that

$$a <^- b. \tag{3.12}$$

If $p_1 = e_1$, $p_2 = 1-e_1$, $q_1 = f_1$, $q_2 = 1-f_1$, then $1 = p_1+p_2$ and $1 = q_1+q_2$ are two decompositions of the identity of the ring $R$, and by Theorem 3.5, it follows

$$b = \begin{bmatrix} a & 0 \\ 0 & b-a \end{bmatrix}_{p\times q}. \tag{3.13}$$

This is equivalent to $a = p_1bq_1$ and $b-a = p_2bq_2$. Multiplying the latter equation by $p_1$ from the left gives $p_1(b-a) = 0$, so $a = p_1b$. Similarly, $a = bq_1$. On the other hand, suppose that there exist idempotents $p_1$ and $q_1$ such that

$$a = p_1b = bq_1. \tag{3.14}$$

Let $x = q_1a^{(1)}p_1$, where $a^{(1)} \in a\{1\}$. We have $a <^- b$, because $axa = aq_1a^{(1)}p_1a = aa^{(1)}a = a$, $ax = bq_1x = bx$ and $xa = xp_1b = xb$.

Obviously, (3.13) implies

$$bR = aR \oplus (b-a)R, \tag{3.15}$$

$$Rb = Ra \oplus R(b-a), \tag{3.16}$$

$$aR \cap (b-a)R = \{0\} = Ra \cap R(b-a). \tag{3.17}$$

Note that condition (3.15) is used as a definition of direct sum partial order. The equivalence of (3.12) and (3.15) was proved in [3] Lemma 3, and equivalence of (3.15)–(3.17) was proved in [9] Theorem 1. Thus, (3.12)–(3.17) are equivalent. Some other characterizations of minus partial order on regular semigroup can be found in [14].

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ELA
http://math.technion.ac.il/iic/ela



Let

$$a\{1\}_b = \{x \in a\{1\} : ax = bx, xa = xb\} \quad \text{and}$$
$$a\{1,2\}_b = \{x \in a\{1,2\} : ax = bx, xa = xb\}.$$

In the next theorem we obtain explicit representations of $a\{1\}_b$ and $a\{1,2\}_b$. For the case when $a$, $b$ are complex matrices, see [10] and [11].

THEOREM 3.9. *Let $a, b \in R^{(1)}$ such that $a <^- b$. Then*

(i) $a\{1\}_b = \{b^{(1)} - b^{(1)}(b-a)b^{(1)} : b^{(1)} \in b\{1\}\}$;
(ii) $a\{1,2\}_b = \{b^{(1)}ab^{(1)} : b^{(1)} \in b\{1\}\} = \{b^{(1,2)}ab^{(1,2)} : b^{(1,2)} \in b\{1,2\}\}$.

*Proof.* (i): Let us denote the set on the right-hand side of (i) by $S$. Since $a <^- b$, we have that $a$, $b$ and $b^{(1)}$ have the representations given by (3.6) and (3.10) respectively. It follows that $x \in S$ if and only if

$$x = b^{(1)} - b^{(1)}(b-a)b^{(1)} = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & 0 & 0 \\ x_{31} & 0 & x'_{33} \end{bmatrix}_{f\times e}, \tag{3.18}$$

for certain elements $x_{13} \in f_1Re_3$, $x_{31} \in f_3Re_1$ and $x'_{33} \in f_3Re_3$ ($x'_{33} = x_{33} - x_{32}(b-a)x_{23}$). A trivial verification shows that $axa = a$, $ax = bx$ and $xa = xb$, i.e., $x \in a\{1\}_b$.

