Electronic Journal of Linear Algebra

Volume 27 Article 2

2014

Spectral characterizations of propeller graphs

Xiagang Liu

Sanming Zhou
smzhou@ms.unimelb.edu.au

Recommended Citation

Liu, Xiagang and Zhou, Sanming. (2014), "Spectral characterizations of propeller graphs", *Electronic Journal of Linear Algebra*, Volume 27.
DOI: https://doi.org/10.13001/1081-3810.1603

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

ELA
http://math.technion.ac.il/iic/ela

SPECTRAL CHARACTERIZATIONS OF PROPELLER GRAPHS*

XIAOGANG LIU† AND SANMING ZHOU†

Abstract. A *propeller graph* is obtained from an ∞-graph by attaching a path to the vertex of degree four, where an ∞-graph consists of two cycles with precisely one common vertex. In this paper, it is proved that all propeller graphs are determined by their Laplacian spectra as well as their signless Laplacian spectra.

Key words. L-spectrum, Q-spectrum, L-DS graph, Q-DS graph, L-cospectral graph, Q-cospectral graph.

AMS subject classifications. 05C50.

1. Introduction. All graphs considered in the paper are undirected and simple. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix whose (i, j)-entry is 1 if v_i and v_j are adjacent and 0 otherwise. Denote by $d_i = d_G(v_i)$ the degree of v_i in G, and by

$$\deg(G) = (d_1, d_2, \dots, d_n)$$

the degree sequence of G. The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix with diagonal entries $d_1, d_2, \dots, d_n$. We call $Q(G) = D(G) + A(G)$ the *signless Laplacian matrix* of G. Denote the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$, respectively. The collection of eigenvalues of $A(G)$ together with multiplicities are called the *A-spectrum* of G. Two graphs are said to be *A-cospectral* if they have the same A-spectrum. A graph is called an *A-DS graph* if it is *determined by its A-spectrum*, meaning that there exists no other graph that is non-isomorphic to it but A-cospectral with it. Similar terminology will be used for $L(G)$ and $Q(G)$. So we can speak of *L-spectrum*, *Q-spectrum*, *L-cospectral graphs*, *Q-cospectral graphs*, *L-DS graphs* and *Q-DS graphs*.

Which graphs are determined by their spectra? This is a classical question in spectral graph theory which was raised by Günthard and Primas [12] in 1956 with

*Received by the editors on April 17, 2012. Accepted for publication on December 15, 2013. Handling Editor: Xingzhi Zhan.

†Department of Mathematics and Statistics, The University of Melbourne, Parkville, VIC 3010, Australia (xiaogliu@student.unimelb.edu.au, smzhou@ms.unimelb.edu.au).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

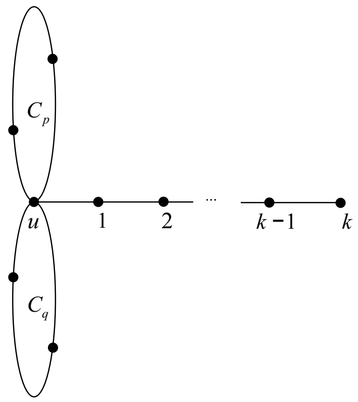

FIG. 1.1. *A propeller graph.*

motivations from chemistry. This problem is also related to complexity theory. It is well-known that the complexity of the problem of determining graph isomorphism is unknown [13]. Since checking whether two graphs are cospectral can be done in polynomial time, the isomorphism problem can be reduced to the one of checking isomorphism between cospectral graphs. Up to now, many graphs have been proved to be determined by their (A, L or/and Q) spectra [2, 3, 5, 8, 10, 11, 17, 19, 20, 21, 22, 24, 28, 29]. However, the problem of determining A-DS (respectively, L-DS, Q-DS) graphs is still far from being completely solved. Therefore, finding new families of DS graphs deserves further attention in order to enrich our database of DS graphs. Unfortunately, even for some simple-looking graphs, it is often challenging to determine whether they are A-DS, L-DS or Q-DS.

In this paper, we give a new family graphs that are both L-DS and Q-DS. We define a *propeller graph* (see Fig. 1.1) as a graph obtained from an ∞-graph by attaching a path to the vertex of degree 4, where an ∞-*graph* is a graph consisting of two cycles with exactly one vertex in common [28]. The main results of this paper are as follows.

THEOREM 1.1. *All propeller graphs are determined by their L-spectra.*

THEOREM 1.2. *All propeller graphs are determined by their Q-spectra.*

Since the L-spectrum of a graph determines that of its complement [18], Theorem 1.1 implies that the complement of any propeller graph is also determined by its L-spectrum.

We will prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

ELA
http://math.technion.ac.il/iic/ela

2. Preliminaries. In this section, we collect some known results that will be used in the proof of Theorems 1.1 and 1.2. Denote by

$$\phi(M) = \phi(M; x) = \det(xI - M) = l_0x^n + l_1x^{n-1} + \cdots + l_n$$

the characteristic polynomial of an $n \times n$ matrix M, where I is the identity matrix of the same size. In particular, for a graph G, we call $\phi(A(G))$ (respectively, $\phi(L(G))$, $\phi(Q(G))$) the *adjacency* (respectively, *Laplacian*, *signless Laplacian*) *characteristic polynomial* of G.

Denote by $n_3(G)$ the number of triangles in G.

LEMMA 2.1. [23] *Let G be a graph with n vertices and m edges, and let* $\deg(G) = (d_1, d_2, \ldots, d_n)$ *be its degree sequence. Then the first four coefficients in $\phi(L(G))$ are:*

$$l_0 = 1, \qquad l_1 = -2m, \qquad l_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2,$$

$$l_3 = \frac{1}{3}\left(-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i^3 - 3\sum_{i=1}^{n} d_i^2 + 6n_3(G)\right).$$

The following result follows from [10] and Lemma 2.1.

LEMMA 2.2. *Let G be a graph. The following can be determined by its L-spectrum:*

(a) *the number of vertices of G;*
(b) *the number of edges of G;*
(c) *the number of components of G;*
(d) *the number of spanning trees of G.*

LEMMA 2.3. [5] *Let u be a vertex of G, $N(u)$ the set of vertices of G adjacent to u, and $C(u)$ the set of cycles of G containing u. Then,*

$$\phi(A(G); x) = x\phi(A(G-u); x) - \sum_{v \in N(u)} \phi(A(G-u-v); x)$$
$$-2\sum_{Z \in C(u)} \phi(A(G - V(Z)); x).$$

LEMMA 2.4. [28] *Let G be a graph with n vertices, m edges and degree sequence* $\deg(G) = (d_1, d_2, \ldots, d_n)$. *If a graph H with degree sequence* $\deg(H) = (d_1 + t_1, d_2 + t_2, \ldots, d_n + t_n)$ *is L-cospectral (respectively, Q-cospectral) with G, then $t_1, t_2, \ldots, t_n$ are integers such that*

$$\sum_{i=1}^{n} t_i = 0 \quad \text{and} \quad \sum_{i=1}^{n}(t_i^2 + 2d_it_i) = 0.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

Denote by P_n and C_n the path and cycle on n vertices, respectively. Let B_n be the matrix of order n obtained from $L(P_{n+1})$ by deleting the row and column corresponding to one end vertex of P_{n+1}, and U_n be the matrix of order n obtained from $L(P_{n+2})$ by deleting the rows and columns corresponding to the two end vertices of P_{n+2}.

