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THE MAXIMUM ORDER OF REDUCED SQUARE $(0,1)$-MATRICES WITH A GIVEN RANK*

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Abstract. The maximum order of a square $(0,1)$-matrix A with a fixed rank r is considered, provided A has no repeated rows or columns. When A is the adjacency matrix of a graph, Kotlov and Lovász [A. Kotlov and L. Lovász. The rank and size of graphs. *J. Graph Theory*, 23:185–189, 1996.] proved that the maximum order equals $\Theta(2^{r/2})$. In this note, it is showed that this result remains correct if A is symmetric, but becomes false if symmetry is not required.

Key words. $(0,1)$-Matrix, Rank, Graph.

AMS subject classifications. 05B20, 05E99, 15B36.

1. Introduction. At the workshop "Directions in Matrix Theory 2011" to which the present special issue is devoted, the first author gave a presentation about the results from [2] on the maximum order of reduced adjacency matrices of graphs with a given rank. At the end of the presentation, there was some discussion on what can be said if the zero diagonal, or the symmetry condition is dropped. Here we give some answers to these more general situations.

Given a $(0,1)$-matrix, we can duplicate a row or column and add zero rows and columns without changing the rank. This motivates the following definition.

Definition 1.1. A (0,1)-matrix is said to be *reduced* if no two rows or columns are dependent.

We define $n(r)$ to be the maximum order of a reduced square $(0,1)$-matrix of rank r. When we restrict to symmetric matrices, or adjacency matrices of graphs, we denote the maximum order by $n_{\rm sym}(r)$ and $n_{\rm gr}(r)$, respectively. Kotlov and Lovász [3] obtain that $n_{\rm gr}(r) = \mathcal{O}(2^{r/2})$, and give a construction showing that $n_{\rm gr}(r) \geq 2^{(r+2)/2} - 2$ if r is even, and $n_{\rm gr}(r) \geq 5 \cdot 2^{(r-3)/2} - 2$ if r is odd. In [1], the authors present a second construction giving the same number of vertices. They conjecture that the mentioned values are equal to the maximum $n_{\rm gr}(r)$, and that all graphs attaining this maximum can be obtained by the two construction methods. In [2], we show that the conjecture

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is correct if the graph contains $\frac{r}{2}K_2$ or $K_3 + \frac{r-3}{2}K_2$ as an induced subgraph.

Here, we show that the same asymptotic upper bound holds for $n_{\rm sym}(r)$ as for $n_{\rm gr}(r)$. For even r we find no better lower bound for $n_{\rm sym}(r)$ than for $n_{\rm gr}(r)$, but for odd r the lower bound for $n_{\rm sym}(r)$ can be improved. In addition, we prove that $n(r) \geq 2^{(r-3)/2}\sqrt{r}$, which shows that the upper bound of Kotlov and Lovász does not hold anymore for the nonsymmetric case.

2. Symmetric $(0,1)$-matrices.

2.1. The upper bound. A symmetric $(0,1)$-matrix can be interpreted as the adjacency matrix of a (simple) graph admitting loops. For the proof of the upper bound $n_{\rm gr}(r) = \mathcal{O}(2^{r/2})$, Kotlov and Lovász defined *twins* in a graph as two vertices with the same set of neighbors. For a loopless graph, twins are necessarily nonadjacent, but if loops are allowed twins can be adjacent vertices with loops. In other words, twins correspond to a pair of repeated rows and columns in the adjacency matrix. Keeping this in mind, we claim that the mentioned proof of Kotlov and Lovász is also valid for graphs admitting loops, since nowhere it is used that the matrix has zero diagonal. Therefore:

THEOREM 2.1. $n_{\rm sym}(r) = \mathcal{O}(2^{r/2})$.

2.2. Constructions. The following recursive construction was given in [3] (see also [1, 2]). Suppose A be a reduced adjacency matrix of a graph on n vertices with rank r, then

$$\begin{bmatrix} A & A & \mathbf{0} & \mathbf{0} \\ A & A & \mathbf{1} & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{1}^\top & 0 & 1 \\ \mathbf{0}^\top & \mathbf{0}^\top & 1 & 0 \end{bmatrix}$$

is the reduced adjacency of a graph on $2n+2$ vertices with rank $r+2$. If we start with K_2 (which has $r = n = 2$) and K_3 (which has $r = n = 3$), then the recursive construction gives graphs with $2^{(r+2)/2} - 2$ vertices if r is even, and $5 \cdot 2^{(r-3)/2} - 2$ vertices if r is odd. Note that there exist no graph whose adjacency matrix has rank 1. Obviously, the mentioned numbers are lower bounds for $n_{\rm sym}(r)$, but for odd r we can do better since we can start the recursive construction with $A = [\,1\,]$ and $n = r = 1$, which leads to reduced symmetric $(0,1)$-matrices of order $3 \cdot 2^{(r-1)/2} - 2$ and rank r. Thus, we have:

