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Louis Deaett
Louis.Deaett@quinnipiac.edu

Ramiro H. Lafuente-Rodriguez

Juan Marin Jr.

Erin Haller Martin

Linda J. Patton

See next page for additional authors

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Authors

Louis Deaett, Ramiro H. Lafuente-Rodriguez, Juan Marin Jr., Erin Haller Martin, Linda J. Patton, Kathryn Rasmussen, and Rebekah B. Johnson Yates

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TRACE CONDITIONS FOR SYMMETRY OF
THE NUMERICAL RANGE[*]

LOUIS DEAETT[†], RAMIRO H. LAFUENTE-RODRIGUEZ[‡], JUAN MARIN JR.[§], ERIN HALLER MARTIN[¶], LINDA J. PATTON[‖], KATHRYN RASMUSSEN[**], AND REBEKAH B. JOHNSON YATES[††]

Abstract. A subset S of the complex plane has n-fold symmetry about the origin (n-sato) if $z \in S$ implies $e^{\frac{2\pi}{n}} z \in S$. The 3×3 matrices A for which the numerical range $W(A)$ has 3-sato have been characterized in two ways. First, $W(A)$ has 3-sato if and only if the spectrum of A has 3-sato while $\operatorname{tr}(A^2 A^*) = 0$. In addition, $W(A)$ has 3-sato if and only if A is unitarily similar to an element of a certain family of generalized permutation matrices. Here it is shown that for an $n \times n$ matrix A, if a specific finite collection of traces of words in A and A^* are all zero, then $W(A)$ has n-sato. Further, this condition is shown to be necessary when $n = 4$. Meanwhile, an example is provided to show that the condition of being unitarily similar to a generalized permutation matrix does not extend in an obvious way.

Key words. Numerical range, Symmetry.

AMS subject classifications. 15A60, 15A15, 15B99, 15A21.

1. Introduction. The *numerical range* of an $n \times n$ matrix A is the subset of the complex plane $\mathbb{C}$ defined by

$$W(A) = \{\langle Av, v \rangle \mid v \in \mathbb{C}^n, \|v\| = 1\}.$$

The numerical range, which is also called the field of values, is a useful tool for studying the properties of a matrix. The Toeplitz-Hausdorff Theorem ([6, 13]) states

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that the numerical range is always convex. It follows directly from the definition
that the numerical range of a matrix is invariant under unitary equivalence. When
a linear map is defined on a finite-dimensional space (the only setting in this paper),
the associated numerical range is compact. These and many other properties of the
numerical range are in [5] and [7].

It is well-known that the numerical range of a 2×2 matrix A is a possibly
degenerate ellipse with foci at the eigenvalues of A. The shapes of numerical ranges
of 3×3 matrices were classified by Kippenhahn [10] in terms of an algebraic curve
associated with the matrix. The four basic shapes include a polygon (for normal
matrices), the convex hull of an ellipse and a point, an ovular shape, and a shape
with one flat portion on its boundary. Concrete tests to determine the shape from
the entries of the 3×3 matrix are in [9]. Work has also been done to generalize
Kippenhahn's result to 4×4 matrices [2]. More generally, [8] includes a necessary
and sufficient condition for a convex subset of $\mathbb{C}$ to be the numerical range of some
$n \times n$ matrix; the result follows from work on linear matrix inequalities. However,
as the size of the matrix increases, it becomes more difficult to classify the different
possible shapes and to develop concrete tests that determine shapes. It can be more
tractable to focus on one type of numerical range condition and develop tests and
results for that condition. In this paper, we will study numerical ranges that have the
strong symmetry condition stated below.

DEFINITION 1.1. A subset S of the complex numbers is said to have *n-fold
symmetry about the origin* if $z \in S$ implies $e^{\frac{2\pi i}{n}} z \in S$.

For convenience, we will write "*n*-sato" as an abbreviation for *n*-fold symmetry
about the origin. If A is a 2×2 matrix such that the origin is the midpoint of the
eigenvalues of A, then clearly $W(A)$ has 2-sato. A set with 3-sato is shown in Figure
1.1; the curve is the boundary of the numerical range of the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

Li and Tsing [12] describe a special class of bounded linear operators whose gen-
eralized numerical ranges have *n*-sato. More specific results concerning 3×3 matrices
whose classical numerical ranges have 3-sato were derived in [11]. The background
about these prior results is discussed in Section 2 of this paper. Generalizations to
4×4 matrices whose numerical ranges have 4-sato are derived in Section 3. Section
4 provides a collection of trace conditions sufficient for the numerical range of an
$n \times n$ matrix to have *n*-sato. A combinatorial analysis of the number of these trace