Lemma 2.5. [16] *Set* $\phi(L(P_0)) = 0$, $\phi(B_0) = 1$, $\phi(U_0) = 1$. *Then*

(a) $\phi(L(P_{n+1})) = (x-2)\phi(L(P_n)) - \phi(L(P_{n-1}))$, $(n \geq 1)$;
(b) $x\phi(B_n) = \phi(L(P_{n+1})) + \phi(L(P_n))$;
(c) $\phi(L(P_n)) = x\phi(U_{n-1})$, $(n \geq 1)$;
(d) $\phi(L(C_n)) = \frac{1}{x}\phi(L(P_{n+1})) - \frac{1}{x}\phi(L(P_{n-1})) + 2(-1)^{n+1}$, $(n \geq 3)$.

Combining these and $\phi(L(P_1); 4) = 4$, we obtain the following formulas.

Proposition 2.6. (a) $\phi(L(P_n); 4) = 4n$; (b) $\phi(B_n; 4) = 2n+1$; (c) $\phi(U_n; 4) = n+1$; (d) $\phi(L(C_n); 4) = 2 + 2(-1)^{n+1}$.

For a vertex v of G, let $L_v(G)$ denote the principal sub-matrix of $L(G)$ formed by deleting the row and column corresponding to v.

Lemma 2.7. [15] *Let G_1 and G_2 be vertex-disjoint graphs. Let G be the graph obtained by taking the union of G_1 and G_2 and then adding an edge between a vertex u of G_1 and a vertex v of G_2. Then*

$$\phi(L(G)) = \phi(L(G_1))\phi(L(G_2)) - \phi(L(G_1))\phi(L_v(G_2)) - \phi(L(G_2))\phi(L_u(G_1)).$$

Lemma 2.8. [7, 26] *Let G be a graph with n vertices, m edges and $n_3(G)$ triangles. Let $T_k = \sum_{i=1}^{n} \nu_i^k$ be the kth Q-spectral moment of G, $k = 0, 1, 2, \ldots$ Then*

$$T_0 = n, \quad T_1 = \sum_{i=1}^{n} d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \quad T_3 = 6n_3(G) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

From Lemma 2.8, we can easily get the following result.

Lemma 2.9. *Let G and H be Q-cospectral graphs. Then*

(a) *G and H have the same number of vertices;*
(b) *G and H have the same number of edges;*
(c) $\sum_{v \in V(G)} d_G(v)^2 = \sum_{v \in V(H)} d_H(v)^2$;
(d) $6n_3(G) + \sum_{v \in V(G)} d_G(v)^3 = 6n_3(H) + \sum_{v \in V(H)} d_H(v)^3$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014
ELA
http://math.technion.ac.il/iic/ela

Let $\mathcal{L}(G)$ denote the line graph of a graph G. Let $\mathcal{S}(G)$ be the *subdivision graph* of G obtained by replacing each edge of G by a path of length two. The Q-spectrum of a graph can be exactly expressed by the A-spectrum of its line and subdivision graphs [4, 7, 9], and the following results can be found in [4, 7, 28].

LEMMA 2.10. *If two graphs G and H are Q-cospectral, then $\mathcal{L}(G)$ and $\mathcal{L}(H)$ are A-cospectral.*

LEMMA 2.11. *Two graphs G and H are Q-cospectral if and only if $\mathcal{S}(G)$ and $\mathcal{S}(H)$ are A-cospectral.*

LEMMA 2.12. [6] *Let G be a graph with n vertices and m edges. Let $n_4(G)$ be the number of subgraphs of G isomorphic to C_4, and x_k the number of vertices of degree k in G. Then*

$$\sum_i \lambda_i^4 = 8n_4(G) + \sum_k kx_k + 4\sum_{k\geq 2}\frac{k(k-1)}{2}x_k.$$

A spanning subgraph of G whose components are trees or odd-unicyclic graphs is called a *TU-subgraph* of G [7]. Suppose that a TU-subgraph G^{TU} of G contain c unicyclic graphs and trees $T_1, T_2, \ldots, T_s$. The weight $W(G^{TU})$ of G^{TU} is defined by

$$W(G^{TU}) = 4^c\prod_{i=1}^{s}(1+|E(T_i)|).$$

Then the coefficients of $\phi(Q(G))$ can be expressed in terms of the weights of TU-subgraphs of G as follows.

LEMMA 2.13. [7] *Let $\phi(Q(G)) = q_0x^n + q_1x^{n-1} + \cdots + q_n$. Then $q_0 = 1$ and*

$$q_j = \sum_{G_j^{TU}}(-1)^jW(G_j^{TU}), \quad j = 1, 2, \ldots, n,$$

where the summation runs over all TU-subgraphs G_j^{TU} of G with j edges.

3. Proof of Theorem 1.1. Throughout this section, we use G to denote a propeller graph with $n = p + q + k - 1$ vertices as shown in Fig. 1.1. To prove Theorem 1.1, we first compute the Laplacian characteristic polynomial of G. Before proceeding, we need the following results.

PROPOSITION 3.1. *Let G_1 and G_2 be vertex-disjoint graphs. Let $G_1 \cdot G_2$ be the coalescence obtained from G_1 and G_2 by identifying a vertex u of G_1 with a vertex v of G_2. Then*

$$\phi(L(G_1\cdot G_2); x) = \phi(L(G_1))\phi(L_v(G_2)) + \phi(L_u(G_1))\phi(G_2) - x\phi(L_u(G_1))\phi(L_v(G_2)).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

Proof. The coalescence $G_1 \cdot G_2$ has Laplacian matrix

$$\begin{pmatrix} L_u(G_1) & \mathbf{u} & O \\ \mathbf{u}^T & d_{G_1}(u) + d_{G_2}(v) & \mathbf{v} \\ O^T & \mathbf{v}^T & L_v(G_2) \end{pmatrix},$$

where $\begin{pmatrix} L_u(G_1) & \mathbf{u} \\ \mathbf{u}^T & d_{G_1}(u) \end{pmatrix}$ and $\begin{pmatrix} d_{G_2}(v) & \mathbf{v} \\ \mathbf{v}^T & L_v(G_2) \end{pmatrix}$ are the Laplacian matrices of G_1 and G_2 respectively, and O is the zero matrix of appropriate size. Then

$$\begin{aligned} \phi(L(G_1 \cdot G_2); x) &= \begin{vmatrix} xI - L_u(G_1) & -\mathbf{u} & O \\ -\mathbf{u}^T & x - d_{G_1}(u) - d_{G_2}(v) & -\mathbf{v} \\ O^T & -\mathbf{v}^T & xI - L_v(G_2) \end{vmatrix} \\ &= \begin{vmatrix} xI - L_u(G_1) & -\mathbf{u} & O \\ -\mathbf{u}^T & x - d_{G_1}(u) & -\mathbf{v} \\ O^T & \mathbf{0} & xI - L_v(G_2) \end{vmatrix} \\ &\quad + \begin{vmatrix} xI - L_u(G_1) & \mathbf{0} & O \\ -\mathbf{u}^T & x - d_{G_2}(v) & -\mathbf{v} \\ O^T & -\mathbf{v}^T & xI - L_v(G_2) \end{vmatrix} \\ &\quad + \begin{vmatrix} xI - L_u(G_1) & \mathbf{0} & O \\ -\mathbf{u}^T & -x & -\mathbf{v} \\ O^T & \mathbf{0} & xI - L_v(G_2) \end{vmatrix}, \end{aligned}$$

and the result follows. ∎

PROPOSITION 3.2. *Let $G_{p,q}$ be an ∞-graph consisting of cycles C_p and C_q with a common vertex u. Then*

$$\begin{aligned} \phi(L(G_{p,q}); x) &= (x-4)\phi(U_{p-1})\phi(U_{q-1}) - 2\phi(U_{q-1})\left(\phi(U_{p-2}) + (-1)^p\right) \\ &\quad - 2\phi(U_{p-1})\left(\phi(U_{q-2}) + (-1)^q\right), \end{aligned} \tag{3.1}$$

$$\phi(L(G_{p,q}); 4) = 2(p+q) - 4pq - 2\left((-1)^q p + (-1)^p q\right). \tag{3.2}$$

