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THE K-TH DERIVATIVES OF THE IMMANANT AND THE χ-SYMMETRIC POWER OF AN OPERATOR*

SÓNIA CARVALHO† AND PEDRO J. FREITAS‡

Abstract. In recent papers, formulas are obtained for directional derivatives, of all orders, of the determinant, the permanent, the m-th compound map and the m-th induced power map. This paper generalizes these results for immanants and for other symmetric powers of a matrix.

Key words. Immanant, Derivative, Compound matrix, Induced power of a matrix.

AMS subject classifications. 15A69, 15A15.

1. Introduction. There is a formula for the derivative of the determinant map on the space of the square matrices of order n, known as the *Jacobi formula*, which has been well known for a long time. In recent work, T. Jain and R. Bhatia derived formulas for higher order derivatives of the determinant ([2]) and T. Jain also had derived formulas for all the orders of derivatives for the map $\wedge^m$ that takes an $n \times n$ matrix to its m-th compound ([5]). Later, P. Grover, in the same spirit of Jain's work, did the same for the permanent map and the for the map $\vee^m$ that takes an $n \times n$ matrix to its m-th induced power. The mentioned authors extended the theory in [1].

This paper follows along the lines of this work. It is known that the determinant map and the permanent map are special cases of a more generalized map, which is the immanant, and the compound and the induced power of a matrix are also generalized by other symmetric powers, related to symmetric classes of tensors. These will be our objects of study.

2. Immanant. We will write $M_n(\mathbb{C})$ to represent the vector space of the square matrices of order n with complex entries. Let $A \in M_n(\mathbb{C})$ and χ be an irreducible

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character of $\mathbb{C}$. We define the immanant determined by χ as:

$$d_\chi(A) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

In other words, $d_\chi : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ is a map taking an $n \times n$ matrix to its immanant. This, being a polynomial map, is differentiable. For $X \in M_n(\mathbb{C})$, we denote by $Dd_\chi A(X)$ the directional derivative of d_χ at A in the direction of X.

We denote by $A_n(i|j)$ the $n \times n$ square matrix that is obtained from A by replacing the i-th row and j-th column with zero entries, except entry (i,j) which we set to 1. We define the *immanantal adjoint* $\operatorname{adj}_\chi(A)$ as the $n \times n$ matrix in which the entry (i,j) is $d_\chi(A_n(i|j))$. This agrees with the definition of permanental adjoint in [7, p. 237], but not with the usual adjugate matrix (we would need to consider the transpose in that case). This is a matter of convention.

We obtain the following result similar to the *Jacobi formula* for the determinant.

THEOREM 2.1. *For each $X \in M_n(\mathbb{C})$,*

$$Dd_\chi(A)(X) = \operatorname{tr}(\operatorname{adj}_\chi(A)^T X).$$

Proof. For each $1 \le j \le n$, let $A(j;X)$ be the matrix obtained from A by replacing the j-th column of A by the j-th column of X and keeping the rest of the columns unchanged. Then the given equality can be restated as

$$Dd_\chi(A)(X) = \sum_{j=1}^{n} d_\chi(A(j;X)). \tag{2.1}$$

On the other hand, we note that $Dd_\chi(A)(X)$ is the coefficient of t in the polynomial $d_\chi(A+tX)$. Using the fact that the immanant is a multilinear function of the columns we obtain the desired result. ∎

Again using the fact that the immanant is a multilinear function, we notice that for any $1 \le i \le n$, we have

$$d_\chi(A) = \sum_{i=1}^{n} a_{ij} d_\chi(A_n(i|j)).$$

Using this and (2.1), we get that

$$Dd_\chi(A)(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} d_\chi(A_n(i|j)). \tag{2.2}$$

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We will generalize the expressions (2.1) and (2.2) for the derivatives of all orders of the immanant.

We now turn to derivatives. Let $V_1,\dots,V_n$ be n vector spaces over $\mathbb{C}$, and let $\phi : V_1 \times \cdots \times V_n \longrightarrow \mathbb{C}$ be a multilinear form. For $A, X^1, \dots X^k \in V_1,\dots,V_n$, the k-th derivative of ϕ at A in the direction of $(X^1,\dots,X^k)$ is given by the expression

$$D^k\phi(A)(X^1,\dots,X^k) := \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\Big|_{t_1=\cdots=t_k=0}\phi(A + t_1X^1 + \cdots + t_kX^k).$$

This is a multilinear function defined on $M_n(\mathbb{C})^k$. In particular, if we consider $\phi = d_\chi$ we have the definition of the k-th derivative of the immanant.

3. First expression for the derivatives of the immanant. We start by introducing some notation. Given a matrix $A \in M_n(\mathbb{C})$, we will represent by $A_{[i]}$ the i-th column of A, $i = \{1,\dots,n\}$.

Let k be a natural number, $1 \le k \le n$. Take $A, X^1,\dots,X^k \in M_n(\mathbb{C})$, and $t_1,\dots,t_k$ k indeterminates. Let $Q_{k,n}$ be the set of strictly increasing maps $\{1,\dots,k\} \to \{1,\dots,n\}$ and $G_{k,n}$ the set of increasing maps.

We will denote by $A(\alpha; X^1,\dots,X^k)$ the matrix of order n obtained from A replacing the $\alpha(j)$ column of A by the $\alpha(j)$ column of X^j. The next theorem gives the first expression for the higher order derivatives of the immanant.