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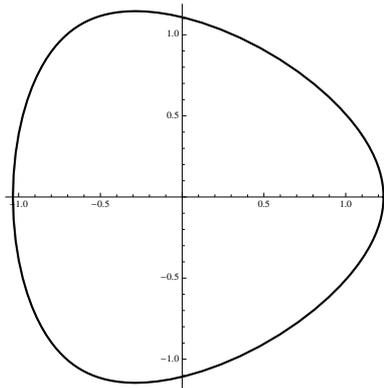
FIG. 1.1. *A numerical range with 3-sato.*

conditions for a given n is also included.

2. Background: n-fold symmetry about the origin. The numerical range has been extensively generalized. One generalization, given in [4], is particularly relevant to symmetry results.

DEFINITION 2.1. Let M and C be $n \times n$ complex matrices. The *C-numerical range* of M is the set

$$W_C(M) = \{\operatorname{tr}(CUMU^*) \,|\, U^*U = I\}.$$

The classical numerical range satisfies the identity $W(M) = W_{E_{11}}(M)$ where E_{11} is the matrix with 1 in the upper left corner and zeroes elsewhere. Consequently, if a result holds for all C-numerical ranges of a matrix A, then it holds for $W(A)$. In [12], it is shown that all of the C-numerical ranges of an $m \times m$ matrix A have n-fold symmetry about the origin if and only if A is unitarily equivalent to a block matrix of the form

$$(2.1) \qquad \begin{bmatrix} 0 & 0 & 0 & 0 & A_{1,n} \\ A_{2,1} & 0 & 0 & 0 & 0 \\ 0 & A_{3,2} & 0 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & A_{n,n-1} & 0 \end{bmatrix}.$$

Thus, for any matrix A that is unitarily equivalent to a matrix of the form (2.1), $W(A)$ (along with many other C-numerical ranges) will have n-sato. The easiest

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version of this result occurs when A is an $n \times n$ matrix and one is questioning whether $W(A)$ has n-sato. In that case, if all C-numerical ranges have n-sato, then A is unitarily equivalent to a matrix of the form above where each block $A_{i,j}$ is simply a complex scalar. In addition, if A is unitarily equivalent to an $n \times n$ matrix of the form above, then $W(A)$ has n-sato because all of the C-numerical ranges of A have n-sato.

It is natural to question whether or not $W(A)$ having n-sato is sufficient to imply that A is unitarily equivalent to a matrix of the form (2.1). That implication is proved in [11] for 3×3 matrices along with other conditions as stated below.

THEOREM 2.2 (Harris et al. [11]). *Let M be any 3×3 matrix. Assume $W(M)$ is not a disk. Then the following are equivalent:*

1. $W(M)$ *has 3-fold symmetry about the origin.*
2. *The spectrum of M has 3-fold symmetry about the origin and $\operatorname{tr}(M^2 M^*) = 0$.*
3. *There exist $p, q, r \in \mathbb{C}$ such that M is unitarily equivalent to the matrix*

$$\begin{bmatrix} 0 & 0 & p \\ q & 0 & 0 \\ 0 & r & 0 \end{bmatrix}.$$

We will show that a version of the equivalence of conditions (1) and (2) generalizes to $n = 4$ but that the $n = 4$ version of condition (1) no longer implies (3). Tests given in [3] determine exactly when a 4×4 matrix M has an elliptic numerical range; a special case of these results is that a nilpotent 4×4 matrix has a circular numerical range (which must be centered at the origin due to nilpotence) if and only if $\operatorname{tr}(M^2 M^*) = 0$ and $\operatorname{tr}(M^3 M^*) = 0$. Since circular numerical ranges centered at the origin obviously have n-fold symmetry about the origin for all n, some of the following results for $n = 4$ can be considered a generalization of the relevant results in [3].

The support function of a matrix A can be used to determine whether $W(A)$ has n-sato.

DEFINITION 2.3. The *support function* for (the numerical range of) an $n \times n$ matrix A is the function $p_A : [0, 2\pi] \to \mathbb{R}$ defined by

$$p_A(\theta) = \sup\{\Re(e^{-i\theta}\langle Av, v \rangle) \mid v \in \mathbb{C}^n, \|v\| = 1\}.$$

Since $H_\theta = \Re(e^{-i\theta} A)$ is a self-adjoint finite matrix, this supremum is attained and $p_A(\theta)$ is the maximum eigenvalue of H_θ.