Proof. Lemma 2.5 implies that

$$\begin{aligned} \phi(L(C_n)) &= \frac{1}{x}\phi(L(P_{n+1})) - \frac{1}{x}\phi(L(P_{n-1})) + 2(-1)^{n+1} \\ &= \frac{1}{x}\big((x-2)\phi(L(P_n)) - \phi(L(P_{n-1}))\big) - \frac{1}{x}\phi(L(P_{n-1})) + 2(-1)^{n+1} \\ &= \frac{x-2}{x}\phi(L(P_n)) - \frac{2}{x}\phi(L(P_{n-1})) + 2(-1)^{n+1} \\ &= (x-2)\phi(U_{n-1}) - 2\phi(U_{n-2}) + 2(-1)^{n+1}. \end{aligned} \tag{3.3}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

Note that $G_{p,q}$ is a coalescence of C_p and C_q. Thus, we obtain (3.1) by using (3.3), $\phi(L_u(C_q)) = \phi(U_{q-1})$, $\phi(L_u(C_p)) = \phi(U_{p-1})$ and Proposition 3.1. (3.2) is an immediate consequence of (3.1) and Proposition 2.6. □

Proposition 3.3. *Let G be a propeller graph with $n = p + q + k - 1$ vertices as shown in Fig. 1.1. Then*

$$\phi(L(G); x) = \phi(L(G_{p,q}))\phi(L(P_k)) - \phi(L(G_{p,q}))\phi(B_{k-1}) - \phi(L(P_k))\phi(U_{p-1})\phi(U_{q-1}), \tag{3.4}$$

$$\phi((L(G); 4) = 2(2k + 1)(p + q - (-1)^q p - (-1)^p q) - 4pq(3k + 1).$$

Proof. We obtain (3.4) by using Lemma 2.7 and $\phi(L_u(G_{p,q})) = \phi(U_{p-1})\phi(U_{q-1})$. From (3.2) and (3.4) and Proposition 2.6, we have

$$\begin{aligned}\phi(L(G); 4) &= \phi(L(G_{p,q}); 4)\phi(L(P_k); 4) - \phi(L(G_{p,q}); 4)\phi(B_{k-1}; 4)\\ &\quad -\phi(L(P_k); 4)\phi(U_{p-1}; 4)\phi(U_{q-1}; 4)\\ &= (2(p + q) - 4pq - 2((-1)^q p + (-1)^p q))(4k - (2k - 1)) - 4kpq\\ &= 2(2k + 1)(p + q - (-1)^q p - (-1)^p q) - 4pq(3k + 1)\end{aligned}$$

as required. □

Note that $\phi(L(P_{n+1})) = (x - 2)\phi(L(P_n)) - \phi(L(P_{n-1}))$ by Lemma 2.5. Solving this recurrence equation, and noting $\phi(L(P_0)) = 0$ and $\phi(L(P_1)) = x$, we obtain that, for $n \geq 1$,

$$\phi(L(P_n)) = \frac{(y + 1)(y^{2n} - 1)}{y^{n+1} - y^n}, \tag{3.5}$$

where y satisfies the characteristic equation $y^2 - (x - 2)y + 1 = 0$ with $x \neq 4$. Substituting (3.5) into (b) and (c) of Lemma 2.5, we obtain

$$\phi(B_n) = \frac{y^{2n+1} - 1}{y^{n+1} - y^n}, \tag{3.6}$$

$$\phi(U_n) = \frac{y^{2n+2} - 1}{y^{n+2} - y^n}. \tag{3.7}$$

Plugging (3.5), (3.6) and (3.7) into (3.1) and then (3.4), and with the help of Maple, we obtain

$$y^n(y - 1)^3(y + 1)^2\phi(L(G)) + 1 - 3y - 4y^2 + 4y^{2n+3} + 3y^{2n+4} - y^{2n+5} = f_L(p, q, k; y), \tag{3.8}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

ELA
http://math.technion.ac.il/iic/ela

where

$$\begin{aligned} f_L(p,q,k;y) = \;& 2(-1)^{1+q}y^{2p+q+2k+3} && +2(-1)^{1+p}y^{2q+p+2k+3} && +2(-1)^{q}y^{2p+q+2k+1} \\ & +2(-1)^{p}y^{p+2q+2k+1} && +3y^{2p+2q+1}+3y^{2p+2q} && +y^{2p+3+2k} \\ & +y^{2q+3+2k} && +3y^{2p+2k+2} && +3y^{2q+2k+2} \\ & +2y^{2p+1+2k} && +2y^{2q+1+2k} && +2(-1)^{q}y^{2p+2+q} \\ & +2(-1)^{p}y^{2q+2+p} && +2(-1)^{1+q}y^{2p+q} && +2(-1)^{1+p}y^{2q+p} \\ & +2(-1)^{p}y^{3+p+2k} && +2(-1)^{q}y^{3+q+2k} && +2(-1)^{1+p}y^{p+2k+1} \\ & +2(-1)^{1+q}y^{q+2k+1} && -2y^{2p+2}-2y^{2q+2} && -3y^{2p+1}-3y^{2q+1} \\ & -y^{2p}-y^{2q}-3y^{2k+3} && +2(-1)^{1+p}y^{2+p} && +2(-1)^{1+q}y^{2+q} \\ & +2(-1)^{p}y^{p} && +2(-1)^{q}y^{q} && -3y^{2k+2}. \end{aligned}$$

Lemma 3.4. *No two non-isomorphic propeller graphs are L-cospectral.*

Proof. Let G and G' be L-cospectral propeller graphs with $n = p+q+k-1$ and $n' = p'+q'+k'-1$ vertices, respectively. Without loss of generality, we let $p \geq q$ and $p' \geq q'$. By (a) and (d) of Lemma 2.2, we have

$$p+q+k = p'+q'+k', \tag{3.9}$$

$$pq = p'q'. \tag{3.10}$$

By (3.8), we then get

$$f_L(p,q,k;y) = f_L(p',q',k';y). \tag{3.11}$$

Clearly, the term in $f_L(p,q,k;y)$ with the smallest exponent is $2(-1)^q y^q$ or $-3y^{2k+2}$, and similarly for $f_L(p',q',k';y)$. From (3.11) we have either $2(-1)^q y^q = 2(-1)^{q'} y^{q'}$ or $-3y^{2k+2} = -3y^{2k'+2}$. In the former case, we have $q = q'$, and so $p = p'$ and $k = k'$ by (3.9) and (3.10). In the latter case, we have $k = k'$, and so $(p,q) = (p',q')$ by (3.9) and (3.10). Therefore, G and G' are isomorphic in each case. ∎

Lemma 3.5. *Let H be a graph that is L-cospectral with the propeller graph G. Then*

$$\deg(H) = (5, 2^{n-2}, 1), (4^2, 2^{n-4}, 1^2), (4, 3^3, 2^{n-7}, 1^3), \text{ or } (3^6, 2^{n-10}, 1^4),$$

where the exponent denotes the number of vertices in H having the corresponding degree.