Theorem 3.1. *For every* $1 \le k \le n$,

$$D^k d_\chi(A)(X^1,\dots,X^k) = \sum_{\sigma\in S_k}\sum_{\alpha\in Q_{k,n}} d_\chi A(\alpha; X^{\sigma(1)},\dots,X^{\sigma(k)}).$$

In particular,

$$D^k d_\chi(A)(X,\dots,X) = k! \sum_{\alpha\in Q_{k,n}} d_\chi A(\alpha; X,\dots,X).$$

Proof. Just like in the case of the first derivative, $D^k d_\chi(A)(X^1,\dots,X^k)$ is the coefficient of $t_1\cdots t_k$ in the expansion of the polynomial $d_\chi(A + t_1X^1 + \cdots + t_kX^k)$. Using the linearity of the immanant function in each column, we obtain the desired equality. ◻

We can re-write the last expression for the k-th derivative of the immanant map using the concept of mixed immanant, generalizing the respective concepts for the determinant and the permanent.

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Definition 3.2. Let $X^1, \ldots, X^n$ be n matrices of order n. We define the *mixed immanant* of $X^1, \ldots, X^n$ as

$$\Delta_\chi(X^1, \ldots, X^n) := \frac{1}{n!} \sum_{\sigma \in S_n} d_\chi(X^{\sigma(1)}_{[1]}, \ldots, X^{\sigma(n)}_{[n]}).$$

If $X^1 = \cdots = X^t = A$, for some $t \le n$ and $A \in M_n(\mathbb{C})$, we denote the mixed immanant by $\Delta_\chi(A; X^{t+1}, \ldots, X^n)$.

If $d_\chi = \det$, then the mixed immanant is called mixed discriminant, denoted by $\Delta(X^1, \ldots, X^n)$.

As with the permanent and the determinant, we have that $\Delta_\chi(A, \ldots, A) = d_\chi(A)$. This is consistent with the abbreviation we introduced in the definition.

Proposition 3.3. *Let $A \in M_n(\mathbb{C})$. We have that*

$$\Delta_\chi(A; X^1, \ldots, X^k) := \frac{(n-k)!}{n!} \sum_{\sigma \in S_k} \sum_{\alpha \in Q_{k,n}} d_\chi A(\alpha; X^{\sigma(1)}, \ldots, X^{\sigma(k)}).$$

Proof. One simply has to observe that each summand in $\Delta_\chi(A; X^1, \ldots, X^k)$ appears $(n-k)!$ times: Once we fix a permutation of the matrices $X^1, \ldots, X^k$, these summands correspond to the possible permutations of the $n-k$ matrices equal to A. □

As an immediate consequence of this result, we can obtain another formula for the derivative of order k of the immanant map. This generalizes formula (26) in [2].

Proposition 3.4.

$$D^k d_\chi(A)(X^1, \ldots, X^k) = \frac{n!}{(n-k)!} \Delta_\chi(A; X^1, \ldots, X^k). \tag{3.1}$$

4. Laplace expansion for immanants. We start by generalizing the Laplace expansion, known for the determinant and the permanent, to all immanants. This expansion was proved first for the determinant and the same arguments used can be used to prove the corresponding expansion for the permanent. These classical Laplace expansions can be found in [6] and in [8].

The similarity of proofs is due to the fact that the determinant and the permanent of $A \oplus B$ is just the product of the determinants, or the permanents, of A and B. However, if χ is any other irreducible character, there is no clear general relation between the immanant of A and the immanant of any submatrix of A. So the Laplace expansion formula for any immanant is a little more complicated.

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Let $1 \leq k \leq n$, $\alpha, \beta \in Q_{k,n}$. We denote by $S_{\alpha,\beta}$ the subset of S_n defined as

$$S_{\alpha,\beta} = \{\sigma \in S_n : \sigma(\operatorname{Im}\alpha) = \operatorname{Im}\beta\}.$$

LEMMA 4.1. *For every $\alpha \in Q_{k,n}$, the set $\{S_{\alpha,\beta} : \beta \in Q_{k,n}\}$ is a partition of S_n.*

Proof.

1. Let $\beta, \gamma \in Q_{k,n}$, we prove that if $\beta \neq \gamma$ then $S_{\alpha,\beta} \cap S_{\alpha,\gamma} = \emptyset$.
   Suppose $\sigma \in S_{\alpha,\beta} \cap S_{\alpha,\gamma}$. Then $\sigma(\operatorname{Im}\alpha) = \operatorname{Im}\beta = \operatorname{Im}\gamma$, and thus, $\operatorname{Im}\beta = \operatorname{Im}\gamma$. Since $\beta, \gamma \in Q_{k,n}$, it follows that $\beta = \gamma$.

2. $S_n = \bigcup_{\beta \in Q_{k,n}} S_{\alpha,\beta}$.

   Take $\pi \in S_n$ with $\pi(\operatorname{Im}\alpha) = \{j_1, \ldots, j_k\}$ and suppose $j_1 < \cdots < j_k$.
   Let $\gamma \in Q_{k,n}$ such that $\gamma(i) = j_i$, for $i = 1, \ldots, k$. Therefore, $\pi \in S_{\alpha,\gamma}$ and $S_n \subseteq \bigcup_{\beta \in Q_{k,n}} S_{\alpha,\beta}$.

The other inclusion is trivial. □

Now, for every $\alpha \in Q_{k,n}$ denote by $\overline{\operatorname{Im}\alpha}$ the complement of $\operatorname{Im}\alpha$, that is,

$$\overline{\operatorname{Im}\alpha} = \{1, 2, \ldots, n\} \setminus \operatorname{Im}\alpha.$$

LEMMA 4.2. *Let $1 \leq k \leq n$ and $\alpha, \beta \in Q_{k,n}$. If $\sigma \in S_{\alpha,\beta}$ then*

$$\sigma(\overline{\operatorname{Im}\alpha}) = \overline{\operatorname{Im}\beta}.$$

Proof. Suppose that $l \in \overline{\operatorname{Im}\alpha}$ and $\sigma(l) = j_l \in \operatorname{Im}\beta$. We have $\sigma \in S_{\alpha,\beta}$ so we can find $i \in \operatorname{Im}\alpha$ such that $\sigma(i) = j_l = \sigma(l)$. But $i \neq l$. This is a contradiction, because $\sigma \in S_n$, and therefore is injective. □

Since $\{S_{\alpha,\beta} : \beta \in Q_{k,n}\}$ is a partition of S_n we have

$$|\{S_{\alpha,\beta} : \beta \in Q_{k,n}\}| = |Q_{k,n}| = \frac{n!}{k!(n-k)!}.$$

In an analogous way, we can prove the same results if we fix β instead of α. That is, if we consider the set $\{S_{\alpha,\beta} : \alpha \in Q_{k,n}\}$.

