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ON SOME PROPERTIES OF THE PSEUDOSPECTRAL RADIUS*

G. KRISHNA KUMAR† AND S.H. LUI‡

Abstract. Pseudospectra provide an analytical and graphical alternative for investigating non-normal matrices and operators, give a quantitative estimate of departure from non-normality and give information about stability. In this paper, we prove that pseudospectral radius is sub-additive and sub-multiplicative for a commuting pair of matrices over the complex field, extending the same result for spectral radius. We discuss the same result for a non-commutative pair of matrices. We also give an analogue of the spectral radius formula for pseudospectrum.

Key words. Pseudospectrum, Pseudospectral radius, Spectral radius.

AMS subject classifications. 15A27, 15A42, 15A60, 47A10, 47A11, 65F15, 65F35.

1. Introduction.

Let $A \in \mathbb{C}^{N\times N}$. Denote the spectrum of A by $\Lambda(A)$ and the spectral radius of A by $r(A)$. Let $\|\cdot\|$ denote the matrix 2-norm. For $\epsilon \geq 0$, the ϵ-pseudospectrum of A is defined as

$$\Lambda_\epsilon(A) := \{\lambda \in \Lambda(A+E) : \|E\| \leq \epsilon\},$$

while the ϵ-pseudospectral radius of A is defined as

$$r_\epsilon(A) := \sup\{|\lambda| : \lambda \in \Lambda_\epsilon(A)\}.$$

The goal of this paper is to extend some of the properties of r to r_ϵ. Let A and B be any square matrices. Three well-known properties of r are

1. $r(AB) = r(BA)$,
2. $r(A) = \lim\limits_{k\to\infty} \|A^k\|^{1/k}$,
3. $r(A) = r^{1/k}(A^k)$ for all $k \in \mathbb{N}$.

We shall see that corresponding results for the pseudospectral radius are

1. $r_\epsilon(AB) \leq r_{\epsilon+\delta}(BA)$, where $\delta = \|BA - AB\|$,

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2. $r_\epsilon(A) = \lim_{k\to\infty} \sup_{\|E\|\le\epsilon} \|(A+E)^k\|^{1/k}$,
3. $\max\{1, r(A)\} \le \lim_{k\to\infty} r_\epsilon^{1/k}(A^k)$.

Suppose $AB = BA$ for some $A, B \in \mathbb{C}^{N\times N}$. Then it is well-known that r is sub-additive and sub-multiplicative, namely,

$$r(A+B) \le r(A) + r(B) \quad \text{and} \quad r(AB) \le r(A)r(B).$$

We shall show that r_ϵ is also sub-additive and sub-multiplicative, with the latter requiring some additional restrictions. A similar result in case of condition spectrum and condition spectral radius can be proved ([3]).

In Section 2, we prove the three properties of the pseudospectral radius mentioned above. In Section 3, we show sub-additivity and sub-multiplicativity of the pseudospectral radius for commuting matrices, to be followed by those for non-commuting matrices in the final section.

2. Some properties of pseudsospectral radius.

THEOREM 2.1. *Let $A, B \in \mathbb{C}^{N\times N}$, $\delta = \|BA - AB\|$ and $\epsilon \ge 0$. Then*

$$r_\epsilon(AB) \le r_{\epsilon+\delta}(BA).$$

Proof. Let $z \in \mathbb{C}$ so that $|z| = r_\epsilon(AB)$. Then there is some $E \in \mathbb{C}^{N\times N}$ with $\|E\| \le \epsilon$ so that $z \in \Lambda(AB+E) = \Lambda(BA + (AB-BA) + E)$. This implies that $z \in \Lambda_{\epsilon+\delta}(BA)$ and so $r_\epsilon(AB) \le r_{\epsilon+\delta}(BA)$. □

Let $A \in \mathbb{C}^{N\times N}$, $\alpha, \beta \in \mathbb{C}$ with $\beta \ne 0$, and $\epsilon \ge 0$. The following properties ([7]) will be used in this paper:

$$r(A) \le r_\epsilon(A) - \epsilon \le \|A\| \quad \text{and} \quad \Lambda_\epsilon(\alpha + \beta A) = \alpha + \beta\, \Lambda_{\frac{\epsilon}{|\beta|}}(A).$$