The following well-known property holds for all compact convex sets. See [14] for

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many related results about convex sets.

PROPOSITION 2.4. *If A and B are $n \times n$ matrices, then the function $p_A(\theta)$ completely determines the numerical range; that is, if $p_A(\theta) = p_B(\theta)$ for all θ, then $W(A) = W(B)$.*

Note that if T is a bounded linear operator on an infinite-dimensional Hilbert space, then $W(A)$ is not necessarily closed; in this case, p_A determines only the closure of $W(A)$.

PROPOSITION 2.5. *If A is an $m \times m$ matrix and $p_A(\theta)$ is $\frac{2\pi}{n}$-periodic, then $W(A)$ has n-fold symmetry about the origin.*

Proof. Assume A is an $m \times m$ matrix such that $p_A(\theta)$ is $\frac{2\pi}{n}$-periodic. This implies that $p_A(\theta) = p_A(\theta + \frac{2\pi}{n})$ for all real θ. The definition of the support function shows that

$$p_A \left(\theta + \frac{2\pi}{n} \right) = p_{e^{-\frac{2\pi i}{n}} A}(\theta).$$

Proposition 2.4 now implies that $W(A) = W(e^{-\frac{2\pi i}{n}} A)$. Since $W(cA) = cW(A)$ for any scalar c, we conclude that $e^{-\frac{2\pi i}{n}} W(A) = W(A)$. $\square$

In the cases we will consider, the support function can be analyzed using the characteristic polynomial of H_θ. For a matrix H, the coefficients of the characteristic polynomial $q(z) = \sum_{j=0}^{n}(-1)^j a_j z^{n-j}$ can be described recursively in terms of traces of powers of H using Newton's Identities [1].

THEOREM 2.6 (Newton's Identities). *Let H be an $n \times n$ matrix with characteristic polynomial*

$$q(z) = \sum_{j=0}^{n}(-1)^j a_j z^{n-j},$$

where $a_0 = 1$. Then, for $m = 1, \ldots, n$,

$$(-1)^m a_m = -\left(\frac{1}{m}\right) \sum_{j=0}^{m-1}(-1)^j \operatorname{tr}\left(H^{m-j}\right) a_j.$$

3. Results about 4×4 matrices. Theorem 3.1 below follows from Theorem 4.2 in the next section. However, the concrete details in the $n = 4$ case motivate the general case and they are used to prove the converse in Theorem 3.3.

THEOREM 3.1. *Let A be a 4×4 matrix with complex entries whose eigenvalues have 4-fold symmetry about the origin. If $\operatorname{tr}(A^2 A^*) = 0$ and $\operatorname{tr}(A^3 A^*) = 0$, then $W(A)$ has 4-fold symmetry about the origin as well.*

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Proof. Assume that A is a 4×4 matrix whose eigenvalues have 4-sato. Then there exists $a \in \mathbb{C}$ such that these eigenvalues are $\lambda_1 = a$, $\lambda_2 = ae^{\frac{\pi i}{2}} = ai$, $\lambda_3 = ae^{\pi i} = -a$, and $\lambda_4 = ae^{\frac{3\pi i}{2}} = -ai$. Results in [3] show that if A is nilpotent with the given trace conditions, then $W(A)$ is a disk which clearly has 4-sato. Hence, we may assume $a \neq 0$. Every matrix is unitarily equivalent to an upper triangular matrix, and thus, the unitary invariance of the numerical range allows us to assume that

$$A = \begin{bmatrix} a & * & * & * \\ 0 & ai & * & * \\ 0 & 0 & -a & * \\ 0 & 0 & 0 & -ai \end{bmatrix},$$

where $*$ denotes an unspecified entry. Note then that

$$\mathrm{tr}(A) = \mathrm{tr}(A^2) = \mathrm{tr}(A^3) = \mathrm{tr}(A^*) = \mathrm{tr}\left((A^*)^2\right) = \mathrm{tr}\left((A^*)^3\right) = 0.$$

Recall that the support function of $W(A)$ is defined by

$$\begin{aligned} p_A(\theta) &= \sup\{\Re(e^{-i\theta}\langle Av, v\rangle) \mid v \in \mathbb{C}^4, \|v\| = 1\} \\ &= \max\{\langle H_\theta v, v\rangle \mid v \in \mathbb{C}^4, \|v\| = 1\} \\ &= \max \sigma(H_\theta), \end{aligned}$$

where

$$H_\theta = \Re\left(e^{-i\theta}A\right) = \frac{e^{-i\theta}A + e^{i\theta}A^*}{2}.$$