Now, we can conclude that for every $\alpha, \beta \in Q_{k,n}$ the value

$$\sum_{\sigma \in S_{\alpha,\beta}} \chi(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}$$

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does not depend on the values of the following entries of the matrix A:

I. Entries a_{ij} with $i \in \operatorname{Im}\alpha$ and $j \in \overline{\operatorname{Im}\beta}$.
II. Entries a_{ij} with $i \in \overline{\operatorname{Im}\alpha}$ and $j \in \operatorname{Im}\beta$.

We now denote by $A\{\alpha|\beta\} = (a^+_{ij})$ the matrix of order n obtained by replacing in the matrix A every entry in I and II by zeros.

LEMMA 4.3. *With the previously established notation, we have that for each* $\alpha, \beta \in Q_{k,n}$*,*

$$\sum_{\sigma \in S_{\alpha,\beta}} \chi(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)} = d_\chi(A\{\alpha|\beta\}).$$

Proof. Using the definition of the immanant and the fact that $S_n = \displaystyle\bigcup_{\gamma \in Q_{k,n}} S_{\alpha,\gamma}$, we have that

$$\begin{aligned} d_\chi(A\{\alpha|\beta\}) &= \sum_{\sigma \in S_n} \chi(\sigma) \prod_{t=1}^{n} a^+_{t\sigma(t)} \\ &= \sum_{\sigma \in \bigcup_{\gamma \in Q_{k,n}} S_{\alpha,\gamma}} \chi(\sigma) \prod_{t=1}^{n} a^+_{t\sigma(t)} \\ &= \sum_{\gamma \in Q_{k,n}} \sum_{\sigma \in S_{\alpha,\gamma}} \chi(\sigma) \prod_{t=1}^{n} a^+_{t\sigma(t)}. \end{aligned}$$

Now take $\delta \in Q_{k,n}$ such that $\delta \neq \beta$ and $\sigma \in S_{\alpha,\delta}$. Then $\displaystyle\prod_{t=1}^{n} a^+_{t\sigma(t)} = 0$, because at least one of the factors is zero, by definition of the matrix $A\{\alpha|\beta\}$. Therefore,

$$\sum_{\sigma \in S_{\alpha,\gamma}} \chi(\sigma) \prod_{t=1}^{n} a^+_{t\sigma(t)} = 0,$$

for every $\gamma \in Q_{k,n} \setminus \{\beta\}$.

Moreover, for $A\{\alpha|\beta\}$, if $\sigma \in S_{\alpha,\beta}$ then $a^+_{t\sigma(t)} = a_{t\sigma(t)}$. So

$$d_\chi(A\{\alpha|\beta\}) = \sum_{\sigma \in S_{\alpha,\beta}} \chi(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)}.$$

This concludes the proof. ☐

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PROPOSITION 4.4. *Let A be an $n \times n$ complex matrix and suppose $1 \leq k \leq n$. Suppose $\alpha \in Q_{k,n}$. Then*

$$d_\chi(A) = \sum_{\beta \in Q_{k,n}} d_\chi(A\{\alpha|\beta\}).$$

Proof.

$$\begin{aligned} d_\chi(A) &= \sum_{\sigma \in S_n} \chi(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)} \\ &= \sum_{\beta \in Q_{k,n}} \sum_{\sigma \in S_{\alpha,\beta}} \chi(\sigma) \prod_{t=1}^{n} a_{t\sigma(t)} \\ &= \sum_{\beta \in Q_{k,n}} d_\chi(A\{\alpha|\beta\}). \end{aligned}$$

This concludes our proof. ☐

Now we construct matrices of order n using matrices of order k and order $n-k$. In general, this could be done by using the usual direct sum of matrices. We introduce a generalization of this concept.

Let $\alpha, \beta \in Q_{k,n}$, and let A be a $k \times k$ matrix and let B be a $(n-k) \times (n-k)$ matrix. Denote by $\bar{\alpha}$ be the unique element of $Q_{n-k,n}$ with $\operatorname{Im} \overline{\alpha} = \overline{\operatorname{Im} \alpha}$.

We define

$$A \bigoplus_{\alpha|\beta} B = (x_{ij}),$$

as a $n \times n$ matrix such that

- $x_{ij} = 0$ if $i \in \operatorname{Im} \alpha$ and $j \notin \operatorname{Im} \beta$;
- $x_{ij} = 0$ if $i \notin \operatorname{Im} \alpha$ and $j \in \operatorname{Im} \beta$;
- $x_{ij} = a_{\alpha^{-1}(i)\beta^{-1}(j)}$ if $i \in \operatorname{Im} \alpha$ and $j \in \operatorname{Im} \beta$;
- $x_{ij} = b_{\overline{\alpha}^{-1}(i)\overline{\beta}^{-1}(j)}$ if $i \notin \operatorname{Im} \alpha$ and $j \notin \operatorname{Im} \beta$.

In a sense, we place A in rows α and columns β and we place B in rows $\bar{\alpha}$ and columns $\bar{\beta}$.

