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CHANGE OF THE *CONGRUENCE CANONICAL FORM
OF 2-BY-2 MATRICES UNDER PERTURBATIONS

VYACHESLAV FUTORNY†, LENA KLIMENKO‡, AND VLADIMIR V. SERGEICHUK§

Abstract. It is constructed the Hasse diagram for the closure ordering on the sets of *congruence classes of 2 × 2 matrices. In other words, it is constructed the directed graph whose vertices are 2 × 2 canonical complex matrices for *congruence and there is a directed path from A to B if and only if A can be transformed by an arbitrarily small perturbation to a matrix that is *congruent to B.

Key words. Closure graph, *Congruence canonical form, Perturbations.

AMS subject classifications. 15A21, 15A63, 47A07.

1. Introduction. We study how arbitrarily small perturbations of a 2 × 2 complex matrix can change its *canonical form for *congruence (matrices A and B are *congruent if $S^*AS = B$ for a nonsingular $S$). We construct the closure graph $G_2$, which is defined for any natural n as follows.

Definition 1.1. The closure graph $G_n$ for *congruence classes of $n \times n$ complex matrices is the directed graph, in which each vertex $v$ represents in a one-to-one manner a *congruence class $C_v$ of $n \times n$ matrices, and there is a directed path from a vertex $v$ to a vertex $w$ if and only if one (and hence each) matrix from $C_v$ can be transformed to a matrix form $C_w$ by an arbitrarily small perturbation.

The graph $G_n$ is the Hasse diagram of the *congruence classes of $n \times n$ matrices with the following partial order: $a \preceq b$ means that $a$ is contained in the closure of $b$. Thus, the graph $G_n$ shows how the *congruence classes relate to each other in the affine space of $n \times n$ matrices.

Since each $n \times n$ matrix is uniquely represented in the form $P + iQ$ in which $P$ and $Q$ are Hermitian matrices, $G_n$ is also the closure graph for *congruence classes.

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of Hermitian matrix pencils $P + \lambda Q$.

Note that the closure graph $G_2$ for *congruence, which we construct in Theorem 2.2, is more complicated than the closure graphs for congruence classes of 2-by-2 and 3-by-3 matrices, which were constructed by the authors in [4], since an arrow between *congruence classes in $G_2$ may depend on the parameters of their matrices.

Unlike perturbations of matrices under congruence and *congruence, perturbations of matrices under similarity and of matrix pencils have been much studied. For a given matrix $A$, den Boer and Thijsse [3] and, independently, Markus and Parilis [17] described the set of all Jordan canonical matrices such that for each $J$ from this set there exists a matrix that is arbitrarily close to $A$ and is similar to $J$. Their description was extended to Kronecker’s canonical forms of pencils by Pokrzywa [18]. Edelman, Elmroth, and Kågström [7] developed a comprehensive theory of closure relations for similarity classes of matrices, for equivalence classes of matrix pencils, and for their bundles. The software StratiGraph [8] constructs their closure graphs. The closure graph for $2 \times 3$ matrix pencils was constructed and studied by Elmroth and Kågström [9].

The term “*congruence orbit” is often used instead of “*congruence class” (see De Terán and Dopico [2]). The problem that we consider can be called “the stratification of orbits of matrices under *congruence” by analogy with the stratification of orbits of matrices under similarity and of matrix pencils [7, 8, 15]. An informal introduction to perturbations of matrices determined up to similarity, congruence, or *congruence is given by Klimenko and Sergeichuk [16].

All matrices that we consider are complex matrices.

2. The closure graph for *congruence classes of 2-by-2 matrices. Define the $n$-by-$n$ matrices:

$$J_n(\lambda) := \begin{bmatrix} \lambda & 1 & 0 \\ \vdots & \ddots & \ddots \\ 0 & \ddots & \lambda \end{bmatrix}, \quad \Delta_n := \begin{bmatrix} 0 & 1 \\ \vdots & \ddots & \ddots \\ 1 & \ddots & 0 \end{bmatrix}.$$

We use the following canonical form for *congruence.

**Proposition 2.1 ([10, Theorem 4.5.21]).** Each square complex matrix is *congruent to a direct sum, uniquely determined up to permutation of summands, of matrices of the form

$$
\begin{bmatrix}
0 & I_m \\
J_m(\lambda) & 0
\end{bmatrix}
(0 \neq \lambda \in \mathbb{C}, |\lambda| < 1), \quad \mu_0 \Delta_n (\mu \in \mathbb{C}, |\mu| = 1), \quad J_k(0).
$$
This canonical form obtained in [11] was based on [21, Theorem 3] and was generalized to other fields in [14]. A direct proof that this form is canonical is given in [12, 13].