Our goal is to show that $p_A(\theta)$ is $\frac{2\pi}{4}$-periodic by showing that the maximum eigenvalue of H_θ is $\frac{2\pi}{4}$-periodic. Let $q_\theta(z) = z^4 - a_1 z^3 + a_2 z^2 - a_3 z + a_4$ be the characteristic polynomial of H_θ. Applying Theorem 2.6 (Newton's Identities) gives the following expressions for the coefficients of $q_\theta(z)$:

$$\begin{aligned} a_1 &= \mathrm{tr}\, H_\theta, \\ a_2 &= \frac{1}{2}\left[(\mathrm{tr}\, H_\theta)^2 - \mathrm{tr}\, H_\theta^2\right], \\ a_3 &= \frac{1}{3}\mathrm{tr}\, H_\theta^3 - \frac{1}{2}\mathrm{tr}\, H_\theta \,\mathrm{tr}\, H_\theta^2 + \frac{1}{6}(\mathrm{tr}\, H_\theta)^3, \text{and} \\ a_4 &= -\frac{1}{4}\mathrm{tr}\, H_\theta^4 + \frac{1}{3}\mathrm{tr}\, H_\theta^3 \,\mathrm{tr}\, H_\theta - \frac{1}{4}(\mathrm{tr}\, H_\theta)^2 \,\mathrm{tr}\, H_\theta^2 + \frac{1}{8}(\mathrm{tr}\, H_\theta^2)^2 + \frac{1}{24}(\mathrm{tr}\, H_\theta)^4. \end{aligned}$$

The definition of H_θ then gives

$$a_1 = \mathrm{tr}\left(\frac{e^{-i\theta}A + e^{i\theta}A^*}{2}\right) = \frac{e^{-i\theta}\mathrm{tr}\, A + e^{i\theta}\mathrm{tr}\, A^*}{2} = 0.$$

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Using the fact that $\operatorname{tr} H_\theta = 0$ and the property that $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ for any $n \times n$ matrices X and Y, we have

$$
\begin{aligned}
a_2 &= \frac{1}{2} \left[(\operatorname{tr} H_\theta)^2 - \operatorname{tr} H_\theta^2 \right] \\
&= -\frac{1}{2} \operatorname{tr} H_\theta^2 \\
&= -\frac{1}{2} \operatorname{tr} \left(\frac{e^{-2\theta i} A^2 + e^{2\theta i} (A^*)^2 + AA^* + A^*A}{4} \right) \\
&= -\frac{1}{2} \left(\frac{e^{-2\theta i} \operatorname{tr} A^2 + e^{2\theta i} \operatorname{tr}(A^*)^2 + \operatorname{tr}(AA^*) + \operatorname{tr}(A^*A)}{4} \right) \\
&= -\frac{1}{4} \operatorname{tr}(AA^*).
\end{aligned}
$$

Next, our hypothesis that $\operatorname{tr}(A^2 A^*) = 0$ and the fact that

$$
\operatorname{tr}\left((A^*)^2 A \right) = \overline{\operatorname{tr}(A^* A^2)} = \overline{\operatorname{tr}(A^2 A^*)} = 0
$$

yield

$$
\begin{aligned}
a_3 &= \frac{1}{3} \operatorname{tr} H_\theta^3 - \frac{1}{2} \operatorname{tr} H_\theta \operatorname{tr} H_\theta^2 + \frac{1}{6} (\operatorname{tr} H_\theta)^3 \\
&= \frac{1}{3} \operatorname{tr} H_\theta^3 \\
&= \frac{1}{3} \operatorname{tr} \left(\frac{e^{-3i\theta} A^3 + 3e^{-i\theta}(A^2 A^*) + 3e^{i\theta}(A^*)^2 A + e^{3i\theta}(A^*)^3}{8} \right) \\
&= \frac{1}{3} \left(\frac{e^{-3i\theta} \operatorname{tr} A^3 + 3e^{-i\theta} \operatorname{tr}(A^2 A^*) + 3e^{i\theta} \operatorname{tr}\left((A^*)^2 A\right) + e^{3i\theta} \operatorname{tr}(A^*)^3}{8} \right) \\
&= 0.
\end{aligned}
$$