If $\alpha = \beta = (1, \ldots, k)$, this is the usual direct sum of A and B, that is

$$A \bigoplus_{(1,\ldots,k)|(1,\ldots,k)} B = \begin{pmatrix} A & O \\ O & B \end{pmatrix}.$$

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Checking the definitions, it is easy to see that

$$X\{\alpha|\beta\} = X[\alpha|\beta] \bigoplus_{\alpha|\beta} X(\alpha|\beta).$$

Now we can state *the Laplace expansion for immanants.*

THEOREM 4.5 (Generalized Laplace Expansion). *Let X be an $n \times n$ complex matrix, let $1 \leq k \leq n$, and α a fixed element in $Q_{k,n}$. Suppose χ is an irreducible character of S_n. Then*

$$d_\chi(X) = \sum_{\beta \in Q_{k,n}} d_\chi(X[\alpha|\beta] \bigoplus_{\alpha|\beta} X(\alpha|\beta)) = \sum_{\beta \in Q_{k,n}} d_\chi(X\{\alpha|\beta\}). \tag{4.1}$$

and

$$d_\chi(X) = \sum_{\beta \in Q_{k,n}} d_\chi(X[\beta|\alpha] \bigoplus_{\beta|\alpha} X(\beta|\alpha)) = \sum_{\beta \in Q_{k,n}} d_\chi(X\{\beta|\alpha\}).$$

EXAMPLE 4.6. Let A be a matrix of order 4. Let $k = 2$ and $\alpha = (1, 2)$. Then, we have

$$\begin{aligned} d_\chi(A) = d_\chi &\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix} + d_\chi \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{12} & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix} \\ + d_\chi &\begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ a_{12} & 0 & 0 & a_{24} \\ 0 & a_{32} & a_{33} & 0 \\ 0 & a_{42} & a_{43} & 0 \end{pmatrix} + d_\chi \begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ 0 & a_{22} & a_{23} & 0 \\ a_{31} & 0 & 0 & a_{34} \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix} \\ + d_\chi &\begin{pmatrix} 0 & a_{12} & 0 & a_{14} \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & 0 & a_{43} & 0 \end{pmatrix} + d_\chi \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix}. \end{aligned}$$

We list some properties of the matrix $X\{\alpha|\beta\}$.

PROPOSITION 4.7. *Let $\alpha, \beta, \alpha', \beta' \in Q_{k,n}$. Then we have*

$$X\{\alpha|\beta\}[\alpha|\beta] = X[\alpha|\beta] \text{ and } X\{\alpha|\beta\}(\alpha|\beta) = X(\alpha|\beta).$$

If $\beta \neq \beta'$, then both matrices $X\{\alpha|\beta\}[\alpha|\beta']$ and $X\{\alpha|\beta\}(\alpha|\beta')$ have a zero column. If $\alpha \neq \alpha'$, then both matrices $X\{\alpha|\beta\}[\alpha'|\beta]$ and $X\{\alpha|\beta\}(\alpha'|\beta)$ have a zero row.

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Proof. These are consequences of the definitions. ◻

We can now check that this formula generalizes the known Laplace formulas for the determinant and the permanent (see [6] and [8]). If $\chi = \varepsilon$ then $d_\varepsilon = \det$. For $\alpha \in Q_{k,n}$, denote $|\alpha| = \alpha(1) + \cdots + \alpha(k)$. Fixing $\alpha \in Q_{k,n}$ and applying the previous properties we have:

$$\begin{aligned}
\det X &= \sum_{\beta \in Q_{k,n}} \det(X\{\alpha|\beta\}) \\
&= (-1)^{|\alpha|} \sum_{\beta \in Q_{k,n}} \sum_{\gamma \in Q_{k,n}} (-1)^{|\gamma|} \det(X\{\alpha|\beta\}[\alpha|\gamma]) \det(X\{\alpha|\beta\}(\alpha|\gamma)) \\
&= (-1)^{|\alpha|} \sum_{\beta \in Q_{k,n}} (-1)^{|\beta|} \det(X\{\alpha|\beta\}[\alpha|\beta]) \det(X\{\alpha|\beta\}(\alpha|\beta)) \\
&= (-1)^{|\alpha|} \sum_{\beta \in Q_{k,n}} (-1)^{|\beta|} \det(X[\alpha|\beta]) \det(X(\alpha|\beta)). \qquad (4.2)
\end{aligned}$$

This is exactly the expression of the the Laplace expansion for determinants. With similar arguments we can prove the same result for the Laplace expansion of the permanent.

5. Second expression for the derivatives of the immanant. We now present a formula where the entries of the matrices $X^1, \ldots, X^k$ are, in a way, separated from the entries of A.

It is easy to express the determinant of a direct sum in terms of determinants of direct summands, and the same happens with the permanent. With immanants, the best one can do is use formula (4.1), which is what we do in this second expression.

Take $X^1, \ldots, X^k$ complex matrices of order n, and, for $\sigma \in S_k$, $\beta \in Q_{k,n}$. Denoting by $\mathbf{0}$ the zero matrix of order n, we define

$$X^\sigma_\beta = \mathbf{0}(\beta; X^{\sigma(1)}, \ldots, X^{\sigma(k)}),$$

the matrix whose $\beta(p)$-th column is equal to $X^{\sigma(p)}_{[\beta(p)]}$ and the remaining columns are zero, for $1 \le p \le k$.

Theorem 5.1.

$$D^k d_\chi A(X^1, \ldots, X^k) = \sum_{\sigma \in S_k} \sum_{\alpha, \beta \in Q_{k,n}} d_\chi(X^\sigma_\beta[\alpha|\beta] \bigoplus_{\alpha|\beta} A(\alpha|\beta)),$$

in particular

$$D^k d_\chi A(X, \ldots, X) = k! \sum_{\alpha, \beta \in Q_{k,n}} d_\chi(X[\alpha|\beta] \bigoplus_{\alpha|\beta} A(\alpha|\beta)).$$

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Proof. We have proved that

$$D^k d_\chi A(X^1, \dots, X^k) = \sum_{\sigma \in S_k} \sum_{\beta \in Q_{k,n}} d_\chi A(\beta; X^{\sigma(1)}, \dots, X^{\sigma(k)}).$$