The vertices of $G_n$ can be identified with the $n \times n$ canonical matrices for congruence since each congruence class contains exactly one canonical matrix.

For each $A \in \mathbb{C}^{n \times n}$ and a small matrix $X \in \mathbb{C}^{n \times n}$,

$$(I + X)^*A(I + X) = A + X^*A + AX + X^*AX$$

and so the congruence class of $A$ in a small neighborhood of $A$ can be obtained by a very small deformation of the real affine matrix space $\{ A + X^*A + AX \mid X \in \mathbb{C}^{n \times n} \}$.

(By the local Lipschitz property [20], if $A$ and $B$ are close to each other and $B = S^*A S$ with a nonsingular $S$, then $S$ can be taken near $I_n$.) The real vector space

$$T(A) := \{ X^*A + AX \mid X \in \mathbb{C}^{n \times n} \}$$

is the tangent space to the congruence class of $A$ at the point $A$. The numbers

$$\dim \mathbb{R} T(A), \quad \text{codim} \mathbb{R} T(A) := 2n^2 - \dim \mathbb{R} T(A)$$

are called the dimension and, respectively, codimension over $\mathbb{R}$ of the congruence class of $A$.

The following theorem proved in Section 3 is the main result of the paper.

**Theorem 2.2.** The closure graph for congruence classes of $2 \times 2$ matrices is

$$(2.2) \quad \dim \mathbb{R} T(A), \quad \text{codim} \mathbb{R} T(A) := 2n^2 - \dim \mathbb{R} T(A)$$

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The following theorem proved in Section 3 is the main result of the paper.

**Theorem 2.2.** The closure graph for congruence classes of $2 \times 2$ matrices is

$$(2.3) \quad \dim \mathbb{R} T(A), \quad \text{codim} \mathbb{R} T(A) := 2n^2 - \dim \mathbb{R} T(A)$$

are called the dimension and, respectively, codimension over $\mathbb{R}$ of the congruence class of $A$. 

$$\begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \tau \\ \tau & \tau \end{pmatrix}$$

$|\mu| = |\nu| = |\tau| = 1, \quad \mu \neq \pm \nu, \quad |\sigma| < 1, \quad \text{dimension}_{\mathbb{R}} = 6.$

$\lambda \in \mathbb{R}, +\mathbb{R}, \quad \Im(\lambda \tau) \geq 0, \quad \tau = \lambda$

$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$

$|\lambda| = 1, \quad \text{dimension}_{\mathbb{R}} = 4.$

$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$

$|\lambda| = 1, \quad \text{dimension}_{\mathbb{R}} = 3.$

$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$

$|\lambda| = 1, \quad \text{dimension}_{\mathbb{R}} = 0.$
Change of the *Congruence Canonical Form of 2-by-2 Matrices Under Perturbations

in which \( \lambda, \mu, \nu, \sigma, \tau \in \mathbb{C}, \mathbb{R}^+ \) denotes the set of nonnegative real numbers, and \( \text{Im}(c) \) denotes the imaginary part of \( c \in \mathbb{C} \). Each *congruence class is given by its canonical matrix, which is a direct sum of blocks of the form (2.1). The graph is infinite: Each vertex except for \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) represents an infinite set of vertices indexed by the parameters of the corresponding canonical matrix. The *congruence classes of canonical matrices that are located at the same horizontal level in (2.3) have the same dimension over \( \mathbb{R} \), which is indicated to the right.

The arrow \( \begin{bmatrix} \lambda & 0 \\ 0 & \nu \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 \\ 0 & \nu \end{bmatrix} \) exists if and only if \( \lambda = \mu a + \nu b \) for some nonnegative \( a, b \in \mathbb{R} \). The arrow \( \begin{bmatrix} \lambda & 0 \\ 0 & \nu \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \tau \nu \\ \tau \nu & 0 \end{bmatrix} \) exists if and only if the imaginary part of \( \lambda \bar{\nu} \) is nonnegative. The arrow \( \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \tau \nu \\ \tau \nu & 0 \end{bmatrix} \) exists if and only if \( \tau = \pm \lambda \). The arrows \( \begin{bmatrix} \lambda & 0 \\ 0 & \pm \lambda \end{bmatrix} \rightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \pm \lambda \end{bmatrix} \) exist if and only if the value of \( \lambda \) is the same in both matrices. The other arrows exist for all values of parameters of their matrices.