THEOREM 2.2. *Let $A \in \mathbb{C}^{N\times N}$ and $\epsilon \ge 0$. Then*

$$\lim_{k\to\infty} \sup_{\|E\|\le\epsilon} \|(A+E)^k\|^{1/k} = r_\epsilon(A).$$

Proof. Let $r_\epsilon(A) = |z|$ for some $z \in \Lambda_\epsilon(A) = \Lambda(A+E)$, where $E \in \mathbb{C}^{N\times N}$ so that $\|E\| \le \epsilon$. Let $(A+E)u = zu$ for some eigenvector u with $|u| = 1$. Then

$$\|(A+E)^k\| \ge |(A+E)^k u| = |z|^k = r_\epsilon^k(A),$$

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and so,

$$\lim_{k\to\infty} \sup_{\|E\|\le\epsilon} \|(A+E)^k\|^{1/k} \ge r_\epsilon(A).$$

Let $\delta > 0$. For each $k \in \mathbb{N}$, there is some $E_k \in \mathbb{C}^{N\times N}$ so that $\|E_k\| \le \epsilon$, and

$$\|(A+E_k)^k\|^{1/k} > L_k - \delta \quad \text{and} \quad L_k = \sup_{\|E\|\le\epsilon} \|(A+E)^k\|^{1/k}.$$

Since $\|E_k\| \le \epsilon$ for every k, there is some $E_\infty \in \mathbb{C}^{N\times N}$ so that $\|E_\infty\| \le \epsilon$ and some subsequence E_{n_j} so that $E_{n_j} \to E_\infty$. Now

$$\begin{aligned}\lim_{k\to\infty} \sup_{\|E\|\le\epsilon} \|(A+E)^k\|^{1/k} &< \lim_{k\to\infty} \lim_{j\to\infty} \|(A+E_{n_j})^k\|^{1/k} + \delta \\ &= \lim_{k\to\infty} \|(A+E_\infty)^k\|^{1/k} + \delta \\ &= r_\epsilon(A) + \delta.\end{aligned}$$

Since δ is arbitrary, we have

$$\lim_{k\to\infty} \sup_{\|E\|\le\epsilon} \|(A+E)^k\|^{1/k} \le r_\epsilon(A).$$

This completes the proof of the theorem. □

THEOREM 2.3. *Let $A \in \mathbb{C}^{N\times N}$. Then*

$$\lim_{k\to\infty} r_\epsilon^{1/k}(A^k) \ge \max\{r(A), 1\}.$$

Proof. For any $\epsilon \ge 0$, since $r_\epsilon(A) \ge r(A) + \epsilon$, we have that for every $k > 0$,

$$r_\epsilon(A^k) \ge r(A^k) + \epsilon = r^k(A) + \epsilon.$$

Suppose $r(A) \ge 1$. Then

$$r_\epsilon^{1/k}(A^k) \ge (r^k(A)+\epsilon)^{1/k} = r(A)\left(1 + \frac{\epsilon}{r^k(A)}\right)^{1/k} \to r(A)$$

as $k \to \infty$.

Suppose $r(A) < 1$. Then

$$r_\epsilon^{1/k}(A^k) \ge \epsilon^{1/k}\left(1 + \frac{r^k(A)}{\epsilon}\right)^{1/k} \to 1$$

as $k \to \infty$.

Combine the results of the above two paragraphs to obtain

$$\lim_{k\to\infty} r_\epsilon^{1/k}(A^k) \ge \max\{r(A), 1\}. \quad \square$$

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3. Sub-additivity and sub-multiplicativity for commuting matrices. Let I be the identity matrix and $\mathcal{I} = \{\alpha I,\ \alpha \in \mathbb{C}\}$. The following result is a modification of one on page 18 in [1].

LEMMA 3.1. *Let Γ denote a bounded semigroup of $\mathbb{C}^{N\times N}$ containing I, and Γ contains no other scalar multiples of I. Let $\epsilon \geq 0$. Then there exists a function $p : \mathbb{C}^{N\times N} \to \mathbb{R}^+$ satisfying the following conditions:*

1. *$r_\epsilon(A) - \epsilon \leq p(A)$ for all $A \in \mathbb{C}^{N\times N}$.*
2. *$p(S) \leq 1$ for all $S \in \Gamma$.*
3. *$p(A + B) \leq p(A) + p(B)$ for all $A, B \in \mathbb{C}^{N\times N} \setminus \mathcal{I}$.*
4. *$p(AB) \leq p(A)p(B)$ for all $A, B \in \mathbb{C}^{N\times N}$.*