Proof. Suppose $\deg(H) = (5+t_1, 2+t_2, 2+t_3, \ldots, 2+t_{n-1}, 1+t_n)$. Since $\deg(G) = (5, 2^{n-2}, 1)$ and H is L-cospectral with G, by (c) of Lemma 2.2,

$$t_1 \geq -4,\; t_2 \geq -1,\; \ldots,\; t_{n-1} \geq -1,\; t_n \geq 0. \tag{3.12}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

ELA
http://math.technion.ac.il/iic/ela

Moreover, by Lemma 2.4, $t_1, t_2, \ldots, t_n$ are integers such that

$$\sum_{i=1}^{n} t_i = 0, \tag{3.13}$$

$$\sum_{i=1}^{n} t_i^2 + 4\sum_{i=2}^{n-1} t_i + 10t_1 + 2t_n = 0. \tag{3.14}$$

So $t_1 = -\sum_{i=2}^{n-1} t_i - t_n$. Plugging this into (3.14) yields

$$t_1^2 + 6t_1 + a = 0, \tag{3.15}$$

where a is given by

$$\sum_{i=2}^{n-1} t_i^2 = a - (t_n^2 - 2t_n). \tag{3.16}$$

Obviously, $a \geq t_n^2 - 2t_n \geq -1$. Solving (3.15) for t_1, we get

$$t_1 = -3 \pm \sqrt{9-a}.$$

Since t_1 is an integer and $-1 \leq a \leq 9$, we see that $a = 0, 5, 8, 9$. We discuss these cases one by one.

Case 1. $a = 0$. Then $t_1 = 0$ as $t_1 \geq -4$ by (3.12). Since $a = 0$, we have $\sum_{i=2}^{n-1} t_i^2 = -(t_n^2 - 2t_n) \geq 0$, which implies $t_n = 0, 1, 2$ as $t_n \geq 0$ by (3.12). Solving the Diophantine equations (3.13) and (3.16) for each t_n, and using (3.12), we obtain all possibilities for $(t_2, \ldots, t_{n-1})$, and hence, $\deg(H)$ as in Table 3.1. (In Tables 3.1–3.4, an exponent under the column $(t_2, \ldots, t_{n-1})$ indicates the number of times the corresponding value appears in this sequence. For example, -1^2 means that -1 appears twice.)

Case 2. $a = 5$. Then $t_1 = -1$ as $t_1 \geq -4$ by (3.12). Since $a = 5$, we have $\sum_{i=2}^{n-1} t_i^2 = 5 - (t_n^2 - 2t_n) \geq 0$, which implies $t_n = 0, 1, 2, 3$ as $t_n \geq 0$ by (3.12). Again, by using (3.12), (3.13) and (3.16), we obtain all possibilities for $(t_2, \ldots, t_{n-1})$ and $\deg(H)$ as shown in Table 3.2.

Case 3. $a = 8$. Then $t_1 = -2$ or $t_1 = -4$, and so (3.13) gives $\sum_{i=2}^{n} t_i = 2$ or 4, respectively. Since $\sum_{i=2}^{n-1} t_i^2 = 8 - (t_n^2 - 2t_n) \geq 0$ and $t_n \geq 0$, in each case we have $t_n = 0, 1, 2, 3, 4$. So we have ten combinations in total. Using (3.12), (3.13) and (3.16), we obtain all possibilities for $(t_2, \ldots, t_{n-1})$ and $\deg(H)$ as shown in Table 3.3.

Case 4. $a = 9$. Then $t_1 = -3$ and so $\sum_{i=2}^{n} t_i = 3$. Since $\sum_{i=2}^{n-1} t_i^2 = 9 - (t_n^2 - 2t_n) \geq 0$ and $t_n \geq 0$, we have $t_n = 0, 1, 2, 3, 4$. Again, by using (3.12), (3.13) and (3.16), we obtain all possibilities for $(t_2, \ldots, t_{n-1})$ and $\deg(H)$ as shown in Table 3.4. ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

ELA
http://math.technion.ac.il/iic/ela

t_1	t_n	$(t_2,\ldots,t_{n-1})$	$\deg(H)$
0	0	(0^{n-2})	$(5,2^{n-2},1)$
0	1	$(-1^1,0^{n-3})$	$(5,2^{n-2},1)$
0	2	Infeasible	

TABLE 3.1
$a=0$

t_1	t_n	$(t_2,\ldots,t_{n-1})$	$\deg(H)$
-1	0	$(2^1,-1^1,0^{n-4})$, $(1^3,-1^2,0^{n-7})$	$(4^2,2^{n-4},1^2)$, $(4,3^3,2^{n-7},1^3)$
-1	1	$(1^3,-1^3,0^{n-8})$, $(2^1,-1^2,0^{n-5})$	$(4,3^3,2^{n-7},1^3)$, $(4^2,2^{n-4},1^2)$
-1	2	$(1^2,-1^3,0^{n-7})$	$(4,3^3,2^{n-7},1^3)$
-1	3	$(-1^2,0^{n-4})$	$(4^2,2^{n-4},1^2)$

TABLE 3.2
$a=5$

t_1	t_n	$(t_2,\ldots,t_{n-1})$	$\deg(H)$
-2	0	$(2^1,1^2,-1^2,0^{n-7})$, $(1^5,-1^3,0^{n-10})$	$(4,3^3,2^{n-7},1^3)$, $(3^6,2^{n-10},1^4)$
-2	1	$(2^1,1^2,-1^3,0^{n-8})$, $(1^5,-1^4,0^{n-11})$	$(4,3^3,2^{n-7},1^3)$, $(3^6,2^{n-10},1^4)$
-2	2	$(2^1,1^1,-1^3,0^{n-7})$, $(1^4,-1^4,0^{n-10})$	$(4,3^3,2^{n-7},1^3)$, $(3^6,2^{n-10},1^4)$
-2	3	$(1^2,-1^3,0^{n-7})$	$(4,3^3,2^{n-7},1^3)$
-2	4	Infeasible	
-4	0	$(2^2,0^{n-4})$, $(2^1,1^3,-1^1,0^{n-7})$ $(1^6,-1^2,0^{n-10})$	$(4^2,2^{n-4},1^2)$, $(4,3^3,2^{n-7},1^3)$ $(3^6,2^{n-10},1^4)$
-4	1	$(3^1,0^{n-4})$, $(2^2,-1^1,0^{n-5})$ $(2^1,1^3,-1^2,0^{n-8})$, $(1^6,-1^3,0^{n-11})$	$(5,2^{n-2},1)$, $(4^2,2^{n-4},1^2)$ $(4,3^3,2^{n-7},1^3)$, $(3^6,2^{n-10},1^4)$
-4	2	$(2^1,1^2,-1^2,0^{n-7})$, $(1^5,-1^3,0^{n-10})$	$(4,3^3,2^{n-7},1^3)$, $(3^6,2^{n-10},1^4)$
-4	3	$(2^1,-1^1,0^{n-4})$, $(1^3,-1^2,0^{n-7})$	$(4^2,2^{n-4},1^2)$, $(4,3^3,2^{n-7},1^3)$
-4	4	(0^{n-2})	$(5,2^{n-2},1)$

TABLE 3.3
$a=8$

t_1	t_n	$(t_2,\ldots,t_{n-1})$	$\deg(H)$
-3	0	$(3^1,0^{n-3})$, $(2^2,-1^1,0^{n-5})$ $(2^1,1^3,-1^2,0^{n-8})$, $(1^6,-1^3,0^{n-11})$	$(5,2^{n-2},1)$, $(4^2,2^{n-4},1^2)$ $(4,3^3,2^{n-7},1^3)$, $(3^6,2^{n-10},1^4)$
-3	1	$(3^1,-1^1,0^{n-4})$, $(2^2,-1^2,0^{n-6})$ $(2^1,1^3,-1^3,0^{n-9})$, $(1^6,-1^4,0^{n-12})$	$(5,2^{n-2},1)$, $(4^2,2^{n-4},1^2)$ $(4,3^3,2^{n-7},1^3)$, $(3^6,2^{n-10},1^4)$
-3	2	$(2^1,1^2,-1^3,0^{n-8})$, $(1^5,-1^4,0^{n-11})$	$(4,3^3,2^{n-7},1^3)$, $(3^6,2^{n-10},1^4)$
-3	3	$(2^1,-1^2,0^{n-5})$, $(1^3,-1^3,0^{n-8})$	$(4^2,2^{n-4},1^2)$, $(4,3^3,2^{n-7},1^3)$
-3	4	$(-1^1,0^{n-3})$	$(5,2^{n-2},1)$

TABLE 3.4
$a=9$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

ELA
http://math.technion.ac.il/iic/ela

LEMMA 3.6. *Suppose the propeller graph G has at most one triangle. If a graph H is L-cospectral with G, then* $\deg(H) = (5, 2^{n-2}, 1)$.