Assume now that $x \in a\{1\}_b$. Then $x \in a\{1\}$, and hence, $x = [x_{ij}]_{f\times e}$, where $x_{11} = x_a$. From $ax = bx$ and $xa = xb$, we obtain $x_{12} = x_{21} = x_{22} = x_{23} = x_{32} = 0$. One can check that $x = b^{(1)} - b^{(1)}(b-a)b^{(1)} \in S$, where

$$b^{(1)} = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & x_{b-a} & 0 \\ x_{31} & 0 & x_{33} \end{bmatrix}_{f\times e}.$$

(ii): The proof of (ii) is similar. We obtain that $x \in a\{1,2\}_b$ is given by:

$$x = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & 0 & 0 \\ x_{31} & 0 & x_{31}ax_{13} \end{bmatrix}_{f\times e},$$

where $x_{13}$ and $x_{31}$ are arbitrary. ∎

The following theorem is generalization of Theorem 3.5.6. in [13] where $a$ and $b$ are complex matrices.

THEOREM 3.10. *Let $a, b \in R^{(1)}$ such that $a <^- b$. Then*

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela



(i) *For any* $a^{(1)} \in a\{1\}_b$ *there exists* $b^{(1)} \in b\{1\}$ *such that*

$$b^{(1)}a = a^{(1)}a, \quad ab^{(1)} = aa^{(1)}; \tag{3.19}$$

(ii) *For any* $b^{(1)} \in b\{1\}$ *there exists* $a^{(1)} \in a\{1\}_b$ *such that (3.19) holds.*

*Proof.* (i): Proof of Theorem 3.9 shows that $a^{(1)} \in a\{1\}_b$ has matrix representation given in (3.18). Then (3.19) holds for

$$b^{(1)} = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & x_{b-a} & x'_{23} \\ x_{31} & x'_{32} & x'_{33} \end{bmatrix}_{f\times e},$$

where $x'_{23} \in f_2Re_3$, $x'_{32} \in f_3Re_2$ and $x'_{33} \in f_3Re_3$ are arbitrary.

(ii): Any $b^{(1)} \in b\{1\}$ is of the form (3.10). The element

$$a^{(1)} = \begin{bmatrix} x_a & 0 & x_{13} \\ 0 & 0 & 0 \\ x_{31} & 0 & x'_{33} \end{bmatrix}_{f\times e},$$

where $x'_{33} \in f_3Re_3$ is arbitrary, has desired properties. ☐

As we pointed out in Remark 3.8, for $a, b \in R^{(1)}$, the condition $a <^- b$ is equivalent to $a = pb = bq$ where $p, q \in R$ are some idempotents. We characterize the class of all such idempotents. The following results are analogous to Theorems 3.5.13–3.5.18 in [13] where it is considered the case when $a$ and $b$ are complex matrices. All of them can be proved using matrix forms (3.6) and identities (3.8).

THEOREM 3.11. *Let* $a, b \in R^{(1)}$ *such that* $a <^- b$. *Then the class of all idempotents* $p \in R$ *satisfying* $a = pb$ *is given by*

$$p = \begin{bmatrix} e_1 & 0 & x_{13}(e_3 - p_{33}) \\ 0 & 0 & x_{23}p_{33} \\ 0 & 0 & p_{33} \end{bmatrix}_{e\times e}, \tag{3.20}$$

*where* $p_{33} \in e_3Re_3$ *is some idempotent and* $x_{13} \in e_1Re_3$, $x_{23} \in e_2Re_3$ *are arbitrary.*

*Proof.* If $p$ is of the given form, then $p$ is an idempotent and $a = pb$. Let $p$ be an idempotent such that $a = pb$. Suppose that $p = [p_{ij}]_{e\times e}$, $i, j = \overline{1, 3}$, with respect to standard decomposition $1 = e_1 + e_2 + e_3$. From $a = pb$, using (3.6) and (3.8), we obtain $p_{11} = e_1$, and $p_{12} = p_{21} = p_{22} = p_{31} = p_{32} = 0$. Condition $p = p^2$ implies $p_{23} = p_{23}p_{33}$, $p_{13} = p_{13} + p_{13}p_{33}$, and $p_{33} = p_{33}^2$. Hence, $p_{13} = x_{13}(e_3 - p_{33})$ and $p_{23} = x_{23}p_{33}$, where $x_{13} \in e_1Re_3$, $x_{23} \in e_2Re_3$ are arbitrary. ☐

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ELA
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In the same manner, we obtain the following theorem.