Proof. Define $q : \mathbb{C}^{N\times N} \to \mathbb{R}^+$ by,

$$q(A) = \begin{cases} \sup\{\|SA\| : S \in \Gamma\} \ \text{ if } \ A \notin \mathcal{I}; \\ |\alpha| \ \text{ if } \ A = \alpha I, \quad \text{some } \alpha \in \mathbb{C}. \end{cases}$$

Then q satisfies

$$\|A\| \leq q(A) \leq K\|A\| \ \text{ for all } \ A \in \mathbb{C}^{N\times N}, \tag{3.1}$$

where $K = \sup\{\|S\| : S \in \Gamma\}$. Since $I \in \Gamma$,

$$q(\alpha A) = |\alpha| q(A) \ \text{ for all } \ \alpha \in \mathbb{C} \ \text{ and } \ A \in \mathbb{C}^{N\times N}, \tag{3.2}$$

$$q(AB) \leq q(A)q(B) \ \text{ for all } \ A, B \in \mathbb{C}^{N\times N}. \tag{3.3}$$

Define $p : \mathbb{C}^{N\times N} \to \mathbb{R}^+$ as

$$p(A) = \sup\{q(AX) : X \in \mathbb{C}^{N\times N} \ \text{ and } \ q(X) \leq 1\}.$$

Claim: $p(A) = q(A)$ for all $A \in \mathbb{C}^{N\times N}$. Since $q(I) = 1$, $q(A) \leq p(A)$ for all $A \in \mathbb{C}^{N\times N}$. Also

$$\begin{aligned} p(A) &= \sup\{q(AX) : X \in \mathbb{C}^{N\times N} \ \text{ and } \ q(X) \leq 1\} \\ &\leq \sup\{q(A)q(X) : X \in \mathbb{C}^{N\times N} \ \text{ and } \ q(X) \leq 1\} \\ &= q(A). \end{aligned}$$

This shows the claim. Now we are ready to prove the four conditions of p.

1. From the above results, it follows that $r_\epsilon(A) - \epsilon \leq \|A\| \leq q(A) = p(A)$.
2. Recall $p(I) = q(I) = 1$. For $S_0 \in \Gamma \setminus \{I\}$,

$$\begin{aligned} q(S_0A) &= \sup\{\|SS_0A\| : S \in \Gamma\} \\ &\leq \sup\{\|SA\| : S \in \Gamma\} \qquad \text{(since } \Gamma \text{ is a semigroup)} \\ &= q(A). \end{aligned}$$

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From the definition of $p(S_0)$ it follows that

$$p(S_0) = \sup_{q(X)\le 1} q(S_0X) \le \sup_{q(X)\le 1} q(X) \le 1.$$

3. Let $A, B \in \mathbb{C}^{N\times N} \setminus \mathcal{I}$. Since $p(A) = q(A)$ for all $A \in \mathbb{C}^{N\times N}$, it is sufficient to prove that $q(A + B) \le q(A) + q(B)$. There are two cases to consider, depending on whether $A + B$ is a scalar multiple of I or not.

   Case 1: $A + B$ is a scalar multiple of I.

$$\begin{aligned} q(A + B) &= \|A + B\| \\ &\le \|A\| + \|B\| \\ &\le q(A) + q(B) \text{ by (3.1).} \end{aligned}$$

   Case 2: $A + B$ is not a scalar multiple of I.

$$\begin{aligned} q(A + B) &= \sup\{\|S(A + B)\| : S \in \Gamma\} \\ &\le \sup\{\|SA\| : S \in \Gamma\} + \sup\{\|SB\| : S \in \Gamma\} \\ &= q(A) + q(B). \end{aligned}$$

4. Using (3.3) and $p(A) = q(A)$ for all $A \in \mathbb{C}^{N\times N}$, it follows $q(AB) \le q(A)q(B)$, and consequently, $p(AB) \le p(A)p(B)$. ☐