Proof. Since H is L-cospectral with G, by Lemma 3.5,

$$\deg(H) = (5, 2^{n-2}, 1), (4^2, 2^{n-4}, 1^2), (4, 3^3, 2^{n-7}, 1^3), \text{ or } (3^6, 2^{n-10}, 1^4).$$

In view of the formula for l_3 in Lemma 2.1, we obtain

$$6n_3(G) - \sum_{v \in V(G)} d_G(v)^3 = 6n_3(H) - \sum_{v \in V(H)} d_H(v)^3. \tag{3.17}$$

Note that $n_3(G) = 1$ or 0 since G contains at most one triangle by our assumption.

Case 1. $\deg(H) = (4^2, 2^{n-4}, 1^2)$. In this case, by (3.17), we have

$$6n_3(G) - (8n + 110) = 6n_3(H) - (8n + 98), \tag{3.18}$$

and so $n_3(H) = -1$ or -2, depending on whether $n_3(G) = 1$ or 0. This is a contradiction because $n_3(H) \geq 0$ by its definition.

Case 2. $\deg(H) = (4, 3^3, 2^{n-7}, 1^3)$. By (3.17), we have

$$6n_3(G) - (8n + 110) = 6n_3(H) - (8n + 92), \tag{3.19}$$

which leads to $n_3(H) = -2$ or -3, again a contradiction.

Case 3. $\deg(H) = (3^6, 2^{n-10}, 1^4)$. Then (3.17) implies

$$6n_3(G) - (8n + 110) = 6n_3(H) - (8n + 86). \tag{3.20}$$

This leads to $n_3(H) = -3$ or -4, which is a contradiction.

Therefore, the only possibility is $\deg(H) = (5, 2^{n-2}, 1)$. □

LEMMA 3.7. *Suppose the propeller graph G has two triangles. If a graph H is L-cospectral to G, then* $\deg(H) = (5, 2^{n-2}, 1)$ *or* $(4^2, 2^{n-4}, 1^2)$, *and the latter occurs only when H is triangle-free.*

Proof. The proof is straightforward by using (3.18), (3.19) and (3.20). □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let G be a propeller graph with at most one triangle. Suppose H is L-cospectral with G. By Lemma 3.6, $\deg(H) = (5, 2^{n-2}, 1)$. Since H is connected by (c) of Lemma 2.2, it follows that H must be a propeller graph. By Lemma 3.4, we conclude that H and G are isomorphic.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

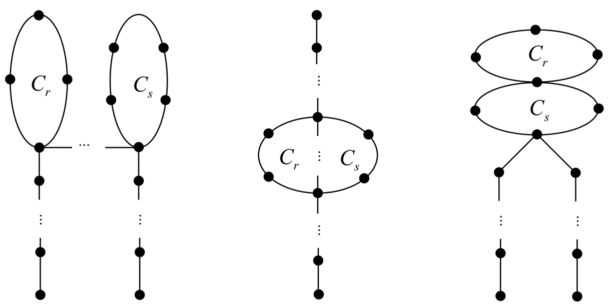

FIG. 3.1. *Proof of Theorem 1.1: Possible cases for H.*

Let G be a propeller graph with two triangles; that is, $p = q = 3$. Suppose H is L-cospectral with G. By Lemma 3.7, $\deg(H) = (5, 2^{n-2}, 1)$ or $(4^2, 2^{n-4}, 1^2)$, and in the latter case H is triangle-free. In the case when $\deg(H) = (5, 2^{n-2}, 1)$, similar to the argument in the first paragraph, it is straightforward to show that H and G are isomorphic.

Consider the case $\deg(H) = (4^2, 2^{n-4}, 1^2)$, where H is triangle-free. Since H is connected by (c) of Lemma 2.2, there are three possibilities for H as shown in Fig. 3.1. However, since H is triangle-free (that is, $r, s \geq 4$), in each case H has more than 9 spanning trees, whilst G has exactly $pq = 9$ spanning trees. This contradicts (d) of Lemma 2.2.

Therefore, H is isomorphic to G and the proof is complete. □

4. Proof of Theorem 1.2. Throughout this section, G is a propeller graph with $n = p + q + k - 1$ vertices as shown in Fig. 1.1. Applying Lemma 2.3 to G, with u the vertex of degree 5 in G, we obtain

$$\begin{aligned}\phi(A(G); x) = {} & x\phi(A(P_{p-1}))\phi(A(P_{q-1}))\phi(A(P_k)) - 2\phi(A(P_{p-2}))\phi(A(P_{q-1}))\phi(A(P_k)) \\ & -2\phi(A(P_{p-1}))\phi(A(P_{q-2}))\phi(A(P_k)) - \phi(A(P_{p-1}))\phi(A(P_{q-1}))\phi(A(P_{k-1})) \\ & -2\phi(A(P_{p-1}))\phi(A(P_k)) - 2\phi(A(P_{q-1}))\phi(A(P_k)). \end{aligned} \tag{4.1}$$

The next lemma follows from (4.1) and $\phi(A(P_n), 2) = n + 1$ [25].

LEMMA 4.1. $\phi(A(G); 2) = -(3k + 2)pq$.

In [25], the adjacency characteristic polynomial of P_n with $n \geq 1$ is given as follows:

$$\phi(A(P_n); x) = \frac{y^{2n+2} - 1}{y^{n+2} - y^n}, \tag{4.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

ELA
http://math.technion.ac.il/iic/ela

where y satisfies $y^2 - xy + 1 = 0$ with $x \neq 2$. Substituting (4.2) into (4.1), by using Maple, we obtain

$$y^n(y^2-1)^3\phi(A(G)) + 1 - 4y^2 - y^{2n+6} + 4y^{2n+4} = f_A(p,q,k;y), \tag{4.3}$$

where $n = p + q + k - 1$ and

$$\begin{aligned} f_A(p,q,k;y) = \ & -2y^{4+2k+p+2q} && -2y^{4+2k+q+2p} && +2y^{2k+2+p+2q} && +2y^{2k+2+q+2p} \\ & +3y^{2p+2q} && +2y^{2+p+2q} && +2y^{2+q+2p} && -2y^{p+2q} \\ & -2y^{q+2p} && -2y^{2+2p} && -2y^{2+2q} && -y^{2p} - y^{2q} \\ & +y^{2k+4+2p} && +y^{2k+4+2q} && +2y^{2k+2+2p} && +2y^{2k+2+2q} \\ & +2y^{2k+4+p} && +2y^{2k+4+q} && -2y^{2k+2+p} && -2y^{2k+2+q} \\ & -2y^{2+p} && -2y^{2+q} && +2y^p + 2y^q && -3y^{2k+4}. \end{aligned}$$

Lemma 4.2. *No two non-isomorphic propeller graphs are A-cospectral.*

Proof. Let G' be a propeller graph with order $n' = p' + q' + k' - 1$. Suppose that G' and G are A-cospectral. Without loss of generality, we may assume $p \geq q$ and $p' \geq q'$. Since cospectral graphs have the same order, we have

$$p + q + k = p' + q' + k'. \tag{4.4}$$

Lemma 4.1 implies

$$(3k+2)pq = (3k'+2)p'q'. \tag{4.5}$$

By (4.3), we have

$$f_A(p,q,k;y) = f_A(p',q',k';y). \tag{4.6}$$

The term in $f_A(p,q,k;y)$ with the smallest exponent is $-3y^{2k+4}$ or $2y^q$, and similarly for $f_A(p',q',k';y)$. From (4.6) we have either $-3y^{2k+4} = -3y^{2k'+4}$ or $2y^q = 2y^{q'}$. In the former case, we have $k = k'$, and so $(p,q) = (p',q')$ by (4.4) and (4.5). In the latter case, we have $q = q'$. Suppose $k \neq k'$. Without loss of generality, let $k' = k + i$ where $i \geq 1$. Substituting back into (4.4), we get $p' = p - i$, and then $(3i + 3k + 2 - 3p)i = 0$, via expressing p', q', k' by p, q, k and i in (4.5). Clearly, $3i + 3k + 2 - 3p \neq 0$, a contradiction. So, $k = k'$, and then $p = p'$. Therefore, G and G' are isomorphic in each case. ∎

Since the subdivision graph of a propeller graph G is also a propeller graph, Lemmas 4.2 and 2.11 immediately imply the following result.