THEOREM 3.12. *Let $a, b \in R^{(1)}$ such that $a <^- b$. Then the class of all idempotents $q \in R$ such that $a = bq$ is given by*

$$q = \begin{bmatrix} f_1 & 0 & 0 \\ 0 & 0 & 0 \\ (f_3 - q_{33})x_{31} & q_{33}x_{32} & q_{33} \end{bmatrix}_{f \times f},$$

*where $q_{33} \in f_3Rf_3$ is some idempotent and $x_{31} \in f_3Rf_1$, $x_{32} \in f_3Rf_2$ are arbitrary.*

REMARK 3.13. Let

$$e'_1 = e_1, \quad e'_2 = e_2, \quad e'_3 = p_{33}, \quad e'_4 = e_3 - p_{33},$$
$$f'_1 = f_1, \quad f'_2 = f_2, \quad f'_3 = q_{33}, \quad f'_4 = f_3 - q_{33},$$

where $p_{33}$ and $q_{33}$ are idempotents appearing in Theorems 3.11 and 3.12. Then

$$1 = e'_1 + \cdots + e'_4, \quad 1 = f'_1 + \cdots + f'_4$$

are decompositions of the identity of the ring $R$. It is clear that

$$a = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e' \times f}, \quad b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e' \times f},$$

$$p = \begin{bmatrix} e'_1 & 0 & 0 & x_{14} \\ 0 & 0 & x_{23} & 0 \\ 0 & 0 & e'_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{e' \times e'},$$

and

$$a = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{e \times f'}, \quad b = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b-a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{e \times f'},$$

$$q = \begin{bmatrix} f'_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x_{32} & f'_3 & 0 \\ x_{41} & 0 & 0 & 0 \end{bmatrix}_{f' \times f'},$$

for some $x_{14} \in e'_1Re'_4$, $x_{23} \in e'_2Re'_3$, $x_{32} \in f'_3Rf'_2$ and $x_{41} \in f'_4Rf'_1$.

COROLLARY 3.14. *Let $a, b \in R^{(1)}$ such that $a <^- b$. Then the class of all idempotents $p$ such that $a = pb$ and $pR = aR$ is given by $\{aa^{(1)} : a^{(1)} \in a\{1\}_b\}$. The*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 23, pp. 1040-1059, December 2012
ELA
http://math.technion.ac.il/iic/ela



*class of all idempotents $q$ such that $a = bq$ and $Rq = Ra$ is given by $\{a^{(1)}a : a^{(1)} \in a\{1\}_b\}$.*

*Proof.* Since $a = e_1 a$ and $e_1 = ax_a$, we have $aR = e_1 R$. By (3.20) we see that $p$ is idempotent such that $a = pb$ and $pR = aR = e_1R$ if and only if $p_{33} = 0$ if and only if

$$p = \begin{bmatrix} e_1 & 0 & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

where $x_{13} \in e_1 R e_3$ is arbitrary. From (3.18) we see that $aa^{(1)}$ where $a^{(1)} \in a\{1\}_b$ has the above form. The second characterization can be obtained in the same manner. □

Corollary 3.15. *Let $a, b \in R^{(1)}$ such that $a <^- b$. Every idempotent $p$ satisfying $a = pb$ can be written as $p = p_1 + p_2$, where $p_1$ is an idempotent such that $a = p_1 b$, $p_1 R = aR$ and $p_2$ is an idempotent such that $p_1p_2 = p_2p_1 = p_2a = p_2b = 0$. Every idempotent $q$ satisfying $a = bq$, can be written as $q = q_1 + q_2$, where $q_1$ is an idempotent such that $a = bq_1$, $Rq_1 = Ra$ and $q_2$ is an idempotent such that $q_1q_2 = q_2q_1 = aq_2 = bq_2 = 0$.*

