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REFINED INERTIAS OF TREE SIGN PATTERNS*

COLIN GARNETT†, D.D. OLESKY‡, AND P. VAN DEN DRIESSCHE†

Abstract. The refined inertia \((n_+, n_-, n_z, 2n_p)\) of a real matrix is the ordered 4-tuple that subdivides the number \(n_0\) of eigenvalues with zero real part in the inertia \((n_+, n_-, n_z)\) into those that are exactly zero \((n_z)\) and those that are nonzero \((2n_p)\). For \(n \geq 2\), the set of refined inertias \(H_n = \{(0, n, 0, 0), (0, n - 2, 0, 2), (2, n - 2, 0, 0)\}\) is important for the onset of Hopf bifurcation in dynamical systems. Tree sign patterns of order \(n\) that require or allow the refined inertias \(H_n\) are considered. For \(n = 4\), necessary and sufficient conditions are proved for a tree sign pattern \((necessarily a path or a star)\) to require \(H_4\). For \(n \geq 3\), a family of \(n \times n\) star sign patterns that allows \(H_n\) is given, and it is proved that if a star sign pattern requires \(H_n\), then it must have exactly one zero diagonal entry associated with a leaf in its digraph.

Key words. Eigenvalues, Tree sign pattern, Refined inertia, Hopf bifurcation.

AMS subject classifications. 15B35, 15A18, 05C50.

1. Introduction. An \(n \times n\) sign pattern is an \(n \times n\) matrix with entries from \(\{+, -, 0\}\). The sign, \(\text{sgn}(a)\), of a real number \(a\) is defined by \(\text{sgn}(a) = +, -, 0\) when \(a > 0, a < 0,\) or \(a = 0\), respectively. The sign pattern of a real matrix \(A = [a_{ij}]\) is the sign pattern \(A = \text{sgn}(A) = [\text{sgn}(a_{ij})]\); matrix \(A\) is called a realization of \(A\). The sign pattern class \(Q(A)\) of the sign pattern \(A\) is the set \(Q(A) = \{A|\text{sgn}(A) = A\}\). The digraph \(D(A)\) of a sign pattern \(A = [a_{ij}]\) has \(n\) vertices, an arc from \(i\) to \(j\) if \(a_{ij} \neq 0\) and a loop at vertex \(i\) if \(a_{ii} \neq 0\). The signed digraph of sign pattern \(A\) is the digraph of \(A\) with \(a_{ij}\) on the arc from \(i\) to \(j\) if \(a_{ij} \neq 0\) and \(a_{ii}\) on the loop at vertex \(i\) if \(a_{ii} \neq 0\).

As defined in [7], the refined inertia \(ri(A)\) of a real \(n \times n\) matrix \(A\) is the ordered 4-tuple \((n_+, n_-, n_z, 2n_p)\) such that \(n_+\) (resp., \(n_-\)) is the number of eigenvalues (including multiplicities) of \(A\) with positive (resp., negative) real part, and \(n_z\) (resp., \(2n_p\)) is the number of zero eigenvalues (resp., nonzero pure imaginary eigenvalues) of \(A\). Here \(n_+ + n_- + n_z + 2n_p = n\). The inertia of \(A\) is \((n_+, n_-, n_z + 2n_p)\), thus the refined inertia subdivides those eigenvalues with zero real part and distinguishes between those that are exactly zero and those that are nonzero.

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A sign pattern $\mathcal{A}$ is sign nonsingular if $n_z = 0$ (i.e., $\det(A) \neq 0$) for all $A \in \mathbb{Q}(\mathcal{A})$; see [3]. An $n \times n$ sign pattern $\mathcal{A}$ is potentially stable if there is a matrix $A \in \mathbb{Q}(\mathcal{A})$ such that $n_- = n$. An $n \times n$ sign pattern $\mathcal{A}$ is sign stable if $n_- = n$ for all $A \in \mathbb{Q}(\mathcal{A})$. Each of these properties is invariant under sign pattern equivalence (i.e., transposition, permutation similarity and signature similarity). Multiplying a matrix $A$ by a positive scalar, one nonzero diagonal entry can be set to have magnitude 1 when refined inertia is considered. Furthermore, for an $n \times n$ irreducible matrix $A \in \mathbb{Q}(\mathcal{A})$, without loss of generality $n - 1$ nonzero off-diagonal entries corresponding to a spanning tree of $D(A)$ can be set to have magnitude 1 by a positive diagonal similarity (see, e.g., [2] Lemma 2.3).

The following observation, which is Lemma 3.4 (iii) in [1], is used to prove some results in Section 2.