Theorem 3.2. *Let $A, B \in \mathbb{C}^{N\times N}$ such that $AB = BA$. Then for $\epsilon \ge 0$,*

$$r_\epsilon(A + B) \le r_\epsilon(A) + r_\epsilon(B).$$

Proof. If $\epsilon = 0$, the result is well known. See [1], for instance. Henceforth, assume $\epsilon > 0$. If both A and B are scalar multiples of I, then the result of the theorem holds trivially. Suppose A or B is a scalar multiple of I. Without loss of generality, assume that $A = \alpha I$ for some $\alpha \in \mathbb{C}$. Then

$$\Lambda_\epsilon(A + B) = \Lambda_\epsilon(\alpha I + B) = \alpha + \Lambda_\epsilon(B) \subseteq \Lambda_\epsilon(A) + \Lambda_\epsilon(B).$$

Consequently,

$$r_\epsilon(A + B) \le r_\epsilon(A) + r_\epsilon(B).$$

Consider the case where both A and B are not scalar multiples of I. For any $\delta > 0$, let

$$U = \frac{A}{r_\epsilon(A) - \epsilon + \delta} \quad \text{and} \quad V = \frac{B}{r_\epsilon(B) - \epsilon + \delta}.$$

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Then both U, V are not scalar multiples of I and $r(U) < 1,\ r(V) < 1$ and $UV = VU$. Thus, the set $\{U^iV^j : i, j \geq 0\}$ is a semigroup under multiplication. We now show that it is bounded.

Since $r(U) < 1$, there is some s so that $r(U) < s < 1$. Given any t satisfying $0 < t < 1 - s$, since $r(U) = \lim_{n\to\infty} \|U^n\|^{1/n}$, there is some N so that for all $n \geq N$,

$$\|U^n\|^{1/n} < r(U) + t.$$

This implies that

$$\|U^n\| < (s + t)^n < 1, \quad n \geq N.$$

This shows that $\{\|U^n\| : \ n \geq 0\}$ is bounded. Similarly, $\{\|V^n\| : \ n \geq 0\}$ is bounded. Since U and V commute, $\{U^iV^j : \ i, j \geq 0\}$ is bounded.

From Lemma 3.1, it follows that there exists a function $p : \mathbb{C}^{N\times N} \to \mathbb{R}^+$ satisfying all four conditions of the lemma. In particular,

$$p\left(\frac{A}{r_\epsilon(A) - \epsilon + \delta}\right) \leq 1 \quad \text{and} \quad p\left(\frac{B}{r_\epsilon(B) - \epsilon + \delta}\right) \leq 1.$$

This gives

$$p(A) \leq r_\epsilon(A) - \epsilon + \delta \quad \text{and} \quad p(B) \leq r_\epsilon(B) - \epsilon + \delta.$$

Thus,

$$r_\epsilon(A + B) - \epsilon \leq p(A + B) \leq p(A) + p(B) \leq (r_\epsilon(A) - \epsilon + \delta) + (r_\epsilon(B) - \epsilon + \delta).$$

Choosing $\delta = \epsilon/2$,

$$r_\epsilon(A + B) \leq r_\epsilon(A) + r_\epsilon(B). \quad \square$$

Next, we show the sub-multiplicativity of r_ϵ. We first examine some trivial cases. The case $\epsilon = 0$ is the classical case. Henceforth, assume $\epsilon > 0$.

1. Suppose $A = \alpha I,\ B = \beta I,\ \alpha, \beta \in \mathbb{C}$. A simple calculation yields $r_\epsilon(AB) = |\alpha\beta| + \epsilon,\ r_\epsilon(A)r_\epsilon(B) = |\alpha\beta| + \epsilon(|\alpha| + |\beta|) + \epsilon^2$. Thus, r_ϵ is sub-multiplicative iff $|\alpha| + |\beta| + \epsilon \geq 1$ iff $r_\epsilon(A) + r_\epsilon(B) - 1 \geq \epsilon$.
2. Suppose $A = 0$. Then for any $B \in \mathbb{C}^{N\times N}$,

$$r_\epsilon(AB) = r_\epsilon(0) = \epsilon \quad \text{and} \quad r_\epsilon(A)r_\epsilon(B) = r_\epsilon(B)\epsilon.$$

Hence, in this case, r_ϵ is sub-multiplicative iff $r_\epsilon(B) \geq 1$ iff $r_\epsilon(A)+r_\epsilon(B)-1 \geq \epsilon$.