Finally, using our hypothesis that $\operatorname{tr}(A^3 A^*) = 0$, and thus that $\operatorname{tr}((A^*)^3 A) = 0$,

$$
\begin{aligned}
a_4 &= -\frac{1}{4} \operatorname{tr} H_\theta^4 + \frac{1}{3} \operatorname{tr} H_\theta^3 \operatorname{tr} H_\theta - \frac{1}{4} (\operatorname{tr} H_\theta)^2 \operatorname{tr} H_\theta^2 + \frac{1}{8} (\operatorname{tr} H_\theta^2)^2 + \frac{1}{24} (\operatorname{tr} H_\theta)^4 \\
&= -\frac{1}{4} \operatorname{tr} H_\theta^4 + \frac{1}{8} (\operatorname{tr} H_\theta^2)^2 \\
&= -\frac{1}{4} \operatorname{tr} \left(\frac{e^{4i\theta} A^4 + 4(A^*)^2 A^2 + 4e^{-2i\theta} A^* A^3 + 4e^{2i\theta} A(A^*)^3 + 2(A^*A)^2 + e^{4i\theta}(A^*)^4}{16} \right) \\
&\quad + \frac{1}{32} (\operatorname{tr}(AA^*))^2 \\
&= -\frac{1}{4} \left(\frac{e^{-4i\theta} \operatorname{tr} A^4 + 4 \operatorname{tr}\left[(A^*)^2 A^2\right] + 2 \operatorname{tr}\left[(A^*A)^2\right] + e^{4i\theta} \operatorname{tr}(A^*)^4}{16} \right) + \frac{1}{32} (\operatorname{tr}(AA^*))^2.
\end{aligned}
$$

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Putting the above together gives

$$q_\theta(z) = z^4 - \frac{1}{4}\operatorname{tr}(AA^*)z^2 - \frac{1}{64}\left(e^{-4i\theta}\operatorname{tr}A^4 + 4\operatorname{tr}\left[(A^*)^2 A^2\right] + 2\operatorname{tr}\left[(A^*A)^2\right]\right.$$
$$\left. + e^{4i\theta}\operatorname{tr}(A^*)^4\right) + \frac{1}{32}(\operatorname{tr}(AA^*))^2.$$

Notice that $q_\theta(z)$ is $\frac{2\pi}{4}$-periodic in θ. Thus, the maximum root of q_θ, i.e., the maximum eigenvalue of H_θ, is $\frac{2\pi}{4}$-periodic. That is, $p_A(\theta)$ is $\frac{2\pi}{4}$-periodic, and hence, $W(A)$ has 4-sato. ☐

The counterexample below shows that the "(1) implies (3)" portion of Theorem 2.2 does not generalize to 4×4 matrices. That is, it is possible for a 4×4 matrix to have a numerical range with 4-sato without being unitarily equivalent to a matrix with the special form

(3.1)
$$V = \begin{bmatrix} 0 & 0 & 0 & p \\ q & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & s & 0 \end{bmatrix}.$$

EXAMPLE 3.2. Let

$$A = \begin{bmatrix} 1 & 1 & \frac{1}{3}\left(-18 - 5\sqrt{14}\right) & 1 \\ 0 & i & 2 & \frac{2}{3}\left(9 + 2\sqrt{14}\right) \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$

The eigenvalues of A clearly have 4-sato. It is straightforward to check that

$$\operatorname{tr}(A^2 A^*) = \operatorname{tr}(A^3 A^*) = 0.$$

Therefore, $W(A)$ has 4-sato by Theorem 3.1. We claim that A is not unitarily equivalent to any matrix of the form (3.1). If A and V were unitarily equivalent, then $A^3(A^*)^2$ would be unitarily equivalent to $V^3(V^*)^2$. The trace of a matrix is invariant under unitary equivalence; however, a direct computation shows that $\operatorname{tr}(A^3(A^*)^2) \neq 0$ while $\operatorname{tr}(V^3(V^*)^2) = 0$. This proves the claim.

The counterexample of Example 3.2 shows that not all of the natural analogs to the three conditions of Theorem 2.2 are equivalent for 4×4 matrices. Nonetheless, it does hold that the condition of $W(A)$ having 4-sato implies certain traces of words in A and A^* must be zero.

THEOREM 3.3. *Assume A is a 4×4 matrix with complex entries whose eigenvalues have 4-fold symmetry about the origin. If $W(A)$ has 4-fold symmetry about the origin, then* $\operatorname{tr}(A^2 A^*) = 0$ *and* $\operatorname{tr}(A^3 A^*) = 0$.