**Remark 2.3.** Let \( M \) be a 2 \( \times \) 2 canonical matrix for *congruence.

- Let \( N \) be another 2 \( \times \) 2 canonical matrix for *congruence. Each neighborhood of \( M \) contains a matrix whose *congruence canonical form is \( N \) if and only if there is a directed path from \( M \) to \( N \) in (2.3) (if \( M = N \), then there is the “lazy” path of length 0 from \( M \) to \( N \)).

- The closure of the *congruence class of \( M \) is equal to the union of the *congruence classes of all canonical matrices \( N \) such that there is a directed path from \( N \) to \( M \) (if \( M = N \) then the “lazy” path exists).

**Remark 2.4.** It is not surprising that \( \text{diag}(\lambda, \pm \lambda) \) and \( \text{diag}(\mu, \nu) \) \((|\lambda| = |\mu| = |\nu| = 1 \text{ and } \mu \neq \pm \nu)\) have different behavior under perturbation: many properties of a nonsingular matrix \( A \) with respect to *congruence are determined by its *cosquare \((A^*)^{-1}A\) (see [13] [13] [19]), the *cosquare of \( \text{diag}(\lambda, \pm \lambda) \) has a multiple eigenvalue, and the *cosquare of \( \text{diag}(\mu, \nu) \) has two distinct eigenvalues.

### 3. Proof of Theorem 2.2

The following lemma is a weak form of [6] Example 2.1 (which is a special case of [6] Theorem 2.2 about \( n \times n \) matrices).

**Lemma 3.1.** Let \( A \) be any 2 \( \times \) 2 matrix. Then all matrices \( A + X \) that are sufficiently close to \( A \) can be simultaneously reduced by some transformation \( S(X)^*(A + X)S(X) \), where \( S(X) \) is nonsingular and continuous on a neighborhood of zero,
to one of the following forms:

\[
\begin{bmatrix}
0 & 0 & + & * & *
\end{bmatrix},
\begin{bmatrix}
\lambda & 0 & + & \varepsilon & 0
\end{bmatrix} (|\lambda| = 1),
\begin{bmatrix}
\lambda & 0 & + & \varepsilon & 0
\end{bmatrix} (\lambda \neq \mu, |\lambda| = |\mu| = 1),
\begin{bmatrix}
0 & 1 & + & 0 & 0
\end{bmatrix} (|\lambda| < 1),
\begin{bmatrix}
0 & \lambda & + & 0 & 0
\end{bmatrix} (|\lambda| = 1).
\]

Each of these matrices has the form \( A_{\text{can}} + D \), in which \( A_{\text{can}} \) is a direct sum of blocks of the form (2.1), the *'s in \( D \) are complex numbers, all \( \varepsilon, \delta_{\lambda}, \delta_{\mu} \) are either real numbers if \( \lambda, \mu \in \mathbb{R} \) or pure imaginary numbers if \( \lambda, \mu \in \mathbb{R} \). (Clearly, \( D \) tends to zero as \( X \) tends to zero.) For each \( A_{\text{can}} + D \), twice the number of its stars plus the number of its entries of the form \( \varepsilon_{\lambda}, \delta_{\lambda}, \delta_{\mu} \) is equal to the codimension over \( \mathbb{R} \) (defined in (2.2)) of the *congruence class of \( A_{\text{can}} \).

Note that the codimensions of congruence and *congruence classes were calculated in [1] [5] and [2] [6], respectively.

By [22] Part III, Theorem 1.7, the boundary of each *congruence class is a union of *congruence classes of strictly lower dimension, which ensures the following lemma.

**Lemma 3.2.** If \( M \rightarrow N \) is an arrow in the closure graph \( G_2 \), then the *congruence class \( C_M \) of \( M \) is contained in the closure of the *congruence class \( C_N \) of \( N \), and so the dimension of \( C_M \) is lower than the dimension of \( C_N \).

For each vertex \( M \) in (2.3), the dimension \( d_M \) over \( \mathbb{R} \) of the *congruence class of \( M \) is indicated in (2.3). It was calculated as follows: By (2.2), \( d_M = 8 - c_M \) in which \( c_M \) is the codimension of the *congruence class of \( M \); \( c_M \) was taken from Lemma 3.1.