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3. Suppose $A = \alpha I,\ \alpha \in \mathbb{C}\backslash\{0\}$ and B is arbitrary. Then $r_\epsilon(AB) = |\alpha|\, r_{\epsilon/|\alpha|}(B)$, $r_\epsilon(A)r_\epsilon(B) = (|\alpha| + \epsilon)r_\epsilon(B)$. Thus, r_ϵ is multiplicative iff

$$r_{\epsilon/|\alpha|}(B) \leq \left(1 + \frac{\epsilon}{|\alpha|}\right) r_\epsilon(B).$$

Now we proceed to the non-trivial case.

THEOREM 3.3. *Let $A, B \in \mathbb{C}^{N\times N} \setminus \mathcal{I}$ such that $AB = BA$. For $0 \leq \epsilon \leq r_\epsilon(A) + r_\epsilon(B) - 1$,*

$$r_\epsilon(AB) \leq r_\epsilon(A)\, r_\epsilon(B).$$

Proof. For $\delta > 0$, argue as in the previous theorem to obtain a function $p : \mathbb{C}^{N\times N} \to \mathbb{R}^+$ satisfying the four conditions in Lemma 3.1. In particular,

$$p(A) \leq r_\epsilon(A) - \epsilon + \delta \quad \text{and} \quad p(B) \leq r_\epsilon(B) - \epsilon + \delta.$$

Thus,

$$r_\epsilon(AB) - \epsilon \leq p(AB) \leq p(A)p(B) \leq (r_\epsilon(A) - \epsilon + \delta)(r_\epsilon(B) - \epsilon + \delta).$$

Since $\delta > 0$ is arbitrary,

$$\begin{aligned} r_\epsilon(AB) &\leq (r_\epsilon(A) - \epsilon)(r_\epsilon(B) - \epsilon) + \epsilon. \\ &= r_\epsilon(A)r_\epsilon(B) - \epsilon(r_\epsilon(A) + r_\epsilon(B) - 1) + \epsilon^2. \\ &\leq r_\epsilon(A)r_\epsilon(B), \end{aligned}$$

because $r_\epsilon(A) + r_\epsilon(B) - 1 \geq \epsilon$. ◻

Using the property $r(A) + \epsilon \leq r_\epsilon(A)$, a simple sufficient condition for the restriction $\epsilon \leq r_\epsilon(A) + r_\epsilon(B) - 1$ in the above theorem is $1 - r(A) - r(B) \leq \epsilon$.

Let $\alpha \in \mathbb{C}$ and $A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$. It is shown in [6] that for $\epsilon \geq 0$, $\Lambda_\epsilon(A) = D(1, \sqrt{|\alpha|\epsilon + \epsilon^2})$, where $D(z, r)$ is the closed disk of radius r with center at z. Thus, $r_\epsilon(A) = 1 + \sqrt{|\alpha|\epsilon + \epsilon^2}$. We now show an example where r_ϵ is not sub-multiplicative for all ϵ sufficiently small.

EXAMPLE 3.1. Let $A = \frac{1}{3}\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$. Then $A^2 = \frac{1}{9}\begin{bmatrix} 1 & 2\alpha \\ 0 & 1 \end{bmatrix}$. We have

$$\Lambda_\epsilon(A) = \frac{\Lambda_{3\epsilon}(3A)}{3} \quad \text{and} \quad \Lambda_\epsilon(A^2) = \frac{\Lambda_{9\epsilon}(9A^2)}{9},$$

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and so,

$$r_\epsilon(A) = \frac{1}{3}\left(1 + \sqrt{3|\alpha|\epsilon + 9\epsilon^2}\right) \quad \text{and} \quad r_\epsilon(A^2) = \frac{1}{9}\left(1 + \sqrt{18|\alpha|\epsilon + 81\epsilon^2}\right).$$

With $B = A$, the above theorem says that $r_\epsilon(A^2) \le r_\epsilon^2(A)$ provided $2r_\epsilon(A) - 1 \ge \epsilon$. A calculation shows that the latter condition is equivalent to

$$|\alpha| \ge \frac{1}{12\epsilon} + \frac{1}{2} - \frac{9\epsilon}{4}.$$

This suggests that sub-multiplicativity may not hold for ϵ small. Indeed, choosing $\alpha = 1$, we find that

$$r_\epsilon(A^2) = \frac{1 + 3\sqrt{2\epsilon + 9\epsilon^2}}{9} = \frac{1 + 3\sqrt{2}\,\sqrt{\epsilon} + O(\epsilon)}{9},$$

while

$$r_\epsilon^2(A) = \frac{1 + 2\sqrt{3\epsilon + 9\epsilon^2} + 3\epsilon + 9\epsilon^2}{9} = \frac{1 + 2\sqrt{3}\,\sqrt{\epsilon} + O(\epsilon)}{9}.$$

Since $3\sqrt{2} \approx 4.24\ldots$ and $2\sqrt{3} \approx 3.46\ldots$, it follows that r_ϵ is not sub-multiplicative for all ϵ sufficiently small.