*Proof.* According to Theorems 3.11, 3.12 and Corollary 3.14, we can take

$$p_1 = \begin{bmatrix} e_1 & 0 & x_{13}(e_3 - p_{33}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}, \quad p_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{23}p_{33} \\ 0 & 0 & p_{33} \end{bmatrix}_{e\times e}, \quad \text{and}$$

$$q_1 = \begin{bmatrix} f_1 & 0 & 0 \\ 0 & 0 & 0 \\ (f_3 - q_{33})x_{31} & 0 & 0 \end{bmatrix}_{f\times f}, \quad q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q_{33}x_{32} & q_{33} \end{bmatrix}_{f\times f}. \quad \square$$

Suppose now that $\mathcal{A}$ is an algebra with identity 1 over a field $K$. Obviously, the algebra $\mathcal{A}$ is a ring $(\mathcal{A}, +, \cdot)$ and the related concepts and results are preserved in the passage from $R$ to $\mathcal{A}$.

Let $A <^- B$, where $A, B \in \mathbb{C}^{n\times n}$ are complex matrices, with ranks $a$ and $b$ respectively, and let $c_1, c_2 \in \mathbb{C}$, $c_2 \neq 0$, $c_1 + c_2 \neq 0$. In [18] it is proved that $c_1A + c_2B$ is invertible if and only if $B$ is invertible and

$$(c_1A + c_2B)^{-1} = (c_1 + c_2)^{-1}B^{-1} + (c_2^{-1} - (c_1 + c_2)^{-1})[(0 \oplus I_{n-a})B(0 \oplus I_{n-a})]^{\dagger},$$

holds, where $(\cdot)^{\dagger}$ is the Moore-Penrose inverse of $(\cdot)$.

The next theorem shows that the same result is valid when $a, b \in \mathcal{A}^{(1)}$. We avoid the use of Moore-Penrose inverse in our formula.

Theorem 3.16. *Let $a, b \in \mathcal{A}^{(1)}$ such that $a <^- b$ and let $c_1, c_2 \in K$, $c_2 \neq 0$,*

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ELA
http://math.technion.ac.il/iic/ela



$c_1 + c_2 \neq 0$. *Then $c_1a + c_2b$ is invertible if and only if $b$ is invertible. Furthermore,*

$$(c_1a + c_2b)^{-1} = c_2^{-1}b^{-1} + ((c_1+c_2)^{-1} - c_2^{-1})b^{-1}ab^{-1} = c_2^{-1}b^{-1} + ((c_1+c_2)^{-1} - c_2^{-1})a^{(1)}, \tag{3.21}$$

*where $a^{(1)} \in a\{1\}_b$.*

*Proof.* Since $a <^- b$, by Theorem 3.5, we have the following representations with respect to standard decompositions:

$$b = \begin{bmatrix} a & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times f}, \quad c_1a + c_2b = \begin{bmatrix} (c_1+c_2)a & 0 & 0 \\ 0 & c_2(b-a) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times f}.$$

Since $c_2 \neq 0$, $c_1 + c_2 \neq 0$, it follows that $b$ is invertible if and only if $c_1a + c_2b$ is invertible if and only if $e_3 = f_3 = 0$. In this case,

$$1 = e_1 + e_2, \quad 1 = f_1 + f_2$$

are two decompositions of the identity of $\mathcal{A}$ and with respect to these decompositions we have

$$b^{-1} = \begin{bmatrix} x_a & 0 \\ 0 & x_{b-a} \end{bmatrix}_{f \times e}, \quad (c_1a + c_2b)^{-1} = \begin{bmatrix} (c_1+c_2)^{-1}x_a & 0 \\ 0 & c_2^{-1}x_{b-a} \end{bmatrix}_{f \times e},$$

so the formula (3.21) can be easily checked. As $b$ is invertible and $a <^- b$, it follows from Theorem 3.9 that $a\{1\}_b = \{b^{-1}ab^{-1}\}$. □

**4. Applications.** In this section, we indicate how the concepts and results from previous sections can be extended to matrices with entries in a ring $R$ and to Banach space operators. We want to point out the universality of the idea of matrix representation of elements of the various structures.