The proof of Theorem 2.2 is divided into two steps.

**Step 1: Let us prove that each arrow in (2.3) is correct.** To make sure that an arrow \( M \rightarrow N \) is correct, we need to prove that the canonical matrix \( M \) can be transformed by an arbitrarily small perturbation to a matrix whose *congruence canonical form is \( N \). Consider each of the arrows of (2.3).

- The arrows \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \nu \end{bmatrix} \), \( \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} \) are correct.

Let \( A := \begin{bmatrix} 0 & 0 \\ 0 & \nu \end{bmatrix} \), or \( \begin{bmatrix} 0 & \pm \tau \end{bmatrix} \). Then \( A \) is *congruent to \( \varepsilon A \), in which \( \varepsilon \) is any positive real number, and each neighborhood of \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) contains \( \varepsilon A \) with a sufficiently small \( \varepsilon \).

- The arrow \( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \lambda & 0 \\ 0 & \nu \end{bmatrix} \) (with given \( \lambda, \mu, \nu \in \mathbb{C} \) such that \( |\lambda| = |\mu| = |\nu| = 1 \)
exists if and only if $\lambda \in \mu \mathbb{R}_+ + \nu \mathbb{R}_+ = \{ \mu a + \nu b | a, b \in \mathbb{R}, a \geq 0, b \geq 0 \}$ (in particular, $[\lambda \ 0] \rightarrow [\lambda \ 0]$ and $[\lambda \ 0] \rightarrow [\lambda \ 0]$ exist).

The arrow $[\lambda \ 0] \rightarrow [\mu \ 0]$ exists if and only if there exists an arbitrarily small perturbation
\[
(3.1) \quad \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + E = \begin{bmatrix} \lambda + \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} \quad \text{of} \quad \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}
\]
that is *congruent to $[\mu \ 0 \ 0 \ 0]$. This means that there exists a nonsingular $S = [x \ y \ z \ t]$ such that
\[
\begin{bmatrix} \bar{x} & \bar{z} \\ \bar{y} & \bar{t} \end{bmatrix} \begin{bmatrix} \mu & 0 \\ 0 & \nu \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + E,
\]
i.e.,
\[
\bar{x} \mu + \bar{z} \nu = \lambda + \varepsilon_{11} \quad \bar{y} \mu + \bar{t} \nu = \varepsilon_{21}
\]
For fixed $\lambda, \mu, \nu$ and an arbitrarily small $\varepsilon_{11}$, the first equation with unknowns $x$ and $z$ has a solution only if $\lambda \in \mu \mathbb{R}_+ + \nu \mathbb{R}_+$.

Conversely, let $\lambda \in \mu \mathbb{R}_+ + \nu \mathbb{R}_+$. Take $\varepsilon_{11} = 0$ and chose $x$ and $z$ for which the first equality in (3.2) holds. Then take arbitrarily small $y, t$ for which $S$ is nonsingular and get arbitrarily small $\varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22}$ for which the other equalities in (3.2) hold.

* The arrow $[\lambda \ 0] \rightarrow [\sigma \ 1]$ ($|\lambda| = 1, |\sigma| < 1$) exists for all $\lambda$ and $\sigma$.

The arrow $[\lambda \ 0] \rightarrow [\sigma \ 1]$ exists if and only if there exists an arbitrarily small perturbation $[\lambda \ 0]$ that is *congruent to $[\sigma \ 1]$. This means that there exists a nonsingular $S = [x \ y \ z \ t]$ such that
\[
\begin{bmatrix} \bar{x} & \bar{z} \\ \bar{y} & \bar{t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & \sigma \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + E,
\]
i.e.,
\[
\bar{x} \sigma + \bar{z} \bar{x} = \lambda + \varepsilon_{11} \quad \bar{x} \sigma + \bar{y} \sigma = \varepsilon_{12}
\]
Suppose that $\bar{x} = u + iv, \sigma = \alpha + \beta i$, and $\lambda + \varepsilon_{11} = a + bi$, in which $u, v, \alpha, \beta, a, b \in \mathbb{R}$. Then the first equation in (3.3) takes the form $(u - vi) + (u + vi)(\alpha + \beta i) = a + bi$, which gives the system
\[
(1 + \alpha)u - \beta v = a \\
\beta u + (\alpha - 1)v = b
\]
with respect to the unknowns \( u \) and \( v \). Its determinant \( \alpha^2 + \beta^2 - 1 \) is nonzero since \(|\sigma| < 1\). Therefore, the first equation in (3.3) holds for some \( x \) and \( z \). Taking arbitrarily small \( y, t \) for which \( S \) is nonsingular, we get arbitrarily small \( \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22} \) for which the other equalities in (3.3) hold.

