Electronic Journal of Linear Algebra

Volume 27 Article 11

2014

Shadowing property of random linear cocyles

Abbas Fakhari
agolmaka@im.ufrj.br

Ali Golmakani

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 190-196, March 2014

ELA
http://math.technion.ac.il/iic/ela

SHADOWING PROPERTY OF RANDOM LINEAR COCYCLES*

ABBAS FAKHARI† AND ALI GOLMAKANI‡

Abstract. In this brief note, it is shown that the random hyperbolicity of a random linear cocycle is equivalent to having the Lipschitz shadowing property.

Key words. Random hyperbolic cocycle, Lipschitz shadowing, Non-zero Lyapunov exponents.

AMS subject classifications. 54H20, 37B40, 37D45.

1. Introduction. The study of the notion of *shadowing* is one of the central tasks of the stability theory in dynamical systems in deterministic and random views. A shadowable dynamics is interpreted commonly as the systems in which the numerical approximation orbit are, in fact, traced by true orbit. It can be useful especially in the numerical solution of differential equations (see Chapter 4 of [11]). During the last years, many studies have been performed to prove the shadowing property of hyperbolic systems in deterministic cases [11] and random cases [3, 5, 7, 9]. But what about the converse? Does the shadowing property imply the hyperbolicity? Although some results have been obtained in deterministic case (see for instance, [12, 13, 14]), it remains to deal with in random case. One of the classical results in the study of hyperbolicity and the shadowing states that a discrete dynamical system generated by a linear operator of a finite dimensional Banach space has the shadowing property if and only if it is hyperbolic [10]. This motivates the study of shadowing property in the case of random linear cocycles. The aim of this note is to investigate the relationship between the shadowing property and hyperbolicity of a random linear cocycle. Namely, we prove that the shadowing property of a random triangular cocycle is equivalent to its hyperbolicity.

We begin by introducing the notation of "Random Linear Cocycle". Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\sigma : \Omega \to \Omega$ be a $\mathbb{P}$-ergodic transformation. Consider a random variable $A : \Omega \to GL(\mathbb{R}, d)$, $d \in \mathbb{N}$ with $\log^+ \|A^{\pm}(\cdot)\| \in \mathcal{L}^1(\mathbb{P})$. A generates

*Received by the editors on October 20, 2012. Accepted for publication on February 24, 2014. Handling Editor: Bryan L. Shader.
†Department of Mathematics, Shahid Beheshti University, G.C. Tehran 19839, Iran (a_fakhari@sbu.ac.ir).
‡Instituto de Matematica, Universidade Federal do Rio de Janeiro, C.P. 68.530, CEP 21.945-970, Rio de Janeiro, Brazil (agolmaka@im.ufrj.br).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 190-196, March 2014

ELA
http://math.technion.ac.il/iic/ela

the linear cocycle [1]

$$A^n(\omega) = \begin{cases} A(\sigma^{n-1}(\omega)) \circ \cdots \circ A(\omega) & \text{if } n > 0, \\ \text{id} & \text{if } n = 0, \\ A(\sigma^{-n}(\omega))^{-1} \circ \cdots \circ A(\sigma^{-1}(\omega))^{-1} & \text{if } n < 0. \end{cases}$$

The cocycle A is *triangular* if for any $\omega \in \Omega$, $A(\omega)$ is triangular.

DEFINITION 1.1. A random variable $g : \Omega \to (0, \infty)$ is called *tempered* if

$$\lim_{n \to \pm\infty} \frac{1}{n} \log g(\sigma^n(\omega)) = 0.$$

For a tempered random variable $\delta > 0$, a sequence $\xi = \{x_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}^d$ is called (ω, δ)-*pseudo orbit* if for any k, $\|A(\sigma^k(\omega))x_k - x_{k+1}\| < \delta(\sigma^k(\omega))$. For a tempered random variable $\epsilon > 0$, the sequence ξ is (ω, ϵ)-*shadowed* by a point $x \in \mathbb{R}^d$ if

$$\|A^k(\omega)x - x_k\| < \epsilon(\sigma^k(\omega)).$$

The following definition can be regarded as a natural extension of the classical deterministic Lipschitz shadowing property to the random case.