**Minus partial order for matrices with entries in a ring.** As before, $R$ denotes a ring with identity 1. The set of all $m \times n$ matrices with entries from $R$ will be denoted by $M_{m\times n}(R)$. For any $A \in M_{m\times n}(R)$, we will denote by $CS(A) := \{A\xi : \xi \in M_{n\times 1}(R)\}$ and $RS(A) := \{\xi A : \xi \in M_{1\times m}(R)\}$ the column space of $A$ and row space of $A$, respectively. If $A \in M_{m\times n}(R)$, we say that $A$ is *regular* matrix if there is a matrix $X \in M_{n\times m}(R)$ such that $AXA = A$, in which case we call $X$ the *generalized inverse* of $A$. If $AXA = A$ and $XAX = X$ then $X$ is a *reflexive generalized inverse* of $A$. Of course, $A\{1\}$ ($A\{1,2\}$) stands for the set of all generalized (reflexive generalized) inverses of $A$. Let $M^{(1)}_{m\times n}(R)$ denotes the set of all regular matrices in $M_{m\times n}(R)$.

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ELA
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According to von Neumann, if $R$ is a regular ring, then $M_{n\times n}(R)$ is also a regular ring. Moreover, every matrix $A \in M_{m\times n}(R)$ over a regular ring $R$ is regular; see Theorem 3.5 in [2].

For $A, B \in M_{m\times n}(R)$, *minus partial order* is defined analogously as in the ring case, and *space pre-order* is defined by $A <^s B$ if

$$CS(A) \subseteq CS(B) \quad \text{and} \quad RS(A) \subseteq RS(B). \tag{4.1}$$

The idea from Remark 2.2 can be applied to $M_{m\times n}(R)$. Let $I_m = E_1 + \cdots + E_r$ and $I_n = F_1 + \cdots + F_s$ be decompositions of the identity of the rings $M_{m\times m}(R)$ and $M_{n\times n}(R)$ respectively, where $I_m \in M_{m\times m}(R)$ and $I_n \in M_{n\times n}(R)$ are identity matrices. For any $X \in M_{m\times n}(R)$ we have

$$X = I_m X I_n = (E_1 + \cdots + E_r)X(F_1 + \cdots + F_s) = \sum_{i=1}^{r}\sum_{j=1}^{s} E_i X F_j.$$

To avoid repetition, we only note that conclusions analogous to those from Remark 2.2 are true in the present case.

As the reader might already have guessed, except Theorem 3.16, all results from the previous section remain valid in the present setting. Some comments are still necessary. Depending on the context, the ring $R$ is replaced by $M_{m\times n}(R)$, $M_{m\times m}(R)$, $M_{n\times m}(R)$ or $M_{n\times n}(R)$. Also, the set $R^{(1)}$ is replaced by $M^{(1)}_{m\times n}(R)$ or $M^{(1)}_{n\times m}(R)$, but in the statements where the regularity of $R$ is assumed, this assumption remains the same (due to the von Neumann result stated above). Conditions $xR\rho yR$ and $Rx\rho Ry$ where $\rho \in \{\subseteq, =\}$ are replaced by $CS(X)\rho CS(Y)$ and $RS(X)\rho RS(Y)$, respectively. Since $R$ has identity, for $X, Y \in M_{m\times n}(R)$, the condition $CS(X) \subseteq CS(Y)$ is equivalent to $X = YZ$ for some $Z \in M_{n\times n}(R)$. Also, $RS(X) \subseteq RS(Y)$ is equivalent to $X = ZY$ for some $Z \in M_{m\times m}(R)$.

Except Theorem 3.16, the proofs of all statements in the present setting proceed along the same lines as the proofs of corresponding statements in the previous setting.