PROPOSITION 2.2. $n_{\rm sym}(r) \geq \begin{cases} 2^{(r+2)/2} - 2 & \textit{if } r \textit{ is even,} \\ 3 \cdot 2^{(r-1)/2} - 2 & \textit{if } r \textit{ is odd.} \end{cases}$

There exist more recursive constructions that make a reduced adjacency matrix

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of rank $r+2$ and order $2n+2$ from one of rank r and order n (see [1, 2]), and if we are not forced to have a zero diagonal we can find even more. For example,

$$\begin{bmatrix} A & A & \mathbf{0} & \mathbf{0} \\ A & A & \mathbf{1} & \mathbf{0} \\ \mathbf{0}^\top & \mathbf{1}^\top & 1 & 1 \\ \mathbf{0}^\top & \mathbf{0}^\top & 1 & 0 \end{bmatrix}$$

also works. So, there will be many symmetric $(0,1)$-matrices that meet the bound given in the above lemma.

3. Square $(0,1)$-matrices. In order to construct large $(0,1)$-matrices of rank r, we consider two sets X and Y of vectors in $V = \{0,1\}^r$. Each vector $v \in V$ is partitioned into two parts:

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

where $v_1 \in \{0,1\}^k$ and $v_2 \in \{0,1\}^{r-k}$ for some $k \in \{0,\ldots,r\}$. The set X consists of all nonzero vectors in V for which $v_2 = 0$, together with the vectors for which $v_1 = 0$ and the weight[1] of v_2 equals 1. The set Y consists of all nonzero vectors in V for which the weight of v_1 is at most 1. Thus, we have $|X| = 2^k + r - k - 1$ and $|Y| = 2^{r-k}(k+1) - 1$. Let M and N be the matrices whose columns are the vectors from X and Y, respectively. Define $A = M^\top N$. Then, clearly A is a $(0,1)$-matrix. Note that both M and N contain r unit columns, so we can write

$$M = [\,I\ M_1\,]\ ,\quad N = [\,I\ N_1\,]\ ,\quad A = \begin{bmatrix} I & N_1 \\ M_1^\top & M_1^\top N_1 \end{bmatrix}.$$

Therefore, A is a reduced matrix with rank r, and A contains a reduced square submatrix of order $n = \min\{|X|,|Y|\}$ with rank r. If we choose k such that n is as large as possible, then we obtain the following values of k and n for $r = 1,\ldots,10$.

r	1	2	3	4	5	6	7	8	9	10
k	1	1	2	3	3	4	5	5	6	6
n	1	2	4	7	9	17	23	34	55	67

Note that for $r = 5$, the above construction gives $n = 9$, whilst in the previous section, we found a symmetric example with $n = 10$. So, we can conclude that $n(5) \geq 10$. For general r, the above construction leads to the following bound.

THEOREM 3.1. *The maximum order $n(r)$ of a reduced square $(0,1)$-matrix with rank r is at least*

$$2^{\frac{r-3}{2}}\sqrt{r}\,.$$

[1]The *weight* of a vector is the number of nonzero coordinates.

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Proof. Assume that $r \geq 2$, and take k such that $k \geq 2^{2k-r}$ and $k+1 < 2^{2(k+1)-r}$. Then the first inequality gives $|X| \leq |Y|$ and $k < r$, and therefore, $n = |X| = 2^k + r - k - 1 \geq 2^k$. The second inequality gives $k > \frac{1}{2}r + \frac{1}{2}\log_2(k+1) - 1$, and $k+1 > \frac{1}{2}r$. This implies

$$n \geq 2^k > 2^{\frac{r}{2}+\frac{1}{2}\log_2(k+1)-1} > 2^{\frac{r-3}{2}}\sqrt{r}\,. \quad \square$$

REFERENCES

[1] S. Akbari, P.J. Cameron, and G.B. Khosrovshahi. Ranks and signatures of adjacency matrices. Preprint, 2004. Availiable at `http://www.maths.qmw.ac.uk/~pjc/preprints/ranksign.pdf`.

[2] W.H. Haemers and M.J.P. Peeters. The maximum order of adjacency matrices of graphs with a given rank. *Des. Codes Cryptogr.*, 2011, DOI: 10.1007/s10623-011-9548-3.

[3] A. Kotlov and L. Lovász. The rank and size of graphs. *J. Graph Theory*, 23:185–189, 1996.