DEFINITION 1.2. A cocycle A has *Lipschitz shadowing property* if there are a tempered random variables $L \geq 2$ and $d_0 > 0$ such that for any tempered random variable $d \leq d_0$, any (ω, d)-pseudo orbit of A can be (ω, Ld)-shadowed by some point of $\mathbb{R}^d$ $\mathbb{P}$-a.e.

DEFINITION 1.3. A cocycle A is *random hypergolic* if there are constant $0 < \lambda < 1$, two tempered random variables C and α and random subbundles $E^s(\omega)$ and $E^u(\omega)$, vary measurably, such that the following hold $\mathbb{P}$-a.e.

- $E^s(\omega) \oplus E^u(\omega) = \mathbb{R}^d$,
- $A(\omega)E^s(\omega) = E^s(\sigma(\omega))$ and $A(\omega)E^u(\omega) = E^u(\sigma(\omega))$,
- $\| \left(A(\omega)|_{E^s(\omega)}\right)^k \|, \| \left(A(\omega)|_{E^u(\omega)}\right)^{-k} \| \leq C(\sigma^k(\omega))\lambda^k,$ for any $k \in \mathbb{N}$,
- $\measuredangle(E^s(\omega), E^u(\omega)) > \alpha(\omega)$.

By changing the inner product, one can get a random inner product $\langle \cdot, \cdot \rangle_\omega$, depends measurably to ω, for which the random subbundles $E^s(\omega)$ and $E^u(\omega)$ are orthogonal and also $C(\omega) = 1$ $\mathbb{P}$-a.e. The obtained random norm $\| \cdot \|_\omega$ satisfies the following assertion

$$\frac{1}{\beta(\omega)} \| \cdot \| \leq \| \cdot \|_\omega \leq \beta(\omega) \| \cdot \|,$$

where β is a tempered random variable [5].

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 190-196, March 2014

ELA
http://math.technion.ac.il/iic/ela

Main Theorem. A random linear cocycle A has the Lipschitz shadowing property if and only if it is random hyperbolic.

2. Proof of the Main Theorem. For the proof of this stage of the Main Theorem, we modify the proof in [5] to get the Lipschitz shadowing property in the linear case. Suppose that $P(\omega) : \mathbb{R}^d \to E^s(\omega)$ and $Q(\omega) : \mathbb{R}^d \to E^u(\omega)$ are the projections on $E^s(\omega)$ and $E^u(\omega)$ along the directions $E^u(\omega)$ and $E^s(\omega)$, respectively. For the proof, as in the deterministic case, it is enough to show that any finite pseudo orbit can be shadowed [5]. Choose a sequence $\{x_k\}_{k=1}^n$ such that $\|A(\sigma^k(\omega))x_k - x_{k+1}\| < d(\sigma^k(\omega))$. There is a sufficiently small tempered random variable $0 < \eta \le \beta d$ such that $A(\omega)(E^\rho_{\eta(\omega)}(\omega)) \subset E^\rho_{\eta(\sigma(\omega))}(\sigma(\omega))$, where $E^\rho_\epsilon(\omega) = \{v \in E^\rho(\omega); \|v\|_w < \epsilon\}$ for $\rho = s, u$. Put

$$[x, y]_\omega = Q(\omega)y + P(\omega)x,$$

and note that $\|[x, y] - y\|_\omega$, $\|[x, y] - x\|_\omega \le \|x - y\|_\omega$ (see [8]). Choose $\tilde{x}_0 = x_0$ and

$$\tilde{x}_k = [x_k, A(\sigma^{k-1}(\omega))\tilde{x}_{k-1}]_{\sigma^k(\omega)}, \qquad x = A^{-n}(\sigma^n(\omega))\tilde{x}_n.$$

It is not difficult to see that $\|A^k(\omega)x - \tilde{x}_k\|_{\sigma^k(\omega)} \le \eta(\sigma^k(\omega))$. On the other hand, $\|x_k - \tilde{x}_k\|_{\sigma^k(\omega)} \le 2\eta(\sigma^k(\omega))$. Hence,

$$\|A^k(\omega)x - x_k\|_{\sigma^k(\omega)} \le \|A^k(\omega)x - \tilde{x}_k\|_{\sigma^k(\omega)} + \|x_k - \tilde{x}_k\|_{\sigma^k(\omega)} \le 3\eta(\sigma^k(\omega)),$$

and so, $\|A^k(\omega)x - x_k\| \le 3\beta^2(\sigma^k(\omega))d(\sigma^k(\omega))$ (see also [4, 5]).