Lemma 3.2 in present setting was originally proved in [6]. In the same paper, it is shown that if $R$ is a regular prime ring and $A, B, C \in M_{m\times n}(R)$, then $CS(C) \subseteq CS(B)$ and $RS(A) \subseteq RS(B)$ if and only if $AB^{(1,2)}C$ is invariant under the choices of $B^{(1,2)} \in B\{1, 2\}$. In Theorem 3.1 in the present setting we only require that $R$ is regular but we consider only invariance of $AB^{(1)}A$.

**Minus partial order for operators on Banach spaces.** We now consider how the concept can be extended to Banach space operators.

Let $\mathcal{B}(X, Y)$ denote the set of all bounded linear operators from Banach space $X$

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ELA
http://math.technion.ac.il/iic/ela



to Banach space $Y$. Also, $\mathcal{B}(X) = \mathcal{B}(X, X)$. We use $\mathcal{N}(A)$ and $\mathcal{R}(A)$ to denote null space and range of $A \in \mathcal{B}(X, Y)$, respectively.

An operator $A \in \mathcal{B}(X, Y)$ is *regular* if there exists an operator $A^{(1)} \in \mathcal{B}(Y, X)$ such that $AA^{(1)}A = A$. The operator $A^{(1)}$ is called *generalized inverse* of $A$. It is well known that $A \in \mathcal{B}(X, Y)$ is regular if and only if $\mathcal{R}(A)$ is closed and complemented in $Y$ and $\mathcal{N}(A)$ is complemented in $X$. If $AA^{(1)}A = A$ and $A^{(1)}AA^{(1)} = A^{(1)}$, then $A^{(1)}$ is *reflexive generalized inverse* of $A$. Of course, $A\{1\}$ and $A\{1, 2\}$ stands for the sets of all generalized and reflexive generalized inverses of $A$. $\mathcal{B}^{(1)}(X, Y)$ denotes the set of all regular operators from $\mathcal{B}(X, Y)$.

For $A, B \in \mathcal{B}(X, Y)$, $A <^- B$ if $A$ is regular and there exists $A^{(1)} \in A\{1\}$ such that $AA^{(1)} = BA^{(1)}$ and $A^{(1)}A = A^{(1)}B$; and $A <^s B$ if $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A)$.

Let $B \in \mathcal{B}^{(1)}(X, Y)$. Note that $\mathcal{R}(B) = \mathcal{R}(BB^{(1)})$ because

$$\mathcal{R}(B) = \mathcal{R}(BB^{(1)}B) \subseteq \mathcal{R}(BB^{(1)}) \subseteq \mathcal{R}(B).$$

Also, $\mathcal{R}(A) \subseteq \mathcal{R}(B) \Leftrightarrow A = BB^{(1)}A$ because $\mathcal{R}(A) \subseteq \mathcal{R}(B) = \mathcal{R}(BB^{(1)}) = \mathcal{N}(I - BB^{(1)})$ so $(I - BB^{(1)})A = 0$ and the opposite direction is trivial. Similarly, $\mathcal{N}(B) = \mathcal{N}(B^{(1)}B)$ and $\mathcal{N}(B) \subseteq \mathcal{N}(A) \Leftrightarrow A = AB^{(1)}B$. Because of this, the conditions $aR\rho bR$ and $Ra\rho Rb$, where $\rho \in \{\subseteq, =\}$, are analogous to the conditions $\mathcal{R}(A)\rho\mathcal{R}(B)$ and $\mathcal{N}(B)\rho\mathcal{N}(A)$, respectively.

A bounded idempotent $E \in \mathcal{B}(X)$ is called a *projection*. If we consider algebra $\mathcal{B}(X)$ as a ring with identity $I_X \in \mathcal{B}(X)$, then the notion of decomposition of the identity of $\mathcal{B}(X)$ makes sense.