By the Laplace expansion for immanants, for every $\beta \in Q_{k,n}$, we have that

$$\begin{aligned} & d_\chi A(\beta; X^{\sigma(1)}, \dots, X^{\sigma(k)}) = \\ &= \sum_{\alpha \in Q_{k,n}} d_\chi (A(\beta; X^{\sigma(1)}, \dots, X^{\sigma(k)})\{\alpha|\beta\}) \\ &= \sum_{\alpha \in Q_{k,n}} d_\chi (A(\beta; X^{\sigma(1)}, \dots, X^{\sigma(k)})[\alpha|\beta] \bigoplus_{\alpha|\beta} A(\beta; X^{\sigma(1)}, \dots, X^{\sigma(k)})(\alpha|\beta)). \end{aligned}$$

Now notice that

$$A(\beta; X^{\sigma(1)}, \dots, X^{\sigma(k)})[\alpha|\beta] = X^\sigma_\beta[\alpha|\beta]$$

and

$$A(\beta; X^{\sigma(1)}, \dots, X^{\sigma(k)})(\alpha|\beta) = A(\alpha|\beta).$$

This concludes the proof. ◻

6. Formulas for the k-th derivative for the m-th χ-symmetric tensor power. In this section, we wish to establish a formula for the k-th derivative of the χ-symmetric tensor power of a matrix. Before we can do this, we need quite a bit of definitions, including the very definition of this matrix.

We start with some classical results that can be found in [7, Chapter 6]. Let χ be an irreducible character of S_m and

$$K_\chi = \frac{\chi(\mathrm{id})}{m!} \sum_{\sigma \in S_m} \chi(\sigma) P(\sigma),$$

where id stands for the identity element of S_m. The map K_χ is a linear operator on $\otimes^m V$, and it is also an orthoprojector. It is called a *symmetriser map*. The range of K_χ is called the *symmetry class of tensors* associated with the irreducible character χ and it is represented by $V_\chi = K_\chi(\otimes^m V)$.

It is well known that the alternating character $\chi(\sigma) = \varepsilon_\sigma$ (sign of the permutation σ) leads to the symmetry class $\wedge^m V$; on the other hand the principal character $\chi(\sigma) \equiv 1$ leads to the symmetry class $\vee^m V$.

Given a symmetriser map K_χ, we denote

$$v_1 * v_2 * \cdots * v_m = K_\chi(v_1 \otimes v_2 \otimes \cdots \otimes v_m).$$

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These vectors belong to V_χ and are called *decomposable symmetrised tensors.*

Let $\Gamma_{m,n}$ be the set of all maps from the set $\{1,\ldots,m\}$ into the set $\{1,\ldots,n\}$. This set can also be identified with the collection of multi-indices $\{(i_1,\ldots,i_m) : i_j \leq n\}$. If $\alpha \in \Gamma_{m,n}$, this correspondence associates to α the m-tuple $(\alpha(1),\ldots,\alpha(m))$. In the set $\Gamma_{m,n}$, we will consider the lexicographic order. The set

$$\{\alpha\sigma : \sigma \in S_m\} \subseteq \Gamma_{m,n}$$

is the orbit of α. The group S_m acts on $\Gamma_{m,n}$ by the action $(\sigma,\alpha) \longrightarrow \alpha\sigma^{-1}$ where $\sigma \in S_m$ and $\alpha \in \Gamma_{m,n}$. The stabiliser of α is the subgroup of S_m defined as

$$G_\alpha = \{\sigma \in S_m : \alpha\sigma = \alpha\}.$$

Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of the vector space V. Then

$$\{e^{\otimes}_\alpha = e_{\alpha(1)} \otimes e_{\alpha(2)} \otimes \cdots \otimes e_{\alpha(m)} : \alpha \in \Gamma_{m,n}\}$$

is a basis of the m-th tensor power of V. So the set of all decomposable symmetrised tensors spans V_χ. However, this set need not be a basis of V_χ, because its elements might not be linearly independent, some of them may even be zero. Let

$$\Omega = \Omega_\chi = \{\alpha \in \Gamma_{m,n} : \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0\}.$$

With simple calculations, we can conclude that

$$\|e^*_\alpha\|^2 = \frac{\chi(\mathrm{id})}{m!} \sum_{\sigma \in G_\alpha} \chi(\sigma). \tag{6.1}$$

So the nonzero decomposable symmetrised tensors are $\{e^*_\alpha : \alpha \in \Omega\}$. Now, let Δ be the system of distinct representatives for the quotient set $\Gamma_{m,n}/S_m$, constructed by choosing the first element in each orbit, for the lexicographic order of indices. It is easy to check that $\Delta \subseteq G_{m,n}$, where $G_{m,n}$ is the set of all increasing sequences of $\Gamma_{m,n}$. Let

$$\overline{\Delta} = \Delta \cap \Omega.$$

It can be proved that the set $\{e^*_\alpha : \alpha \in \overline{\Delta}\}$ is linearly independent. We have already seen that the set $\{e^*_\alpha : \alpha \in \Omega\}$, spans V_χ, so there is a set $\widehat{\Delta}$, such that $\overline{\Delta} \subseteq \widehat{\Delta} \subseteq \Omega$ and

$$\{e^*_\alpha : \alpha \in \widehat{\Delta}\},$$

is a basis for V_χ. It is also known that this basis is orthogonal if χ is a linear character (see [7, p. 167]).