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Proof. Let A be a 4×4 matrix with complex entries whose eigenvalues have 4-sato. Assume $W(A)$ has 4-sato. Let $p_A(\theta)$ be the support function of A and $q_\theta(z)$ be the characteristic polynomial of H_θ. Recall that $p_A(\theta)$ is the maximum root of $q_\theta(z)$. In addition, the definition of $p_A(\theta)$ immediately implies that the minimum root of $q_\theta(z)$ is $-p_A(\theta + \pi)$. Therefore,

$$(3.2) \qquad q_\theta(z) = (z - p_A(\theta))(z + p_A(\theta + \pi))(z^2 + b_1(\theta)z + b_2(\theta))$$

for some $b_1(\theta), b_2(\theta) \in \mathbb{C}$. Since $W(A)$ has 4-fold symmetry about the origin, $p_A(\theta)$ is $\frac{\pi}{2}$-periodic, giving $p_A(\theta + \pi) = p_A(\theta)$. Substituting this into (3.2) yields

$$
\begin{aligned}
q_\theta(z) &= (z - p_A(\theta))(z + p_A(\theta))(z^2 + b_1(\theta)z + b_2(\theta)) \\
&= z^4 + b_1(\theta)z^3 + (b_2(\theta) - p_A^2(\theta))z^2 - p_A^2(\theta)b_1(\theta)z - p_A^2(\theta)b_2(\theta).
\end{aligned}
$$

Recall from the proof of Theorem 3.1 that

$$
\begin{aligned}
q_\theta(z) = {}& z^4 - \frac{1}{4}\operatorname{tr}(AA^*)z^2 + \frac{1}{8}\left[e^{-i\theta}\operatorname{tr}(A^2 A^*) + e^{i\theta}\operatorname{tr}((A^*)^2 A) \right] z \\
& - \frac{1}{4}\operatorname{tr}\left(\frac{e^{4i\theta}A^4 + 4(A^*)^2 A^2 + 4e^{-2i\theta}A^* A^3 + 4e^{2i\theta}A(A^*)^3 + 2(A^*A)^2 + e^{4i\theta}(A^*)^4}{16} \right) \\
& + \frac{1}{32}(\operatorname{tr}(AA^*))^2.
\end{aligned}
$$

Matching coefficients immediately yields $b_1(\theta) = 0$. Consequently $e^{-i\theta}\operatorname{tr}(A^2 A^*) + e^{i\theta}\operatorname{tr}((A^*)^2 A) = 0$ for all θ. As this is a trigonometric polynomial that is identically 0, its coefficients $\operatorname{tr}(A^2 A^*)$ and $\operatorname{tr}((A^*)^2 A)$ must be 0.

Continuing to match coefficients gives $b_2(\theta) - p_A^2(\theta) = -\frac{1}{4}\operatorname{tr}(AA^*)$. Since $p_A(\theta)$ is $\frac{\pi}{2}$-periodic, so is $p_A^2(\theta)$. Thus, as the right hand side of the above equation is independent of θ, $b_2(\theta)$ must be $\frac{\pi}{2}$-periodic as well. Hence,

$$
\begin{aligned}
-p_A^2(\theta)b_2(\theta) = {}& -\frac{1}{4}\operatorname{tr}\left(\frac{e^{4i\theta}A^4 + 4(A^*)^2 A^2 + 4e^{-2i\theta}A^* A^3 + 4e^{2i\theta}A(A^*)^3 + 2(A^*A)^2 + e^{4i\theta}(A^*)^4}{16} \right) \\
& + \frac{1}{32}(\operatorname{tr}(AA^*))^2
\end{aligned}
$$

is also $\frac{\pi}{2}$-periodic. Consequently, $e^{-2i\theta}\operatorname{tr}(A^* A^3) + e^{2i\theta}\operatorname{tr}(A(A^*)^3)$ is $\frac{\pi}{2}$-periodic. Therefore,

$$
\begin{aligned}
e^{-2i\theta}\operatorname{tr}(A^* A^3) + e^{2i\theta}\operatorname{tr}(A(A^*)^3) &= e^{-2i\theta - i\pi}\operatorname{tr}(A^* A^3) + e^{2i\theta + i\pi}\operatorname{tr}(A(A^*)^3) \\
&= -\left(e^{-2i\theta}\operatorname{tr}(A^* A^3) + e^{2i\theta}\operatorname{tr}(A(A^*)^3) \right).
\end{aligned}
$$

Hence, the trigonometric polynomial $e^{-2i\theta}\operatorname{tr}(A^* A^3) + e^{2i\theta}\operatorname{tr}(A(A^*)^3) = 0$ for all θ and we conclude that $\operatorname{tr}(A^* A^3) = \operatorname{tr}(A(A^*)^3) = 0$. ∎

4. Sufficiency of trace conditions for $n \times n$ matrices. To state the general theorem describing how the condition that the trace of each of a certain set of words

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in A and A^* is zero implies that $W(A)$ has n-sato, we define a special collection of words as follows.