Lemma 4.1. *If $E_1, E_2 \in \mathcal{B}(X)$ are two projections such that $E_1E_2 = E_2E_1 = 0$, then $E_1 + E_2$ is projection and $\mathcal{R}(E_1 + E_2) = \mathcal{R}(E_1) \oplus \mathcal{R}(E_2)$.*

*Proof.* It is clear that $E_1 + E_2$ is a projection and that $\mathcal{R}(E_1 + E_2) \subseteq \mathcal{R}(E_1) + \mathcal{R}(E_2)$. If $x \in \mathcal{R}(E_1)$, then $0 = E_2E_1x = E_2x$, so $x = (E_1 + E_2)x \in \mathcal{R}(E_1 + E_2)$. Similarly, $\mathcal{R}(E_2) \subseteq \mathcal{R}(E_1 + E_2)$. If $x \in \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$, then $x = E_1x = E_2x$ and hence, $x = E_1^2x = E_1E_2x = 0$. ☐

Corollary 4.2. *Let*

$$I_X = E_1 + \cdots + E_n \tag{4.2}$$

*be a decomposition of the identity of $\mathcal{B}(X)$ and let $S \subseteq \{1, 2, \ldots, n\}$. Then $\sum_{i\in S} E_i$ is a projection and*

$$\mathcal{R}\left(\sum_{i\in S} E_i\right) = \bigoplus_{i\in S} \mathcal{R}(E_i). \tag{4.3}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA
http://math.technion.ac.il/iic/ela



*In particular,*

$$X = \bigoplus_{i=1}^{n} \mathcal{R}(E_i). \tag{4.4}$$

*Proof.* Without loss of generality, we can assume that $S = \{1, 2, \ldots, k\}$, $1 \le k \le n$. Using Lemma 4.1, (4.3) follows easily by induction on $k$. Equation (4.4) follows from (4.3) by $S = \{1, 2, \ldots, n\}$. ∎

Note that from (4.3) it follows that $\bigoplus_{i \in S} \mathcal{R}(E_i)$ is closed and complemented in $X$. Also, it follows that $\mathcal{N}(E_j) = \bigoplus_{i=1, i \neq j}^{n} \mathcal{R}(E_i)$.

Let $I_X = F_1 + \cdots + F_n$ and $I_Y = E_1 + \cdots + E_m$ be decompositions of the identity of $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, respectively. Suppose that $A \in \mathcal{B}(X, Y)$. Then

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} E_i A F_j, \tag{4.5}$$

and so, for $x = x_1 + \cdots + x_n \in \mathcal{R}(F_1) \oplus \cdots \oplus \mathcal{R}(F_n) = X$, we have

$$Ax = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j, \tag{4.6}$$

where operator $A_{ij} : \mathcal{R}(F_j) \to \mathcal{R}(E_i)$ is defined by $A_{ij} x_j := E_i A F_j x_j = E_i A F_j x$. It is clear that $A_{ij} \in \mathcal{B}(\mathcal{R}(F_j), \mathcal{R}(E_i))$.

Now suppose that $A_{ij} \in \mathcal{B}(\mathcal{R}(F_j), \mathcal{R}(E_i))$ and let $A : X \to Y$ be defined by (4.6). It follows that

$$\begin{aligned} \|Ax\| &\le mM(\|x_1\| + \cdots + \|x_n\|) = mM(\|F_1 x\| + \cdots + \|F_n x\|) \\ &\le mM(\|F_1\| + \cdots + \|F_n\|)\|x\|, \end{aligned}$$

where $M = \max\{\|A_{ij}\| : i = \overline{1, m}, j = \overline{1, n}\}$. Therefore, $A \in \mathcal{B}(X, Y)$ if and only if $A_{ij} \in \mathcal{B}(\mathcal{R}(F_j), \mathcal{R}(E_i))$. In this case, (4.6), i.e., (4.5), can be rewritten in the matrix form

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(F_1) \\ \vdots \\ \mathcal{R}(F_n) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(E_1) \\ \vdots \\ \mathcal{R}(E_m) \end{bmatrix}.$$