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In general, if χ does not have degree one, there are no known orthonormal bases of $V_\chi(S_m)$ formed by decomposable symmetrised tensors. Let $\mathcal{E} = (v_\alpha : \alpha \in \widehat{\Delta})$ be the orthonormal basis of the m-th χ-symmetric tensor power of the vector space V obtained by applying the Gram-Schmidt orthonormalization procedure to $\mathcal{E}$. Let B be the $t \times t$ change of basis matrix, from $\mathcal{E}$ to $\mathcal{E}' = (e^*_\alpha : \alpha \in \widehat{\Delta})$. This means that for each $\alpha \in \widehat{\Delta}$,

$$v_\alpha = \sum_{\gamma \in \widehat{\Delta}} b_{\gamma\alpha} e^*_\gamma.$$

We note that this matrix B *does not depend on the choice of the orthonormal basis of* V, since the set $\widehat{\Delta}$ is independent of the vectors, and has a natural order (the lexicographic order), which the basis $\mathcal{E}$ inherits. Moreover, the Gram-Schmidt process only depends on the numbers $\langle e^*_\alpha, e^*_\beta \rangle$, and, by [7, p. 163], these are given by formula

$$\langle e^*_\alpha, e^*_\beta \rangle = \frac{\chi(\mathrm{id})}{m!} \sum_{\sigma \in S_m} \chi(\sigma) \prod_{t=1}^{m} \langle e_{\alpha(t)}, e_{\beta\sigma(t)} \rangle.$$

Hence, they only depend on the values of $\langle e_i, e_j \rangle = \delta_{ij}$ and thus are independent of the vectors themselves.

It is known that if T is a linear operator on V, then V_χ is invariant for its m-th fold tensor power $\otimes^m T$. Thus, the χ-symmetric power T, denoted by $K_\chi(T)$ is defined as the restriction of $\otimes^m T$ to V_χ. There is a close connection of this χ-symmetric tensor power of T and the immanant, as the following result already shows (this is a rephrasing of [7, p. 230]).

THEOREM 6.1. *Suppose χ is an irreducible character of the group S_m. Let $E = \{e_1, \ldots, e_n\}$ be an orthonormal basis of the inner product space V. Let $T \in L(V, V)$ be the unique linear operator such that $M(T, E) = A$.*

If $\alpha, \beta \in \Gamma_{m,n}$, then

$$\langle K_\chi(T)(e^*_\alpha), e^*_\beta \rangle = \frac{\chi(\mathrm{id})}{m!} d_\chi(A^T[\alpha|\beta]). \tag{6.2}$$

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As we came across the immanant of a transpose, we prove that $d_\chi(A^T) = d_\chi(A)$.

$$\begin{aligned}
d_\chi(A^T) &= \sum_{\sigma\in S_m} \chi(\sigma) \prod_{i=1}^{m} (A^T)_{i\sigma(i)} \\
&= \sum_{\sigma\in S_m} \chi(\sigma) \prod_{i=1}^{m} a_{\sigma(i)i} \qquad (i = \sigma^{-1}(j)) \\
&= \sum_{\sigma\in S_m} \chi(\sigma) \prod_{j=1}^{m} a_{j\sigma^{-1}(j)} \qquad (\sigma = \tau^{-1}) \\
&= \sum_{\tau\in S_m} \chi(\tau^{-1}) \prod_{j=1}^{m} a_{j\tau(j)} \qquad (\tau \text{ is conjugate to } \tau^{-1}) \\
&= \sum_{\tau\in S_m} \chi(\tau) \prod_{j=1}^{m} a_{j\tau(j)} \\
&= d_\chi(A).
\end{aligned}$$

Now we want to define $K_\chi(A)$, the m-th χ-symmetric tensor power of the matrix A. A natural way to do this is to fix an orthonormal basis E in V, and consider the linear endomorphism T such that $A = M(T, E)$. Then the basis $\mathcal{E}$ is orthonormal and one can define

$$K_\chi(A) := M(K_\chi(T), \mathcal{E})$$

The matrix $K_\chi(A)$ has order $t = |\widehat{\Delta}|$, with $|Q_{m,n}| \le t$.

It is important to notice that *this matrix does not depend on the choice of the orthonormal basis E of V*. This is an immediate consequence of the formula (6.2): For $\alpha, \beta \in \widehat{\Delta}$, the (α, β) entry of $K_\chi(A)$ is

$$\begin{aligned}
\langle K_\chi(T)v_\beta, v_\alpha\rangle &= \sum_{\gamma,\delta\in\widehat{\Delta}} \left\langle b_{\gamma\beta} K_\chi(T)e^*_\gamma, b_{\delta\alpha} e^*_\delta \right\rangle \\
&= \sum_{\gamma,\delta\in\widehat{\Delta}} b_{\gamma\beta}\overline{b_{\delta\alpha}} \left\langle K_\chi(T)e^*_\gamma, e^*_\delta \right\rangle \\
&= \frac{\chi(\mathrm{id})}{m!} \sum_{\gamma,\delta\in\widehat{\Delta}} b_{\gamma\beta}\overline{b_{\delta\alpha}} d_\chi(A[\delta|\gamma]^T) \\
&= \frac{\chi(\mathrm{id})}{m!} \sum_{\gamma,\delta\in\widehat{\Delta}} b_{\gamma\beta}\overline{b_{\delta\alpha}} d_\chi(A[\delta|\gamma]).
\end{aligned}$$

This definition admits, as special cases, the m-th compound and the m-th induced power of a matrix, as defined in [7, p. 236]. The matrix $K\chi(A)$ is called the *induced matrix* in [7, p 235], in the case when the character has degree one.

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Denote by $\mathrm{imm}_\chi(A)$ the square matrix with rows and columns indexed by $\widehat{\Delta}$, whose (γ,δ) entry is $d_\chi(A[\gamma|\delta])$ (one could call the elements of this matrix *immanantal minors* indexed by $\widehat{\Delta}$, the usual minors are obtained by considering the alternating character, in which case $\widehat{\Delta} = Q_{m,n}$). With this definition, we can rewrite the previous equation as

$$K_\chi(A) = \frac{\chi(\mathrm{id})}{m!} B^* \,\mathrm{imm}_\chi(A) B.$$

Finally, denote by $\mathrm{miximm}_\chi(X^1,\dots,X^n)$ the square matrix having rows and columns indexed by $\widehat{\Delta}$, whose (γ,δ) entry is $\Delta_\chi(X^1[\gamma|\delta],\dots,X^n[\gamma|\delta])$. With this definition, $\mathrm{miximm}_\chi(A,\dots,A) = \mathrm{imm}_\chi(A)$. We use the same shorthand as with the mixed immanant: For $k \le n$,

$$\mathrm{miximm}_\chi(A; X^1,\dots,X^k) := \mathrm{miximm}_\chi(A,\dots,A,X^1,\dots X^k)$$

Before our main formula, we recall a general result about derivatives which follows from the definition.