Before proceed to the converse of the Main Theorem, let us recall "Oseledec's Multiplication Ergodic Theorem", in the sense of random dynamical systems [1]. The theorem says that there is a σ-invariant subset $\Gamma \subset \Omega$ of $\mathbb{P}$-full measure with the following property: There are a fixed number m, positive integers d_i and real numbers $-\infty < \lambda_m < \cdots < \lambda_1 < \infty$, $1 \le i \le m$, such that for any $\omega \in \Gamma$, there exists a splitting $E^m(\omega) \oplus \cdots \oplus E^1(\omega)$ of T_xM satisfying the two following conditions

- $\dim E^i(\omega) = d_i$, for any $1 \le i \le m$,
- $v \in E^i(\omega) \Rightarrow \lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| = \lambda_i$.

Put

$$E^s(\omega) = \bigoplus_{\lambda_i<0} E^i(\omega), \quad E^c(\omega) = \bigoplus_{\lambda_i=0} E^i(\omega) \quad \text{and} \quad E^u(\omega) = \bigoplus_{\lambda_i>0} E^i(\omega).$$

The subbundles $E^s(\omega)$, $E^0(\omega)$ and $E^u(\omega)$ are well-defined and measurably A-invariant. If $E^0(\omega) = 0$, then for some tempered random variables C and α, the cocycle A has a random hyperbolic decomposition as in the Definition 1.3 (see [2, 8]). In other words,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 190-196, March 2014

ELA
http://math.technion.ac.il/iic/ela

the random hyperbolicity are precisely equivalent to the non-existence of zero Lyapunov exponents. Hence, to prove the converse of the Main Theorem for triangular cocycles, it is sufficient to prove the non-existence of zero Lyapunov exponents.

Now, let us follow the proof of the Main Theorem in proving the converse. First, we prove it for a triangular cocycle. It does get a bit notations. Let $A(\omega) = (a_{ij}(\omega))$ with $a_{ij}(\omega) = 0$ if $j > i$. Then $A^n(\omega) = (a^n_{ij}(\omega))$ with $a^n_{ij}(\omega) = 0$ if $j > i$, and

$$a^n_{ii}(\omega) = a_{ii}(\sigma^{n-1}(\omega)) \cdots a_{ii}(\omega).$$

In the case of triangular matrices, the existence of the limits

$$\lim_{n\to\infty} \frac{1}{n} \log |a^n_{ii}(\omega)|$$

for $\mathbb{P}$-a.e $\omega \in \Omega$ is a consequence of Birkkhoff's Ergodic Theorem. However, the fact of $\mathbb{P}$-integrability of the functions $\log \|A^{\pm}(\cdot)\|$ implies that these limits are exactly the Lyapunov exponents [2].

Theorem 2.1. *If A is a random triangular cocycle with the Lipschitz shadowing property, then the Lyapunov exponent of A are all non-zero.*

Proof. By contradiction, suppose that A has Lipschitz shadowing property with random variable $L \geq 2$ and $d_0 > 0$, however, for some $1 \leq i \leq d$,

$$\lim_{n\to\infty} \frac{1}{n} \log |a^n_{ii}(\omega)| = 0.$$

For simplicity, suppose that $i = 1$ and put $a(\omega) = a_{11}(\omega)$. By the ergodicity of $\mathbb{P}$, the Lyapunov exponent along the fibre ω is also zero $\mathbb{P}$-a.e. Put

$$\gamma_n = \sum_{k=1}^{n} d_0(\sigma^k(\omega))/2|a^k(\omega)|$$

and define a $(\omega, d_0/2)$-pseudo orbit $\xi(\omega) = \{x_n\}_{n\in\mathbb{Z}}$ as follows:

$$x^1_n = \begin{cases} a^n(\omega)x_0 & \text{if } n \leq 0, \\ a^n(\omega)\gamma_n & \text{if } n \geq 0, \end{cases}$$

and $x^j_n = 0$ for $2 \leq j \leq d$. We have

$$\|A(\sigma^n(\omega))x_n - x_{n+1}\| = |a(\sigma^n(\omega))a^n(\omega)\gamma_n - a^{n+1}(\omega)\gamma_{n+1}| \leq d_0(\sigma^n(\omega))/2.$$