We can now restate Theorems 3.5 and 3.6 in the new setting. Suppose that $A, B \in \mathcal{B}^{(1)}(X, Y)$, $A <^- B$ and let $E_i$ and $F_i$ be defined by (3.7). Since $\mathcal{R}(E_1) = \mathcal{R}(AH) =$

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$\mathcal{R}(A)$, $\mathcal{R}(E_2) = \mathcal{R}((B-A)H) = \mathcal{R}(B-A)$ and $\mathcal{R}(F_3) = \mathcal{R}(I_X - HB) = \mathcal{N}(HB) = \mathcal{N}(B)$, we have

$$
\begin{aligned}
A &= \begin{bmatrix} C_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(F_1) \\ \mathcal{R}(F_2) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{R}(E_3) \end{bmatrix}, \\
B &= \begin{bmatrix} C_A & 0 & 0 \\ 0 & C_{B-A} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(F_1) \\ \mathcal{R}(F_2) \\ \mathcal{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(B-A) \\ \mathcal{R}(E_3) \end{bmatrix},
\end{aligned} \tag{4.7}
$$

for some operators $C_A \in \mathcal{B}(\mathcal{R}(F_1), \mathcal{R}(A))$ and $C_{B-A} \in \mathcal{B}(\mathcal{R}(F_2), \mathcal{R}(B-A))$. Theorem 3.6 actually says that $C_A$ and $C_{B-A}$ are invertible. The matrix forms in the above are originally obtained in [16] but in the different way. It is now clear that all results coming after Theorem 3.6 are still valid in the present setting.

The crucial assumption in Theorems 3.1 and 3.3 is that $R$ is a regular ring. We cannot require the same condition for Banach space operators. But, it is shown in [16] that these theorems still hold in the present setting. The only requirement is that $A$ and $B$ are regular operators.

Suppose that $A, B \in \mathcal{B}^{(1)}(X,Y)$ and $A <^- B$. Therefore, $\mathcal{R}(B) = \mathcal{R}(BH) = \mathcal{R}(E_1 + E_2) = \mathcal{R}(E_1) \oplus \mathcal{R}(E_2) = \mathcal{R}(A) \oplus \mathcal{R}(B-A)$. Conversely, suppose that

$$
\mathcal{R}(B) = \mathcal{R}(A) \oplus \mathcal{R}(B-A). \tag{4.8}
$$

Then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, so $A = BB^{(1)}A$. Hence, $A - AB^{(1)}A = BB^{(1)}A - AB^{(1)}A = (B-A)B^{(1)}A$. Also, $A(I-B^{(1)}B) = (A-B)(I-B^{(1)}B)$. Since $\mathcal{R}(A) \cap \mathcal{R}(B-A) = \{0\}$, we have $A = AB^{(1)}A = AB^{(1)}B$. By Lemma 3.4, we conclude that $A <^- B$.

Suppose now that $X$ and $Y$ are finite dimensional vector spaces, i.e., suppose that $A$ and $B$ are $m \times n$ complex matrices. Note that (4.8) is equivalent to $\text{rank}(B) = \text{rank}(A) + \text{rank}(B-A)$, so we obtain (3.4). Let us look at (4.7). Decompositions $\mathbb{C}^n = \mathcal{R}(F_1) \oplus \mathcal{R}(F_2) \oplus \mathcal{N}(B)$ and $\mathbb{C}^m = \mathcal{R}(A) \oplus \mathcal{R}(B-A) \oplus \mathcal{R}(E_3)$ induce changes of bases in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Thus, there exist non-singular matrices $S_1$ and $T$ such that

$$
A = S_1 \begin{bmatrix} C_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} T \quad \text{and} \quad B = S_1 \begin{bmatrix} C_A & 0 & 0 \\ 0 & C_{B-A} & 0 \\ 0 & 0 & 0 \end{bmatrix} T.
$$

Setting

$$
S = S_1 \begin{bmatrix} C_A & 0 & 0 \\ 0 & C_{B-A} & 0 \\ 0 & 0 & I \end{bmatrix},
$$

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we obtain representations (1.2) from the beginning of the paper.

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