Next, we show that the results of the above theorems do not hold if the matrices are not commutative.

Example 3.2. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It is easy to check that $AB \neq BA$ and

$$r_\epsilon(A + B) = 1 + \epsilon$$

and so $r_\epsilon(A + B) \le r_\epsilon(A) + r_\epsilon(B)$ iff $\epsilon \ge 1/3$.

Example 3.3. Take A and B as in the above example. Then

$$r_\epsilon(A) = \sqrt{\epsilon + \epsilon^2} = r_\epsilon(B), \qquad r_\epsilon(AB) = 1 + \epsilon.$$

Hence $r_\epsilon(AB) \le r_\epsilon(A)r_\epsilon(B)$ iff $\epsilon \ge 1$. The condition for sub-multiplicativity in Theorem 3.3 is $\epsilon \ge 1/3$.

Remark 3.1. Consider Example 3.2. We have

$$r_\epsilon(A + B) \le r_\epsilon(A) + r_\epsilon(B) \ \text{ for all } \epsilon \ge 1/3,$$

and

$$r_\epsilon(AB) \le r_\epsilon(A)r_\epsilon(B) \ \text{ for all } \epsilon \ge 1.$$

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Thus, the converse of the result is not true: The matrices need not be commutative to satisfy sub-additivity (Theorem 3.2) and sub-multiplicativity (Theorem 3.3) if ϵ is suitably chosen.

REMARK 3.2. The results proved in this section are also true for a pair of commuting elements from a complex unital Banach algebra.

4. Sub-multiplicativity and sub-additivity for non-commuting matrices. The goal of the present section is to extend the results of the previous section to matrices which do not commute.

Let $A, B \in \mathbb{C}^{N\times N}$ such that A, B are non-commutative. In this case, we need to look for a pair (A', B') which is not a scalar multiple of I and close to (A, B) such that the pair (A', B') is commutative. Define

$$\rho := \min_{\substack{A', B' \notin \mathcal{I} \\ A'B' = B'A'}} \max \{\|A - A'\|, \|B - B'\|\}.$$

Since the map $A \mapsto \Lambda_\epsilon(A)$ is upper semi-continuous, $A \mapsto r_\epsilon(A)$ is a continuous map, we have

$$|r_\epsilon(A) - r_\epsilon(A')| \leq f(A, A', \epsilon, \rho), \tag{4.1}$$

$$|r_\epsilon(B) - r_\epsilon(B')| \leq g(B, B', \epsilon, \rho), \tag{4.2}$$

$$|r_\epsilon(A + B) - r_\epsilon(A' + B')| \leq h(A, A', B, B', \epsilon, \rho), \tag{4.3}$$

$$|r_\epsilon(AB) - r_\epsilon(A'B')| \leq k(A, A', B, B', \epsilon, \rho) \tag{4.4}$$

for some continuous functions f, g, h, k. Consider an arbitrary $\delta > 0$ and let

$$U := \frac{A'}{r_\epsilon(A') - \epsilon + \delta} \quad \text{and} \quad V := \frac{B'}{r_\epsilon(B') - \epsilon + \delta}.$$

Then both U, V are not scalar multiples of I and $r(U) < 1$, $r(V) < 1$ and $UV = VU$. Thus, the set $\{U^i V^j : i, j \geq 0\}$ is a bounded semigroup under multiplication. From Lemma 3.1, it follows that there exists a function $p : \mathbb{C}^{N\times N} \to \mathbb{R}^+$ satisfying all four conditions of the lemma. In particular,

$$p\left(\frac{A'}{r_\epsilon(A') - \epsilon + \delta}\right) \leq 1 \quad \text{and} \quad p\left(\frac{B'}{r_\epsilon(B') - \epsilon + \delta}\right) \leq 1.$$