Hence, $\{x_n\}_{n\in\mathbb{Z}}$ is a $(\omega, d_0/2)$-pseudo orbit for A. Suppose that the pseudo orbit $(\omega, Ld_0/2)$-shadowed by some point x. Then

$$\|A^n(\omega)x - x_n\| < L(\sigma^n(\omega))d_0(\sigma^n(\omega))/2,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 190-196, March 2014

ELA
http://math.technion.ac.il/iic/ela

and hence,

$$|x - \gamma_n| \leq \frac{1}{|a^n(\omega)|}\|A^n(\omega)x - x_n\| \leq L(\sigma^n(\omega))\frac{d_0(\sigma^n(\omega))}{2|a^n(\omega)|}.$$

Now, the inequality

$$|x - \sum_{k=1}^{n} \frac{d_0(\sigma^k(\omega))}{2|a^k(\omega)}| \leq L(\sigma^n(\omega))\frac{d_0(\sigma^n(\omega))}{2|a^n(\omega)|}$$

inductively implies that

$$\frac{d_0(\sigma^n(\omega))}{|a^n(\omega)|} \geq \frac{c}{L(\sigma^n(\omega)) - 1}\prod_{k=1}^{n-1}\left(\frac{L(\sigma^k(\omega))}{L(\sigma^k(\omega)) - 1}\right)$$

for some positive constant c. The recent inequality leads to the following contradiction:

$$\begin{aligned} 0 = \lim_{n\to\infty}\frac{1}{n}\log\frac{d_0(\sigma^n(\omega))}{|a^n(\omega)|} &\geq \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log\left(\frac{L(\sigma^k(\omega))}{L(\sigma^k(\omega)) - 1}\right) \\ &= \int_\Omega \log\frac{L(\omega)}{L(\omega) - 1}\, d\mathbb{P} > 0, \end{aligned}$$

where the last equality holds by “Birkhoff’s Ergodic Theorem”. $\square$

To give the proof in the general situation, i.e., the case where A isn’t necessarily triangular, we first give a cohomology between A and a triangular cocycle. We benefit a method known *Lyapunov-Perron triangularization* (see also [6]). Then, using Theorem 2.1, we conclude the proof in the general case.

By Gram-Schmidt decomposition, any $M \in GL(\mathbb{R}, d)$ is represented uniquely in the form $M = G(M)T(M)$ where $G(M)$ is an orthogonal matrix and $T(M)$ a triangular matrix. Put $H = \Omega \times O(\mathbb{R}, d)$, where $O(\mathbb{R}, d)$ is the set of orthogonal matrices. For any $z = (\omega, U) \in H$, put $\Theta^n(z) = (\theta^n(\omega), G(A^n_\omega U))$. Let $\mathcal{M}_{\mathbb{P}}(\Theta)$ denotes all Θ-invariant probability measures on H whose marginal on Ω coincide with $\mathbb{P}$ (such measures can be characterized in term of their disintegrations μ_ω by $A(\omega)(\mu_\omega) = \mu_{\theta(\omega)}$a.s.). Let $\mathcal{E}_{\mathbb{P}}(\Theta) \subset \mathcal{M}_{\mathbb{P}}(\Theta)$ be the set of ergodic measures. Define random cocycle B over the base space H as follows

$$B(\omega, U) = G(A(\omega)U)^{-1}A(\omega)U = T(A(\omega)U)^{-1}.$$

By the definition,

$$B^n(\omega, U) = G(A^n(\omega)U)^{-1}A^n(\omega)U = T(A^n(\omega)U)^{-1}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 190-196, March 2014

ELA
http://math.technion.ac.il/iic/ela

In particular, B defines a triangular cocycle. The lemma below completes the proof of the Main Theorem.