Lemma 4.3. *No two non-isomorphic propeller graphs are Q-cospectral.*

Lemma 4.4. *Let G be a propeller graph. Then $\lambda_2(G) < 2$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

ELA
http://math.technion.ac.il/iic/ela

Proof. Let u be the vertex of degree 4 in G. By the Interlacing Theorem [14] for the A-spectrum, we obtain

$$\lambda_2(G) \leq \lambda_1(G-u) = \lambda_1\left(P_{q-1} \cup P_{p-1} \cup P_k\right) < 2,$$

where the last inequality holds because the largest eigenvalue for the A-spectrum of a path is less than 2. ☐

Corollary 4.5. *Let G be a propeller graph. Then $\lambda_2\left(\mathcal{S}(G)\right) < 2$.*

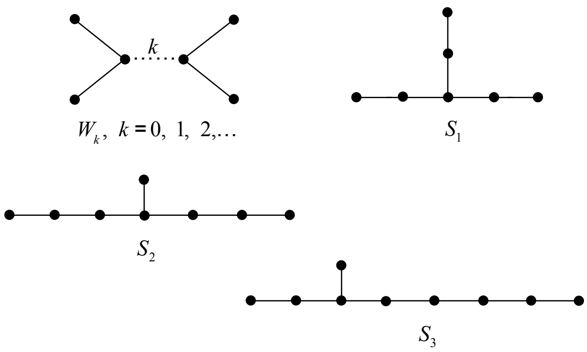

Fig. 4.1. *Smith graphs W_k, S_1, S_2 and S_3.*

A connected graph which satisfies $\lambda_1 = 2$ is called a *Smith graph* [27]. All Smith graphs are known in [27]. They are cycles C_n $(n \geq 3)$ and the graphs depicted in Fig. 4.1, where in W_k, k is the length of the path joining the middle vertices of the two copies of P_3. (Note that $W_0 = K_{1,4}$.)

Lemma 4.6. *Let H be a graph that is Q-cospectral with the propeller graph G. Then H does not contain two vertex-disjoint cycles as its subgraph.*

Proof. Since H is Q-cospectral with G, by Lemma 2.11, $\mathcal{S}(H)$ is A-cospectral to $\mathcal{S}(G)$. This together with Corollary 4.5 implies $\lambda_2(\mathcal{S}(H)) = \lambda_2(\mathcal{S}(G)) < 2$. Since the largest eigenvalue for the A-spectrum of a cycle is 2, it follows that $\mathcal{S}(H)$ does not contain two vertex-disjoint cycles. Since $\mathcal{S}(H)$ is the subdivision graph of H, the same result holds for H. ☐

Lemma 4.7. *Let H be a graph that is Q-cospectral with the propeller graph G. Then*

$$\begin{aligned}\deg(H) = {} & (5, 2^{n-2}, 1),\ (4^2, 2^{n-4}, 1^2),\ (4, 3^3, 2^{n-7}, 1^3),\ (3^6, 2^{n-10}, 1^4),\\ & (4, 3^2, 2^{n-4}, 0),\ \textit{or}\ (3^5, 2^{n-7}, 1, 0).\end{aligned} \tag{4.7}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014
ELA
http://math.technion.ac.il/iic/ela

Proof. Suppose $\deg(H) = (5 + t_1, 2 + t_2, 2 + t_3, \ldots, 2 + t_{n-1}, 1 + t_n)$. Since the connectivity of H cannot be determined by its Q-spectrum, H may contain just isolated vertices as its components. Thus,

$$t_1 \geq -5,\ t_2 \geq -2,\ \ldots,\ t_{n-1} \geq -2,\ t_n \geq -1.$$

The rest of the proof is similar to that of Lemma 3.5, and hence, we omit details. ☐

LEMMA 4.8. *Let H be a graph that is Q-cospectral with the propeller graph G. Then H is a propeller graph.*

Proof. Since H is Q-cospectral with G, by Lemma 2.9,

$$6n_3(G) + \sum_{v\in V(G)} d_G(v)^3 = 6n_3(H) + \sum_{v\in V(G)} d_H(v)^3. \tag{4.8}$$

Since G is a propeller graph, by Lemma 4.7, the degree sequence of H is given in (4.7). We consider the cases for $\deg(H)$ one by one. Note that $n_3(G) = 0$, 1 or 2.

Case 1. $\deg(H) = (5, 2^{n-2}, 1)$. It is straightforward to show that H is a propeller graph.

Case 2. $\deg(H) = (4^2, 2^{n-4}, 1^2)$. In this case, by (4.8) we have $6n_3(G) + (8n+110) = 6n_3(H) + (8n + 98)$. Hence, $n_3(H) = 2, 3, 4$ depending on whether $n_3(G) = 0, 1, 2$ respectively.

By Lemma 4.6 and $\deg(H) = (4^2, 2^{n-4}, 1^2)$, there are three possibilities for H as shown in Fig. 4.2. Note that for the Q-spectrum the multiplicity of 0 gives the number of bipartite components [7]. Clearly, for H_1, there is an eigenvalue 0 in its Q-spectrum, but there is no eigenvalue 0 in the Q-spectrum of G, since $n_3(G) = 1$, that is, G is not bipartite. This is a contradiction, because G and H are not Q-cospectral.

If H is isomorphic to H_2, then Lemma 2.10 implies that the line graphs $\mathcal{L}(G)$ and $\mathcal{L}(H_2)$ are A-cospectral, that is $\sum_i \lambda_i(\mathcal{L}(G))^4 = \sum_i \lambda_i(\mathcal{L}(H_2))^4$. However, by Lemma 2.12, this cannot happen by the following computation:

$$\sum_i \lambda_i(\mathcal{L}(H_2))^4 = \begin{cases} 310, & \text{if } l = 1 \text{ and } t = 1; \\ 6n + 276, & \text{if } l \geq 2 \text{ and } t = 1; \\ 6n + 276, & \text{if } l = 1 \text{ and } t \geq 2; \\ 6n + 284, & \text{if } l \geq 2 \text{ and } t \geq 2; \end{cases}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

ELA
http://math.technion.ac.il/iic/ela

$$\sum_i \lambda_i(\mathcal{L}(G))^4 = \begin{cases} 368, & \text{if } p = q = 4 \text{ and } k = 1; \\ 6n + 332, & \text{if } p = q = 4 \text{ and } k \geq 2; \\ 6n + 312, & \text{if } p > q = 4 \text{ and } k = 1; \\ 6n + 324, & \text{if } p > q = 4 \text{ and } k \geq 2; \\ 6n + 304, & \text{if } p \geq q > 4 \text{ and } k = 1; \\ 6n + 316, & \text{if } p \geq q > 4 \text{ and } k \geq 2. \end{cases} \quad (4.9)$$

If H is isomorphic to H_3, similarly to the above case, $\sum_i \lambda_i(\mathcal{L}(H_3))^4$ is computed as follows:

$$\sum_i \lambda_i(\mathcal{L}(H_3))^4 = \begin{cases} 328, & \text{if } l = 1 \text{ and } t = 1; \\ 6n + 300, & \text{if } l \geq 2 \text{ and } t = 1; \\ 6n + 300, & \text{if } l = 1 \text{ and } t \geq 2; \\ 6n + 308, & \text{if } l \geq 2 \text{ and } t \geq 2. \end{cases}$$

Again, $\sum_i \lambda_i(\mathcal{L}(G))^4 \neq \sum_i \lambda_i(\mathcal{L}(H_3))^4$, a contradiction.