- The arrow \( \left[ \begin{array}{c} \lambda \\ 0 \\ 0 \end{array} \right] \rightarrow \left[ \begin{array}{c} 0 \\ \tau \end{array} \right] \) (\(|\lambda| = |\tau| = 1\)) exists if and only if \( \text{Im}(\lambda \tau) \geq 0 \).

The arrow \( \left[ \begin{array}{c} \lambda \\ 0 \\ 0 \end{array} \right] \rightarrow \tau \left[ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right] \) exists if and only if there exists an arbitrarily small perturbation \( \left[ \begin{array}{c} \delta \\ 0 \\ 0 \end{array} \right] + E \) of \( \left[ \begin{array}{c} \delta \\ 0 \\ 0 \end{array} \right] \) that is *congruent to \( \tau \left[ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right] \). This means that there exists a nonsingular \( S = \left[ \begin{array}{c} z \\ \varepsilon \end{array} \right] \) such that

\[
\begin{bmatrix}
\bar{x} & \bar{y} \\
\bar{z} & i
\end{bmatrix}^T \begin{bmatrix}
0 & 1 \\
1 & i
\end{bmatrix} \begin{bmatrix}
x & y \\
z & t
\end{bmatrix} = \begin{bmatrix}
\lambda & 0 \\
0 & 0
\end{bmatrix} + E,
\]

i.e.,

\[
\begin{align*}
\bar{x}x + \bar{y}z + \bar{z}zi &= \bar{\tau}(\lambda + \varepsilon_{11}) \\
\bar{t}x + \bar{y}z + \bar{z}zi &= \bar{\tau}\varepsilon_{12} \\
\bar{t}y + \bar{y}t + \bar{t}zi &= \bar{\tau}\varepsilon_{21} \\
\bar{t}y + \bar{y}t + \bar{t}zi &= \bar{\tau}\varepsilon_{22}.
\end{align*}
\]

Consider the first equation in (3.4). Since \( \bar{\tau}(\lambda + \varepsilon_{11}) \neq 0, z \neq 0 \) too. Thus,

\[
\text{Im}(\bar{\tau}(\lambda + \varepsilon_{11})) = \text{Im}(\bar{\tau}x + \bar{\tau}z + \bar{\tau}zi) = \bar{\tau}z > 0
\]

and so \( \text{Im}(\bar{\tau}\lambda) \geq 0 \).

Conversely, if \( \text{Im}(\bar{\tau}\lambda) \geq 0 \), then we put \( \varepsilon_{11} = 0 \) and take \( x, z \) such that the first equation in (3.4) holds. Taking arbitrarily small \( y, t \) for which \( S \) is nonsingular, we get arbitrarily small \( \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22} \) for which the other equalities in (3.3) hold.

- The arrow \( \left[ \begin{array}{c} \lambda \\ 0 \\ 0 \end{array} \right] \rightarrow \left[ \begin{array}{c} 0 \\ \tau \\ \tau \end{array} \right] \) (\(|\lambda| = |\tau| = 1\)) exists if and only if \( \lambda = \pm \tau \).

The arrow \( \left[ \begin{array}{c} \lambda \\ 0 \\ 0 \end{array} \right] \rightarrow \tau \left[ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right] \) exists if and only if there exists an arbitrarily small perturbation \( \left[ \begin{array}{c} \delta \\ 0 \\ 0 \end{array} \right] + E \) of \( \left[ \begin{array}{c} \delta \\ 0 \\ 0 \end{array} \right] \) that is *congruent to \( \tau \left[ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right] \). This means that there exists a nonsingular \( S \) such that

\[
S^T \begin{bmatrix}
0 & 1 \\
1 & i
\end{bmatrix} S = \begin{bmatrix}
\lambda & 0 \\
0 & -\lambda
\end{bmatrix} + E
\]