DEFINITION 4.1. Given an $n \times n$ matrix A, let Ω_A be the collection of words in A and A^* of length at most n such that

    1. the number of occurrences of both A and A^* is nonzero, and

    2. the number of occurrences of A is different from the number of occurrences of A^*.

For example, if A is a 5×5 matrix, then $A^5 \notin \Omega_A$ and $A^* A A^* A \notin \Omega_A$ but $A^* A^* A^* A A \in \Omega_A$.

THEOREM 4.2. *Let A be an $n \times n$ matrix whose eigenvalues have n-fold symmetry about the origin. If $tr(X) = 0$ for all $X \in \Omega_A$, then $W(A)$ has n-fold symmetry about the origin as well.*

Proof. Note that $W(A)$ is determined entirely by the support function $p_A(\theta)$, which is the maximum eigenvalue of H_θ. Thus, in order to prove $W(A)$ has n-sato, it suffices to show $p_A(\theta)$ is $\frac{2\pi}{n}$-periodic in θ. Since the value of $p_A(\theta)$ is determined (as are *all* of the eigenvalues of H_θ) by the characteristic polynomial of H_θ, namely $q_\theta(z)$, it suffices to show that $q_\theta(z)$ has that periodicity, for which it clearly suffices to show that this is true of each coefficient of $q_\theta(z)$ individually.

By Newton's Identities, each coefficient of $q_\theta(z)$ can be expressed in terms of products of traces of powers of H_θ. For example, when $n = 5$, the constant term in q_θ is $-a_5$ where

$$a_5 = \frac{1}{5} \operatorname{tr}(H_\theta^5) - \frac{1}{4} \operatorname{tr} H_\theta^4 \operatorname{tr} H_\theta + \frac{1}{6} \operatorname{tr} H_\theta^3 (\operatorname{tr} H_\theta)^2 - \frac{1}{6} \operatorname{tr} H_\theta^3 (\operatorname{tr} H_\theta^2) + \frac{1}{12} (\operatorname{tr} H_\theta)^3 \operatorname{tr} H_\theta^2$$

$$+ \frac{1}{8} (\operatorname{tr} H_\theta^2)^2 \operatorname{tr} H_\theta + \frac{1}{120} (\operatorname{tr} H_\theta)^5.$$

Note that for $1 \le k \le n$,

$$(4.1) \qquad \operatorname{tr}(H_\theta^k) = \operatorname{tr}\left(\left(\frac{e^{-i\theta} A + e^{i\theta} A^*}{2} \right)^k \right) = \frac{1}{2^k} \operatorname{tr}(e^{-i\theta} A + e^{i\theta} A^*)^k.$$

When the rightmost term in the equation above is expanded, the result is a sum of terms where each is the product of three factors, namely $1/2^k$, the trace of some product of k factors each of which is A or A^*, and $e^{-id\theta}$, where d is the number of times A occurs in that product minus the number of times A^* occurs. For $1 \le d \le n-1$, the hypothesis that $\operatorname{tr}(X) = 0$ for all $X \in \Omega_A$ implies that the trace of any term in which both A and A^* appear is zero. In addition, those terms containing only $\operatorname{tr}(A^d)$ will be accompanied by a factor of $e^{-id\theta}$. When $1 \le d \le n-1$, it follows from our assumptions about the symmetry of the eigenvalues that $\operatorname{tr}(A^d) = 0$. When $d = n$, we have $e^{-id\theta}$

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is $\frac{2\pi}{n}$-periodic in θ. Hence, any term containing $\operatorname{tr}(A^d)$ has the required periodicity. Similarly, any term containing only $\operatorname{tr}\big((A^*)^d\big)$ has the required periodicity. Hence, all of the terms in the expansion of (4.1) either are zero or appear with a coefficient of $e^{in\theta}$ or $e^{i0\theta} = 1$. It follows that $\operatorname{tr}(H_\theta^k)$ is $\frac{2\pi}{n}$-periodic. Consequently, every coefficient of q_θ is $\frac{2\pi}{n}$-periodic. This implies the maximum root of q_θ is $\frac{2\pi}{n}$-periodic, which shows that $W(A)$ has n-sato. $\square$

Note that $\operatorname{tr}(A^k) = 0$ for $k = 1, \dots, n-1$ implies the spectrum of A has n-sato. Hence, the hypothesis that $\sigma(A)$ has n-sato could be dropped if the definition of Ω_A is modified to include A^k for $1 \le k \le n-1$.