Lemma 6.2. *If f and g are k-differentiable maps between vector spaces such that $f\circ g$ is well defined, and g is linear, then*

$$D^k(f\circ g)(A)(X^1,\dots,X^k) = D^k f(g(A))(g(X^1),\dots,g(X^k)).$$

Theorem 6.3. *According to our previous notation, we have*

$$D^k K_\chi(A)(X^1,\dots,X^k) = \frac{\chi(\mathrm{id})}{(m-k)!} B^* \,\mathrm{miximm}_\chi(A; X^1,\dots,X^k) B$$

and, using the notation we have already established, the (α,β) entry of this matrix is

$$\frac{\chi(\mathrm{id})}{m!} \sum_{\gamma,\delta\in\widehat{\Delta}} b_{\gamma\beta}\overline{b_{\delta\alpha}} \sum_{\sigma\in S_k} \sum_{\rho,\tau\in Q_{k,m}} d_\chi(X[\delta|\gamma]^\sigma_\tau[\rho|\tau] \bigoplus_{\rho|\tau} A[\delta|\gamma](\rho|\tau)).$$

Proof. Notice that the map $A \mapsto A[\delta|\gamma]$ is linear, so we can apply Lemma 6.2 to compute the derivatives of the entries of the matrix $K_\chi(A)$. The (α,β) entry of the k-th derivative of the m-th χ-symmetric tensor power of A, i.e., the (α,β) entry of the matrix $D^k K_\chi(A)(X^1,\dots,X^k)$ is:

$$\frac{\chi(\mathrm{id})}{k!} \sum_{\gamma,\delta\in\widehat{\Delta}} b_{\gamma\beta}\overline{b_{\delta\alpha}} D^k d_\chi(A[\delta|\gamma])(X^1[\delta|\gamma],\dots,X^k[\delta|\gamma]).$$

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To abbreviate notation, for fixed $\gamma, \delta \in \widehat{\Delta}$, we will write $C := A[\delta|\gamma]$, and $Z^i := X^i[\delta|\gamma]$, $i = 1, \dots, k$. Using formula (3.1), we get

$$\begin{aligned} D^k d_\chi(A[\delta|\gamma])(X^1[\delta|\gamma], \dots, X^k[\delta|\gamma]) &= D^k d_\chi(C)(Z^1, \dots, Z^k) \\ &= \frac{m!}{(m-k)!} \Delta_\chi(C; Z^1, \dots, Z^k). \end{aligned}$$

So the (α, β) entry of $D^k K_\chi(A)(X^1, \dots, X^k)$ is

$$\frac{\chi(\mathrm{id})}{m!} \sum_{\gamma,\delta \in \widehat{\Delta}} b_{\gamma\beta} \overline{b_{\delta\alpha}} \frac{m!}{(m-k)!} \Delta_\chi(C; Z^1, \dots, Z^k) =$$

$$\frac{\chi(\mathrm{id})}{(m-k)!} \sum_{\gamma,\delta \in \widehat{\Delta}} b_{\gamma\beta} \overline{b_{\delta\alpha}} \Delta_\chi(A[\delta|\gamma]; X^1[\delta|\gamma], \dots, X^k[\delta|\gamma]).$$

According to the definition of $\mathrm{miximm}_\chi(A; X^1, \dots, X^k)$, we have

$$D^k K_\chi(A)(X^1, \dots, X^k) = \frac{\chi(\mathrm{id})}{(m-k)!} B^* \,\mathrm{miximm}_\chi(A; X^1, \dots, X^k) B.$$

This establishes the first formula. For the entries of the matrix, we use the formula in Theorem 5.1:

$$D^k d_\chi(C)(Z^1, \dots, Z^k) = \sum_{\sigma \in S_k} \sum_{\rho,\tau \in Q_{k,m}} d_\chi(Z^\sigma_\tau[\rho|\tau] \bigoplus_{\rho|\tau} C(\rho|\tau)),$$

recalling that

$$Z^\sigma_\tau = \mathbf{0}(\tau; Z^{\sigma(1)}, \dots, Z^{\sigma(k)}),$$

where $\mathbf{0}$ denotes the zero matrix of order m.

So, the (α, β) entry of the k-th derivative of $K_\chi(A)$ is:

$$\frac{\chi(\mathrm{id})}{m!} \sum_{\gamma,\delta \in \widehat{\Delta}} b_{\gamma\beta} \overline{b_{\delta\alpha}} \sum_{\sigma \in S_k} \sum_{\rho,\tau \in Q_{k,m}} d_\chi(Z^\sigma_\tau[\rho|\tau] \bigoplus_{\rho|\tau} C(\rho|\tau)) =$$

$$\frac{\chi(\mathrm{id})}{m!} \sum_{\gamma,\delta \in \widehat{\Delta}} b_{\gamma\beta} \overline{b_{\delta\alpha}} \sum_{\sigma \in S_k} \sum_{\rho,\tau \in Q_{k,m}} d_\chi(X[\delta|\gamma]^\sigma_\tau[\rho|\tau] \bigoplus_{\rho|\tau} A[\delta|\gamma](\rho|\tau)).$$

This concludes our proof. □

The formula obtained for the higher order derivatives of $K_\chi(A)(X^1, \dots, X^k)$ generalizes the expressions obtained by Bhatia, Jain and Grover ([2], [3]). We will prove this for the derivative of the m-th compound, establishing that, from the formula in

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Theorem 6.3, one can establish formula (2.5) in [5], from which the main formula for the derivative of the m-th compound of A is obtained.