This gives

$$p(A') \leq r_\epsilon(A') - \epsilon + \delta \quad \text{and} \quad p(B') \leq r_\epsilon(B') - \epsilon + \delta.$$

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Thus,

$$\begin{aligned} r_\epsilon(A+B)-\epsilon &\le r_\epsilon(A'+B')-\epsilon+h \le p(A'+B')+h \le p(A')+p(B')+h \\ &\le (r_\epsilon(A')-\epsilon+\delta)+(r_\epsilon(B')-\epsilon+\delta)+h \\ &\le (r_\epsilon(A)+f-\epsilon+\delta)+(r_\epsilon(B)+g-\epsilon+\delta)+h \\ &= r_\epsilon(A)+r_\epsilon(B)+f+g+h-\epsilon. \end{aligned} \tag{4.5}$$

In the last equality, we have taken $\delta = \epsilon/2$. Whenever A, B commute, we have $f = g = h = 0$ and we end up with the result proved in Theorem 3.2.

We also have

$$\begin{aligned} r_\epsilon(AB)-\epsilon &\le r_\epsilon(A'B')+k-\epsilon \le p(A'B')+k \\ &\le p(A')p(B')+k \le (r_\epsilon(A')-\epsilon+\delta)(r_\epsilon(B')-\epsilon+\delta)+k \\ &\le (r_\epsilon(A)+f-\epsilon+\delta)(r_\epsilon(B)+g-\epsilon+\delta)+k \\ &= (r_\epsilon(A)+f-\epsilon)(r_\epsilon(B)+g-\epsilon)+k. \end{aligned} \tag{4.6}$$

The last equality follows from setting $\delta = 0$. Again, whenever A, B commute, we have $f = g = k = 0$ and recover the result proved in Theorem 3.3.

We now look into the case where $A, B \in \mathbb{C}^{N\times N}$ are almost commutative, i.e., $\|AB - BA\| \le \theta$ for some sufficiently small $\theta > 0$. The following results are available in the literature for almost commuting matrices.

1. In [5], the author find $A, B \in \mathbb{C}^{N\times N}$ such that $\|AB - BA\| \le \theta$ for some $\theta \ge 0$ and A, B may not be near to any commuting pair.
2. Let $A, B \in \mathbb{C}^{N\times N}$ such that $\|A\| \le 1, \|B\| \le 1$ and $\|AB - BA\| \le \theta$ for some $\theta \ge 0$. Using non-standard analysis, the authors in [4] proved that there exists a commuting pair A', B' with $\|A'\| \le 1, \|B'\| \le 1$ such that $\|A-A'\| \le f_N(\theta)$ and $\|B - B'\| \le f_N(\theta)$. It is shown that the constant $f_N(\theta)$ is dependent on the pair A, B and N, the order of the matrices, such that $f_N(\theta) \to 0$ as $\theta \to 0$.

   Thus, (1), (2) together show that finding a quantity independent of the order of the matrix and depending only on the constant θ is not possible. In case A is self-adjoint, it is possible to find a constant which is independent of the order of the matrices.
3. In [2], the authors proved the following result. Let $A, B \in \mathbb{C}^{N\times N}$ with $A = A^*$ and $\|AB - BA\| \le \dfrac{2\theta^2}{N-1}$ for some small $\theta \ge 0$. Then there exist $A', B' \in \mathbb{C}^{N\times N}$ with $A'^* = A'$ such that $A'B' = B'A'$, $\|A-A'\| \le \theta$ and $\|B-B'\| \le \theta$.

Lemma 4.1. *Let A be a square matrix and ϵ, c be non-negative reals. Then*

$$r_\epsilon(A) + c \le r_{\epsilon+c}(A).$$

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Proof. Let $z = \alpha e^{i\theta}$ with $\alpha \geq 0$ and $\theta \in \mathbb{R}$ so that $|z| = r_\epsilon(A)$. Then there is some non-zero vector u and matrix E with $\|E\| \leq \epsilon$ so that $(A + E)u = zu$. Thus,

$$(A + E + ce^{i\theta} I)u = (\alpha + c)e^{i\theta} u,$$

meaning that $(\alpha + c)e^{i\theta} \in \Lambda(A + E + ce^{i\theta} I)$, or $\alpha + c \leq r_{\epsilon+c}(A)$. □

The following theorem extends the result proved in Section 3 to almost commuting matrices. We make use of the above results available in the literature.