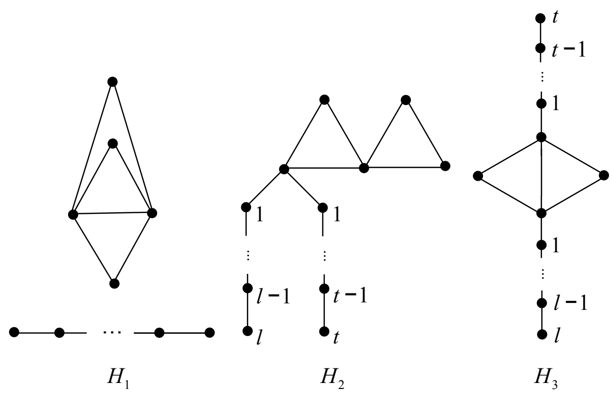

Fig. 4.2. *Proof of Lemma 4.8: Case 2.*

Case 3. $\deg(H) = (4, 3^3, 2^{n-7}, 1^3)$. In this case, by (4.8), we have $6n_3(G) + (8n + 110) = 6n_3(H) + (8n + 92)$. Hence, $n_3(H) = 3, 4, 5$ depending on whether $n_3(G) = 0, 1, 2$ respectively. Again, by Lemma 4.6 and $\deg(H) = (4, 3^3, 2^{n-7}, 1^3)$, there are two possibilities for H as shown in Fig. 4.3. If H is isomorphic to H_4, then $\mathcal{S}(H)$ contains a subgraph isomorphic to a disjoint union of a cycle and the Smith graph S_1. This contradicts the fact $\lambda_2(\mathcal{S}(H)) = \lambda_2(\mathcal{S}(G)) < 2$.

If H is isomorphic to H_5, then Lemma 2.10 implies that the line graphs $\mathcal{L}(G)$ and $\mathcal{L}(H_5)$ are A-cospectral, that is $\sum_i \lambda_i(\mathcal{L}(G))^4 = \sum_i \lambda_i(\mathcal{L}(H_5))^4$. By Lemma 2.12,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

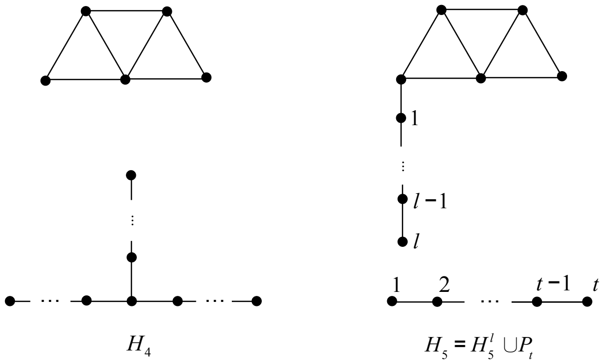

FIG. 4.3. *Proof of Lemma 4.8: Case 3.*

we have

$$\sum_i \lambda_i(\mathcal{L}(H_5))^4 = \begin{cases} 368, & \text{if } l = 1 \text{ and } t = 2; \\ 6n + 316, & \text{if } l = 1 \text{ and } t \geq 3; \\ 6n + 324, & \text{if } l \geq 2 \text{ and } t = 2; \\ 6n + 320, & \text{if } l \geq 2 \text{ and } t \geq 3. \end{cases}$$

By the above computation and (4.9), there exist three equal cases:

Case 3.1. 368: H_5 with $l = 1$, $t = 2$ and G with $p = q = 4$, $k = 1$. With the help of Maple, we have

$$\phi(Q(H_5); x) = x^8 - 18x^7 + 128x^6 - 468x^5 + 948x^4 - 1054x^3 + 584x^2 - 120x;$$
$$\phi(Q(G); x) = x^8 - 18x^7 + 128x^6 - 468x^5 + 948x^4 - 1056x^3 + 592x^2 - 128x.$$

Clearly, $\phi(Q(H_5)) \neq \phi(Q(G))$, a contradiction.

Case 3.2. $6n + 316$: H_5 with $l = 1$, $t \geq 3$ and G with $p \geq q > 4$, $k \geq 2$. Note that H_5 contains an eigenvalue 0 in its Q-spectrum. Then p and q must be even numbers no less than 6. By Lemma 2.13, we have $q_{n-1}(G) = (-1)^{n-1}pqn$, and $q_{n-1}(H_5) = (-1)^{n-1}(60n - 360)$. Then $q_{n-1}(G) = q_{n-1}(H_5)$ implies $36n = 60n - 360$ or $48n = 60n - 360$, since $q_{n-1}(G) > q_{n-1}(H_5)$ with $p \geq q \geq 8$. In the former case, we have $n = 15$. That is, H_5 has 15 vertices with $l = 1$, $t = 9$, and G has 15 vertices with $p = q = 6$, $k = 4$. Note that for a bipartite graph G', $\phi(Q(G')) = \phi(L(G'))$ [7]. Thus, $\phi(Q(H_5)) = \phi(Q(H_5^1))\phi(L(P_9))$, and $\phi(Q(G)) = \phi(L(G))$. By Maple, we obtain

$$\phi(Q(H_5^1); x) = x^6 - 16x^5 + 96x^4 - 276x^3 + 396x^2 - 262x + 60. \tag{4.10}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

Substituting $x = (y+1)^2/y$ into (4.10), then plugging (3.5) and (4.10) into the expression of $\phi(Q(H_5))$, and with the help of Maple, we obtain

$$y^{15}(y-1)^3(y+1)^2\phi(Q(H_5)) + 1 - 3y - 4y^2 + 4y^{33} + 3y^{34} - y^{35} = f_Q(H_5; y),$$

where

$$\begin{aligned} f_Q(H_5; y) = {} & 2y^{30} + 2y^{29} + 2y^{28} + 2y^{25} + 2y^{24} + 2y^{23} - 4y^{20} - 3y^{19} + y^{18} \\ & -y^{17} + 3y^{16} + 4y^{15} - 2y^{12} - 2y^{11} - 2y^{10} - 2y^7 - 2y^6 - 2y^5. \end{aligned}$$

Substituting $p = q = 6$ and $k = 4$ back into (3.8), we have

$$\begin{aligned} f_L(6,6,4; y) = {} & -4y^{29} + 4y^{27} + 3y^{25} + 3y^{24} + 2y^{23} + 6y^{22} + 4y^{21} + 4y^{20} - 4y^{18} \\ & +4y^{17} - 4y^{15} - 4y^{14} - 6y^{13} - 2y^{12} - 3y^{11} - 3y^{10} - 4y^8 + 4y^6. \end{aligned}$$

Thus, $f_Q(H_5; y) \neq f_L(6,6,4; y)$. This contradicts $\phi(Q(H_5)) = \phi(Q(G))$.

In the latter case, we have $n = 30$. That is, H_5 has 30 vertices with $l = 1$, $t = 24$, and G has 30 vertices with $p = 8$, $q = 6$, $k = 17$. Using the similar method to the former case, we have $f_Q(H_5; y) \neq f_L(8,6,17; y)$, which also contradicts $\phi(Q(H_5)) = \phi(Q(G))$.