If we take $\chi = \varepsilon$, the alternating character, then

$$K_\chi(A)(X^1, \ldots, X^k) = \wedge^m(A)(X^1, \ldots, X^k).$$

In this case, $\widehat{\Delta} = Q_{m,n}$ and the basis $\{e^\wedge_\alpha : \alpha \in Q_{m,n}\}$ is orthogonal and it is easy to see (by direct computation or using formula (6.1)) that every vector has norm $1/\sqrt{m!}$. So the matrix B of order $\binom{n}{m}$ is diagonal and its diagonal entries are equal to $\sqrt{m!}$.

We present two properties that we use in our computations.

I. For any square matrices $X \in M_k(\mathbb{C})$, $Y \in M_{n-k}(\mathbb{C})$ and functions $\alpha, \beta \in Q_{k,n}$,

$$\det X \bigoplus_{\alpha|\beta} Y = (-1)^{|\alpha|+|\beta|} \det X \det Y.$$

This is a consequence of formula (4.2). We again notice that if $\gamma \neq \beta$, the matrices

$$(X \bigoplus_{\alpha|\beta} Y)[\alpha|\gamma] \text{ and } (X \bigoplus_{\alpha|\beta} Y)(\alpha|\gamma)$$

have a zero column. Now, using the Laplace expansion for the determinant along α,

$$\begin{aligned}
\det X \bigoplus_{\alpha|\beta} Y &= (-1)^{|\alpha|} \sum_{\gamma \in Q_{k,n}} (-1)^{|\gamma|} \det((X \bigoplus_{\alpha|\beta} Y)[\alpha|\gamma]) \det((X \bigoplus_{\alpha|\beta} Y)(\alpha|\gamma)) \\
&= (-1)^{|\alpha|+|\beta|} \det((X \bigoplus_{\alpha|\beta} Y)[\alpha|\beta]) \det((X \bigoplus_{\alpha|\beta} Y)(\alpha|\beta)) \\
&= (-1)^{|\alpha|+|\beta|} \det X \det Y.
\end{aligned}$$

II. For $\alpha, \beta \in Q_{m,n}$ and $\rho, \tau \in Q_{k,m}$, we have

$$\sum_{\sigma \in S_k} \det(X[\alpha|\beta]^\sigma_\tau[\rho|\tau]) = k!\Delta(X^1[\alpha|\beta][\rho|\tau], \ldots, X^k[\alpha|\beta][\rho|\tau]).$$

To check this, consider the columns of the matrices involved. Remember that

$$X[\alpha|\beta]^\sigma_\tau = \mathbf{0}(\tau, X^{\sigma(1)}[\alpha|\beta], \ldots, X^{\sigma(k)}[\alpha|\beta]).$$

For given $\sigma \in S_k$ and $j \in [k]$, we have:

$$\begin{aligned}
\text{the } (i,j) \text{ entry of } X[\alpha|\beta]^\sigma_\tau[\rho|\tau] &= \text{the } (\rho(i), \tau(j)) \text{ entry of } X[\alpha|\beta]^\sigma_\tau \\
&= \text{the } (\rho(i), \tau(j)) \text{ entry of } X^{\sigma(j)}[\alpha|\beta]_{\tau(j)} \\
&= \text{the } (i,j) \text{ entry of } X^{\sigma(j)}[\alpha|\beta][\rho|\tau].
\end{aligned}$$

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Therefore,

$$X[\alpha|\beta]^\sigma_\tau[\rho|\tau] = (X^{\sigma(1)}[\alpha|\beta][\rho|\tau]_{[1]} \,\ldots\, X^{\sigma(k)}[\alpha|\beta][\rho|\tau]_{[k]})$$

and the matrices that appear in the first sum are the same as the ones that appear in the mixed discriminant.

We are now ready to prove the result. If we replace, in Theorem 6.3, $d_\chi = \det$, we have that that the (α, β) entry of $D^k \wedge^m (A)(X^1, \ldots, X^k)$ is

$$\begin{aligned}
&\frac{1}{m!} \sum_{\gamma,\delta\in Q_{m,n}} b_{\gamma\beta}\overline{b_{\delta\alpha}} \sum_{\sigma\in S_k} \sum_{\rho,\tau\in Q_{k,m}} \det(X[\delta|\gamma]^\sigma_\tau[\rho|\tau] \bigoplus_{\rho|\tau} A[\delta|\gamma](\rho|\tau))\\
&= \frac{1}{m!} m! \sum_{\sigma\in S_k} \sum_{\rho,\tau\in Q_{k,m}} \det(X[\alpha|\beta]^\sigma_\tau[\rho|\tau] \bigoplus_{\rho|\tau} A[\alpha|\beta](\rho|\tau))\\
&= \sum_{\sigma\in S_k} \sum_{\rho,\tau\in Q_{k,m}} (-1)^{|\rho|+|\tau|} \det(A[\alpha|\beta](\rho|\tau)) \det(X[\alpha|\beta]^\sigma_\tau[\rho|\tau])\\
&= k! \sum_{\rho,\tau\in Q_{k,m}} (-1)^{|\rho|+|\tau|} \det(A[\alpha|\beta](\rho|\tau)) \Delta(X^1[\alpha|\beta][\rho|\tau], \ldots, X^k[\alpha|\beta][\rho|\tau])
\end{aligned}$$

Recall (Definition 3.2) that $\Delta(B_1, \ldots, B_n)$ is the mixed discriminant. The formula we obtained is formula (2.5) in [5], if you take into account that in this paper the roles of the letters k and m are interchanged.

Using similar arguments one can obtain the formula for the k-th derivative of $\vee^m(A)(X^1, \ldots, X^k)$ in [3].

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