LEMMA 2.2. *Theorem 2.1 remains true if we drop the condition of triangularity of A.*

Proof. Suppose that A has the Lipschitz shadowing property. We first show that the cocycle B obtained by the triangularization method, described above, has the same property. For this, let $\xi = \{x_n\}$ be a (ω, U, d)-pseudo orbit of B. Put $y_n = G(A^n(\omega)U)x_n$. A straightforward calculation shows that $\zeta = \{y_n\}$ is a (ω, d)-pseudo orbit for A. Hence, ζ can be (ω, Ld) shadows by a point y. It isn't difficult to see that ξ is (ω, U, Ld) shadowed by $x = U^{-1}y$.

Now, choosing a measure $\mu \in \mathcal{E}_{\mathbb{P}}(\Theta)$ and applying Theorem 2.1, one can deduces that the Lyapunov exponents of B are all non-zero on a set $\tilde{H} \subset H$ with $\mu(\tilde{H}) = 1$. Suppose that $\tilde{\Omega} \subset \Omega$ be the projection of $\tilde{H}$ over the first component, so $\mathbb{P}(\tilde{\Omega}) = 1$. For $\omega \in \tilde{\Omega}$ and $v \in \mathbb{R}^d$, choosing $z \in \tilde{H}$ with $z = (\omega, U)$, one has that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| &= \lim_{n\to\infty} \frac{1}{n} \log \|G(A^n(\omega)U)B^n(z)U^{-1}v\| \\ &= \lim_{n\to\infty} \frac{1}{n} \log \|B^n(z)(U^{-1}v)\| \\ &\neq 0.\end{aligned}$$

That is, the Lyapunov exponents of A are all non-zero on $\tilde{\Omega}$. ∎

Acknowledgment. Ali Golmakani thanks *Instituto de Matematica, Universidade Federal do Rio de Janeiro* for its kind hospitality during the preparation of this paper. The authors were partially supported by the following fellowships: A. Fakhari by grant from IPM (No. 92370127) and A. Golmakani by a PosDoc fellowship from CAPES/Brasil.

REFERENCES

[1] L. Arnold. *Random Dynamical Systems.* Springer-Verlag, Berlin, 1998.

[2] L. Barreira and Y. Pesin. *Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents.* Cambridge University Press, 2007

[3] S.N. Chow and E.S. Van Vleck. A shadowing lemma approach to global error analysis for initial value ODEs. *SIAM J. Sci. Comput.*, 15:959–976, 1994.

[4] M. Gundlach. Random homoclinic orbits. *Random & Computational Dynamics*, 3(1-2):1–33, 1995.

[5] M. Gundlach and Y. Kifer. Random hyperbolic systems. In: H. Crauel and M. Gundlach (editors), *Stochastic Dynamics*, Springer, New York, 117–145, 1999.

[6] R. Johnson, K. Palmer, and G. Sell. Ergodic properties of linear dynamical systems. *SIAM J. Math. Anal.*, 18:1-33, 1987.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 190-196, March 2014

ELA
http://math.technion.ac.il/iic/ela

[7] H. Lianfa, Z. Yujun, and Z. Hongwen. Shadowing in random dynamical systems. *Discrete Contin. Dyn. Syst.*, 12(2):355–362, 2005.

[8] P.D. Liu and M.P. Qian. *Smooth Ergodic Theory of Random Dynamical Systems.* Lecture Notes in Math., Vol. 1606, Springer-Verlag, Berlin, 1995.

[9] P.D. Liu, M.P. Qianand, and F.C. Tang. Pseudo-orbit tracing property for random diffeomorphisms. *Proc. Roy. Soc. Edinburgh*, 126A:1027–1033, 1996.

[10] J. Ombach. The shadowing lemma in the linear case. *Universitatis Iagellonicae Acta Mathematica*, Fasciculus xxxi, 1994.

[11] S.Yu. Pilyugin. *Shadowing in Dynamical Systems.* Lecture Notes in Math., Vol. 1706, Springer-Verlag, Berlin, 1999.

[12] S.Yu. Pilyugin, A. Rodionova, and K. Sakai. Orbital and weak shadowing in dynamical systems. *Discrete Contin. Dyn. Syst.*, 9(2):287–308, 2003.

[13] K. Sakai. C^1-stably shadowable chain components. *Ergodic Theory Dynam. Systems*, 28:987–1029, 2008.

[14] K. Sakai. Pseudo-orbit tracing property and strong transversality of diffeomorphisms on closed manifolds. *Osaka J. Math.*, 31:373–386, 1994.