**Observation 1.1.** [1] Suppose $\mathcal{A}$ is a sign pattern that has a realization $A$ with $ri(A) = (n_+, n_-, n_z, 2n_p)$ that allows a full rank Jacobian matrix. If $n_p \geq 1$, then there exist $A_1, A_2 \in \mathbb{Q}(\mathcal{A})$ such that the refined inerties of $A_1$ and $A_2$ are $(2 + n_+, n_-, n_z, 2(n_p - 1))$ and $(n_+, 2 + n_-, n_z, 2(n_p - 1))$, respectively.

Hopf bifurcation is of interest in the study of dynamical systems. To connect Hopf bifurcation in a dynamical system to refined inertia, for $n \geq 2$ let $\mathbb{H}_n = \{(0, n, 0, 0), (0, n - 2, 0, 2), (2, n - 2, 0, 0)\}$, as defined in [1]. A sign pattern $\mathcal{A}$ requires refined inertia $\mathbb{H}_n$ if $\mathbb{H}_n = \{ri(A) \mid A \in \mathbb{Q}(\mathcal{A})\}$. A sign pattern $\mathcal{A}$ allows refined inertia $\mathbb{H}_n$ if $\mathbb{H}_n \subseteq \{ri(A) \mid A \in \mathbb{Q}(\mathcal{A})\}$. Consider an $n$-dimensional dynamical system linearized about an equilibrium with Jacobian matrix having sign pattern $\mathcal{A}$. Let $a$ be a parameter of the system and $A(a_i)$ be the Jacobian matrix with $a = a_i$. If $a_1 < a_2 < a_3$ or $a_1 > a_2 > a_3$ and $ri(A(a_1)) = (0, n, 0, 0)$, $ri(A(a_2)) = (0, n - 2, 0, 2)$, and $ri(A(a_3)) = (2, n - 2, 0, 0)$, then $\mathcal{A}$ allows $\mathbb{H}_n$ and Hopf bifurcation may occur giving rise to periodic solutions. The same idea applies for dynamical systems with magnitude restrictions on some entries of $\mathcal{A}$; for examples from different applications, see [1].

Clearly, if $\mathcal{A}$ requires $\mathbb{H}_n$ then $\mathcal{A}$ is potentially stable and sign nonsingular with $\text{sgn}(\det(A)) = \text{sgn}((-1)^n)$ for all $A \in \mathbb{Q}(\mathcal{A})$. Furthermore, if $\mathcal{A}$ requires $\mathbb{H}_n$ then $\mathcal{A}$ is not sign stable and $-\mathcal{A}$ is not potentially stable.

Some results for the requires $\mathbb{H}_n$ problem can be found in [1]. It is shown in [1] Theorem 2.1] that a $3 \times 3$ sign nonsingular sign pattern allows $\mathbb{H}_3$ if and only if it requires $\mathbb{H}_3$. Theorem 2.3 in [1] states that if a $4 \times 4$ sign nonsingular sign pattern requires a negative trace and allows $\mathbb{H}_4$, then it requires $\mathbb{H}_4$. It is conjectured in [1] that no $n \times n$ sign pattern requires $\mathbb{H}_n$ for $n \geq 8$ and an example of a $7 \times 7$ sign pattern that requires $\mathbb{H}_7$ is given.
In this paper, we focus on (irreducible) tree sign patterns, i.e., sign patterns $A$ for which $D(A)$ is a doubly directed tree. In Section 2, we characterize the $4 \times 4$ tree sign patterns that require $H_4$. Using one of these sign patterns, in Section 3 we describe a set of $n \times n$ star sign patterns that allow $H_n$ for $n \geq 3$. With results from [5], we prove that an $n \times n$ star sign pattern that requires $H_n$ must have exactly one zero diagonal entry associated with a leaf in its digraph. In Section 4, we extend a result from [1] on reducible sign patterns that require $H_n$ and give a new example of a surprising reducible pattern that allows $H_9$. Some concluding remarks are given in Section 5.

2. Tree sign patterns. If $D(A)$ is a doubly directed path, then $A$ is called a path sign pattern. If $D(A)$ is a doubly directed star, then $A$ is called a star sign pattern. In any realization $A = [a_{ij}]$ of a path sign pattern, without loss of generality assume that adjacent vertices on the path are numbered $1, 2, \ldots, n$, and entries $a_{i,i+1} = 1$ for $i = 1, \ldots, n - 1$. In any realization $A = [a_{ij}]$ of a star sign pattern $A$, without loss of generality take vertex 1 in $D(A)$ as the center vertex and the $n - 1$ entries $a_{1,i} = 1$ for $i = 2, \ldots, n$.

Results from [1] can be used to show the following characterization.