Equating the determinants of both sides, we find that \( -\tau^2 \det(S^*S) \) is arbitrarily close to \( -\lambda^2 \). Since

\[
\det(S^*S) = \overline{\det S} \det S
\]

is a real positive number, \( |\tau|^2 \det(S^*S) \) is arbitrarily close to \( |\lambda|^2 \). Since \(|\lambda| = |\tau| = 1\), \( \det(S^*S) \) is arbitrarily close to 1. Hence, \( -\tau^2 = -\lambda^2 \), and so \( \lambda = \pm \tau \).
Change of the *Congruence Canonical Form of 2-by-2 Matrices Under Perturbations

Conversely, let $\lambda = \pm \tau$. Since

$$
\begin{bmatrix}
1 & 1 \\
1/2 & -1/2
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 1/2 \\
1 & -1/2
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix},
$$

$[\frac{\lambda}{0} 0]$ is *congruent to $\pm [\frac{0}{1} 1]$. Its arbitrarily small perturbation $\pm [\frac{0}{1} 1] (\varepsilon \in \mathbb{R}, \varepsilon > 0)$ is *congruent to $\pm [\frac{0}{1} 1]$ via diag($\sqrt{\varepsilon}, 1/\sqrt{\varepsilon}$). Therefore, $[\frac{\lambda}{0} 0] \rightarrow [\frac{0}{1} 1]$, and so $[\frac{\lambda}{0} 0] \rightarrow [\frac{0}{1} 1]$.

**Step 2: Let us prove that we have not missed arrows in (2.3).** We write $M \Rightarrow N$ if the closure graph $G_2$ does not have the arrow $M \rightarrow N$; i.e., if each matrix obtained from $M$ by an arbitrarily small perturbation is not *congruent to $N$. Lemma 3.2 ensures that we need to prove only the absence of the arrows

$$
[\frac{\lambda}{0} 0] \rightarrow [\frac{\mu}{0} 0], \quad [\frac{\lambda}{0} 0] \rightarrow [\frac{0}{\sigma} 0], \quad [\frac{\lambda}{0} 0] \rightarrow [\frac{0}{\tau} 1].
$$

- $[\frac{\lambda}{0} 0] \rightarrow [\frac{\mu}{0} 0]$ and $[\frac{\lambda}{0} 0] \rightarrow [\frac{0}{\sigma} 0]$ ([|\lambda| = |\mu| = |\sigma| = 1, $\mu \neq \pm \nu$, $|\sigma| < 1$).

Suppose that there is an arbitrarily small perturbation $A := [\frac{\lambda}{0} 0] + E$ of $[\frac{\lambda}{0} 0]$ that is *congruent to $B := [\frac{\mu}{0} 0]$ or $C := [\frac{0}{\sigma} 0]$. Then $A^{-1} \cdot A$ is similar to $B^{-*} \cdot B$ or $C^{-*} \cdot C$, which is impossible since the eigenvalues of $A^{-*} \cdot A$ are arbitrarily close to $\lambda^{-1} = \lambda^2$, whereas $B^{-*} \cdot B = \text{diag} (\mu^2, \nu^2)$ and $C^{-*} \cdot C = \text{diag} (\sigma, \sigma^{-1})$.

- $[\frac{\lambda}{0} 0] \rightarrow [\frac{0}{\tau} 1]$ ($|\lambda| = |\tau| = 1$).

Let $[\frac{\lambda}{0} 0] \rightarrow [\frac{0}{1} 1]$; i.e., there exists an arbitrarily small perturbation $A := [\frac{\lambda}{0} 0] + E$ of $[\frac{\lambda}{0} 0]$ that is *congruent to $B := \lambda^{-1} \cdot [\frac{0}{1} 1]$. This means that there exists a nonsingular $S$ such that

$$
S^{-*} \left( [\frac{1}{0} 0] + E \right) S = \lambda^{-1} \cdot [\frac{0}{1} 1].
$$

Equating the determinants of both sides, we find that

$$
r(1 + \varepsilon) = -\lambda^{-1} \cdot \lambda, \quad r : \det (S^{-1} \cdot S) > 0,
$$

in which $\varepsilon$ is arbitrarily small. Since $-\lambda^{-1} \cdot \lambda^2 = 1$, we have $(\lambda^{-1} \cdot \lambda)^2 = -1$, and so $\lambda^{-1} \cdot \lambda = \pm i$. Then rank$(B + B^*) = 1$, which is impossible since $A + A^*$ is *congruent to $B + B^*$ and rank$(A + A^*) = 2$. 


REFERENCES