THEOREM 4.2. *Let $A, B \in \mathbb{C}^{N\times N}$ such that $\|A\| \leq 1, \|B\| \leq 1$ and $\|AB - BA\| \leq \theta$ for some $\theta \geq 0$. Then there exist functions g, h such that for ϵ fixed, both $g(\theta, \epsilon)$ and $h(\theta, \epsilon)$ go to zero whenever θ goes to zero and*

$$r_\epsilon(A + B) \leq r_{\epsilon+g(\theta,\epsilon)}(A) + r_{\epsilon+g(\theta,\epsilon)}(B), \quad \epsilon \geq 0.$$

For all $\epsilon > 0$ satisfying $r_{\epsilon+h(\theta,\epsilon)}(A) + r_{\epsilon+h(\theta,\epsilon)}(B) \geq \epsilon + 1 + h(\theta, \epsilon)/\epsilon$, then

$$r_\epsilon(AB) \leq r_{\epsilon+h(\theta,\epsilon)}(A)\ r_{\epsilon+h(\theta,\epsilon)}(B).$$

Proof. From [4], there exists $f(\theta)$ such that $\|A - A'\| \leq f(\theta)$, $\|B - B'\| \leq f(\theta)$ and $f(\theta)$ goes to zero as θ goes to zero. (To simplify the notation, we have suppressed the dependence of all functions on N.) Since the map $A \mapsto r_\epsilon(A)$ is continuous, equations (4.1) to (4.4) imply

$$\begin{aligned} |r_\epsilon(A) - r_\epsilon(A')| &\leq \tilde{g}(\theta, \epsilon), \\ |r_\epsilon(B) - r_\epsilon(B')| &\leq \tilde{g}(\theta, \epsilon), \\ |r_\epsilon(A + B) - r_\epsilon(A' + B')| &\leq \tilde{g}(\theta, \epsilon), \\ |r_\epsilon(AB) - r_\epsilon(A'B')| &\leq \tilde{g}(\theta, \epsilon) \end{aligned}$$

for some $\tilde{g}(\theta, \epsilon)$ with $\tilde{g}(\theta, \epsilon)$ going to zero whenever θ goes to zero for ϵ fixed. The last two assertions follow from the fact that matrix addition and multiplication are continuous operations. From (4.5),

$$r_\epsilon(A + B) \leq r_\epsilon(A) + r_\epsilon(B) + 3\tilde{g}(\theta, \epsilon).$$

By Lemma 4.1,

$$r_\epsilon(A + B) \leq r_{\epsilon+g(\theta,\epsilon)}(A) + r_{\epsilon+g(\theta,\epsilon)}(B),$$

where $g = 3\tilde{g}/2$. Of course, $g(\theta, \epsilon)$ goes to zero whenever θ goes to zero with ϵ fixed.

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Using (4.6) and Lemma 4.1, there is some function h, with the property that for any fixed ϵ, $h(\theta,\epsilon) \to 0$ whenever $\theta \to 0$, so that

$$\begin{aligned} r_\epsilon(AB) &\le (r_\epsilon(A) + h(\theta,\epsilon) - \epsilon)\ (r_\epsilon(B) + h(\theta,\epsilon) - \epsilon) + h(\theta,\epsilon) + \epsilon \\ &\le (r_{\epsilon+h(\theta,\epsilon)}(A) - \epsilon)\ (r_{\epsilon+h(\theta,\epsilon)}(B) - \epsilon) + h(\theta,\epsilon) + \epsilon \\ &= r_{\epsilon+h(\theta,\epsilon)}(A)\ r_{\epsilon+h(\theta,\epsilon)}(B) - \epsilon(r_{\epsilon+h(\theta,\epsilon)}(A) + r_{\epsilon+h(\theta,\epsilon)}(B) - \epsilon - 1 - h(\theta,\epsilon)/\epsilon) \\ &\le r_{\epsilon+h(\theta,\epsilon)}(A)\ r_{\epsilon+h(\theta,\epsilon)}(B), \end{aligned}$$

since $r_{\epsilon+h(\theta,\epsilon)}(A) + r_{\epsilon+h(\theta,\epsilon)}(B) - \epsilon - 1 - h(\theta,\epsilon)/\epsilon \ge 0$. □

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