Case 3.3. $6n+324$: H_5 with $l \geq 2$ and $t = 2$ and G with $p > q = 4$ and $k \geq 2$. Similarly to Case 3.2, p must be even numbers no less than 6, and Lemma 2.13 implies that $q_{n-1}(G) = (-1)^{n-1}4pn$, and $q_{n-1}(H_5) = (-1)^{n-1}120$. Clearly, $q_{n-1}(G) \neq q_{n-1}(H_5)$, a contradiction.

Case 4. $\deg(H) = (3^6, 2^{n-10}, 1^4)$. In this case, (4.8) yields $6n_3(G) + (8n+110) = 6n_3(H) + (8n+86)$. Hence, $n_3(H) = 4, 5, 6$ depending on whether $n_3(G) = 0, 1, 2$ respectively. By Lemma 4.6, there is no feasible H satisfying $\deg(H) = (3^6, 2^{n-10}, 1^4)$.

Case 5. $\deg(H) = (4, 3^2, 2^{n-4}, 0)$. In this case, there is an eigenvalue 0 in the Q-spectrum of H. This implies that G must be bipartite and so $n_3(G) = 0$. By (4.8), we have $6n_3(G) + (8n+110) = 6n_3(H) + (8n+86)$, which gives $n_3(H) = 4$. Clearly, by Lemma 4.6, there are no feasible H satisfying $\deg(H) = (4, 3^2, 2^{n-4}, 0)$.

Case 6. $\deg(H) = (3^5, 2^{n-7}, 1, 0)$. Similar to Case 5, we have $n_3(G) = 0$. Again, by (4.8), we have $n_3(H) = 4$. Lemma 4.6 implies that there is no feasible H satisfying $\deg(H) = (3^5, 2^{n-7}, 1, 0)$. □

Proof of Theorem 1.2. The result follows from Lemmas 4.3 and 4.8 immediately. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

5. Conclusion. In this paper, we proved that any propeller graph is determined by its L-spectrum as well as its Q-spectrum. Along the way we showed that no two non-isomorphic propeller graphs are A-cospectral (Lemma 4.2). We expect that this result could be used to prove some propeller graphs are A-DS. On the other hand, not every propeller graph is determined by its A-spectrum. For example, in [1, p. 12] and [21, p. 1226], two A-cospectral mates are given. And we expect that there are more graphs that are A-cospectral with propeller graphs. It would be an interesting question to characterize which graphs are A-cospectral with propeller graphs.

Acknowledgment. We appreciate the anonymous referees for their comments and suggestions. X. Liu is supported by MIFRS and MIRS of the University of Melbourne and the Natural Science Foundation of China (No. 11361033). S. Zhou is supported by a Future Fellowship (FT110100629) of the Australian Research Council.

REFERENCES

[1] N.L. Biggs. *Algebraic Graph Theory*, second edition. Cambridge University Press, Cambridge, 1993.
[2] R. Boulet. Spectral characterizations of sun graphs and broken sun graphs. *Discrete Math. Theor. Comput. Sci.*, 11:149–160, 2009.
[3] R. Boulet and B. Jouve. The lollipop graph is determined by its spectrum. *Electron. J. Combin.*, 15:R74, 2008.
[4] D.M. Cvetković. New theorems for signless Laplacian eigenvalues. *Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur., Sci. Math.*, 137(33):131–146, 2008.
[5] D.M. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs - Theory and Applications*, third edition. Johann Ambrosius Barth, Heidelberg. 1995.
[6] D.M. Cvetković and P. Rowlinson. Spectra of unicyclic graphs. *Graphs Combin.*, 3:7–23, 1987.
[7] D.M. Cvetković, P. Rowlinson, and S.K. Simić. Signless Laplacians of finite graphs. *Linear Algebra Appl.*, 423:155–171, 2007.
[8] D.M. Cvetković, P. Rowlinson, and S.K. Simić. *An Introduction to the Theory of Graph Spectra.* Cambridge University Press, Cambridge, 2010.
[9] D.M. Cvetković and S.K. Simić. Towards a spectral theory of graphs based on signless Laplacian, I. *Publ. Inst. Math. (Beograd) (N.S.)*, 85(99):1–15, 2009.
[10] E.R. van Dam and W.H. Haemers. Which graphs are determined by their spectrum?. *Linear Algebra Appl.*, 373:241–272, 2003.
[11] E.R. van Dan and W.H. Haemers. Developments on spectral characterizations of graphs. *Discrete Math.*, 309:576–586, 2009.
[12] Hs.H. Günthard and H. Primas. Zusammenhang von Graphtheorie und Mo-Theotie von Molekeln mit Systemen konjugierter Bindungen. *Helv. Chim. Acta*, 39:1645–1653, 1956.
[13] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W.H. Freeman and Company, San Francisco, 1979.
[14] C. Godsil and G. Royle. *Algebraic Graph Theory.* Springer-Verlag, Inc., New York, 193–194, 2001.
[15] J.-M. Guo. On the second largest Laplacian eigenvalue of trees. *Linear Algebra Appl.*, 404:251–261, 2005.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 19-38, January 2014

[16] J.-M. Guo. A conjecture on the algebraic connectivity of connected graphs with fixed girth. *Discrete Math.*, 308:5702–5711, 2008.

[17] W.H. Haemers, X.-G. Liu, and Y.-P. Zhang. Spectral characterizations of lollipop graphs. *Linear Algebra Appl.*, 428:2415–2423, 2008.

[18] A.K. Kelmans. The number of trees in a graph I, II. *Automation and Remote Control*, 26:2118-2129, 1965 and 27:233–241, 1966. Translated from *Avtomatika i Telemekhanika* 26:2194–2204, 1965 and 27:56–65, 1966 (in Russian).

[19] M.-H. Liu and B.-L. Liu. Some results on the Laplacian spectrum. *Comput. Math. Appl.*, 59:3612–3616, 2010.

[20] X.-G. Liu and S.-J. Wang. Laplacian spectral characterization of some graph products. *Linear Algebra Appl.*, 437:1749–1759, 2012.

[21] P.-L. Lu, X.-G. Liu, Z.-T. Yuan, and X.-Y. Yong. Spectral characterizations of sandglass graphs. *Appl. Math. Lett.*, 22:1225–1230, 2009.

[22] M. Mirzakhah and D. Kiani. The sun graph is determined by its signless Laplacian spectrum. *Electron. J. Linear Algebra* 20:610–620, 2010.

[23] C.S. Oliveira, N.M.M. de Abreu, and S. Jurkiewilz. The characteristic polynomial of the Laplacian of graphs in (a, b)-linear classes. *Linear Algebra Appl.*, 365:113–121, 2002.

[24] G.R. Omidi and K. Tajbakhsh. Starlike trees are determined by their Laplacian spectrum. *Linear Algebra Appl.*, 422:654–658, 2007.

[25] F. Ramezani, N. Broojerdian, and B. Tayfeh-Rezaie. A note on the spectral characterization of θ-graphs. *Linear Algebra Appl.*, 431:626–632, 2009.

[26] S.K. Simić and Z. Stanić. Q-integral graphs with edge-degrees at most five. *Discrete Math.*, 308:4625–4634, 2008.

[27] J.H. Smith. Some properties of spectrum of graphs. In: R. Guy, H. Hanani, N. Sauer, and J. Schönheim (editors), *Combinatorial Structures and Their Applications*, Gordon and Breach, Science Publ., Inc., New York, 403–406, 1970.

[28] J.-F. Wang, Q.-X. Huang, F. Belardo, and E.M. Li Marzi. On the spectral characterizations of ∞-graphs. *Discrete Math.*, 310:1845–1855, 2010.

[29] J. Zhou and C.-J. Bu. Laplacian spectral characterization of some graphs obtained by product operation. *Discrete Math.*, 312:1591–1595, 2012.