It is natural to ask if the sufficient conditions of Theorem 4.2 are necessary. In fact, Theorem 3.3 shows that this question has an affirmative answer in the $n = 4$ case. To see this, note that requiring every word in Ω_A to have a zero trace is equivalent to requiring this only of the words in a particular subset of Ω_A. This is due to the cyclic invariance of the trace, together with the fact that the trace of any matrix is equal to the complex conjugate of the trace of its adjoint. For example, in the $n = 5$ case, we have

$$\operatorname{tr}(AAA^*AA^*) = \operatorname{tr}(A^*AAA^*A) = \operatorname{tr}(AA^*AAA^*)$$
$$= \operatorname{tr}(A^*AA^*AA) = \operatorname{tr}(AA^*AA^*A).$$

Hence, we may group the words of Ω_A into equivalence classes, where two words are equivalent if and only if one of them is obtained from the other by first applying some circular permutation followed by possibly taking the adjoint. Each such equivalence class then has the property that every word it contains has a zero trace when any one of them does. In the $n = 4$ case, there are only two such equivalence classes, one containing the word A^2A^* and the other containing A^3A^*. Hence, Theorem 3.3 may be viewed as saying that the sufficient condition of Theorem 4.2 is in fact necessary in the $n = 4$ case.

To explore the question of necessity for larger n, it may be useful to let $\overline{\Omega}_A$ be a "reduced" set of words from Ω_A, consisting of a single word from each equivalence class of Ω_A as described above. It is then possible to replace Ω_A with $\overline{\Omega}_A$ in the hypothesis of Theorem 4.2. To understand the words in $\overline{\Omega}_A$ more concretely, it may be valuable to find how many such words there are, as well as how to describe them combinatorially.

To this end, assume A is $n \times n$ and note that the words in Ω_A of length k are in correspondence with the different ways of arranging k factors, each either A or A^*, around a circle. Hence, the number of non-equivalent words may be derived beginning with the number of necklaces with k-beads, each red or black. This number is given

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(see, e.g., [15, Theorem 10.5]) as

$$\frac{1}{k} \sum_{d|k} \varphi\left(\frac{k}{d}\right) 2^d,$$

where φ is Euler's totient function. However, this number would count some words in A and A^* that are excluded by the definition of Ω_A. In particular, words should not be counted if they fail to include both A and A^*, nor if they receive an equal contribution from each of A and A^*. In the former case, there are exactly two such words, while the latter case occurs only when k is even, and accounts for a number of words equal to

$$\frac{1}{k} \sum_{2d|k} \varphi\left(\frac{k}{2d}\right) \binom{2d}{d}.$$

Hence, the number of equivalence classes of words in Ω_A is equal to

$$\frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} \left(\sum_{d|k} \varphi\left(\frac{k}{d}\right) 2^d - \sum_{2d|k} \varphi\left(\frac{k}{2d}\right) \binom{2d}{d} - 2 \right),$$

where the factor of $1/2$ comes from the equivalence of each word with its adjoint, and the second of the inner summations is vacuous when k is odd. Using this formula, we see how the size of $\overline{\Omega}_n$ grows with n.

TABLE 4.1

Assuming A is a $n \times n$ matrix, the table below displays the growth in the number of words in A and A^* contained in $\overline{\Omega}_A$, a proper subset of Ω_A. The condition that all of these words have zero trace is sufficient to guarantee that the trace of every word in Ω_A is zero, and hence that the hypothesis of Theorem 4.2 is satisfied.

n	3	4	5	6	7	8	9	10	11	12	13	14
Number of words in $\overline{\Omega}_A$	1	2	5	9	18	30	59	99	192	327	642	1109

As we have shown for an $n \times n$ matrix A, the traces of all words in $\overline{\Omega}_A$ being zero (along with other natural hypotheses) is sufficient for $W(A)$ to have n-sato. It is clear from the growth rate of the cardinality of $\overline{\Omega}_A$ that computing all these traces quickly becomes computationally costly. Determining whether these conditions are necessary for $n > 4$ or if some of them can be collapsed remains an open question.

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