**Theorem 2.1.** [1] A $3 \times 3$ tree sign pattern requires $H_3$ if and only if it is potentially stable and sign nonsingular, but not sign stable.

If the digraph of a $4 \times 4$ sign pattern $D(A)$ is a doubly directed tree, then $A$ is either a path sign pattern or a star sign pattern.

**Observation 2.2.** If $A$ is a $4 \times 4$ sign pattern that requires a positive determinant, then $A \in Q(A)$ can have one of only six possible refined inertias, namely the three refined inertias in $H_4$, $(4, 0, 0, 0)$, $(2, 0, 0, 2)$ or $(0, 0, 0, 4)$.

The potentially stable $4 \times 4$ path and star sign patterns are listed in [6] and [8]. Beginning with these sign patterns, we show that up to equivalence there are exactly 5 path sign patterns and 5 star sign patterns that require $H_4$.

2.1. Path sign patterns of order 4. Up to equivalence, the following are the only $4 \times 4$ path sign patterns that are potentially stable, sign nonsingular, not sign stable, and for which its negative is not potentially stable (see [6] and [8]):

$$P_1 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & - \\ 0 & 0 & - & - \end{bmatrix}.$$
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\[
P_1 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \quad P_5 = \begin{bmatrix} - & + & 0 & 0 \\ + & + & 0 & + \\ 0 & - & - & + \\ 0 & 0 & + & 0 \end{bmatrix}.
\]

Since \( P_1, P_2, P_3, \) and \( P_4 \) have only nonpositive entries on the diagonal, Theorem 2.3 in [1] applies; i.e., if any one of them allows \( H_4 \), then it also requires \( H_4 \). The following result is immediate from this theorem and the table of realizations below.

**Theorem 2.3.** The path sign patterns \( P_1, P_2, P_3, \) and \( P_4 \) require \( H_4 \).

<table>
<thead>
<tr>
<th>Realization of Sign Pattern</th>
<th>ri ((0, 4, 0, 0))</th>
<th>ri ((0, 2, 0, 2))</th>
<th>ri ((2, 2, 0, 0))</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & a & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \in Q(P_1) \] | \( a = 0.76 \) | some \( a \in (0.76, 0.77) \) | \( a = 0.77 \) |
| \[
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \in Q(P_2) \] | \( a = 3.24 \) | some \( a \in (3.23, 3.24) \) | \( a = 3.23 \) |
| \[
\begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & a & -1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \in Q(P_3) \] | \( a = 1.65 \) | some \( a \in (1.65, 1.66) \) | \( a = 1.66 \) |
| \[
\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & a & -1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \in Q(P_4) \] | \( a = 0.5 \) | \( a = 1 \) | \( a = 2 \) |

The next result shows that the above is also true for \( P_5 \), which is a superpattern of a sign pattern equivalent to \( P_2 \).

**Theorem 2.4.** The path sign pattern \( P_5 \) requires \( H_4 \).

**Proof.** To consider refined inertia, any realization of \( P_5 \) can be normalized to

\[
M = \begin{bmatrix} -a & 1 & 0 & 0 \\ d & b & 1 & 0 \\ 0 & -e & -c & 1 \\ 0 & 0 & f & 0 \end{bmatrix} \in Q(P_5),
\]

where \( a, b, c, d, e, f \in \mathbb{R}^+ \). The characteristic polynomial of \( M \) is \( x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 \) with

\[
\begin{align*}
p_1 &= a + c - b \\
p_2 &= ac + e - ab - bc - d - f \\
p_3 &= ae + bf - abc - cd - af \\
p_4 &= abf + fd.
\end{align*}
\]
Define the map \( \chi : \mathbb{R}^6 \to \mathbb{R}^4 \) by \( \chi(a, b, c, d, e, f) = (p_1, p_2, p_3, p_4) \). The Jacobian matrix of the map \( \chi \), i.e., \( \left[ \frac{\partial (p_1, \ldots, p_4)}{\partial (a, \ldots, f)} \right] \) is

\[
Jac_{\chi} = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
 c - b & -a - c & a - b & -1 & -1 \\
 e - f - bc & f - ac & -ab - d & -c & a - b \\
 bf & af & 0 & f & d + ab
\end{bmatrix}.
\]

The 4 \( \times \) 4 submatrix consisting of columns 1, 2, 4 and 5 has determinant \( fe \), and hence, \( Jac_{\chi} \) has rank 4. Since \( -P_5 \) is not potentially stable [6, 8], \( P_5 \) does not allow refined inertia \( (4, 0, 0, 0) \), and thus by Observation 1.1 it also does not allow \((2, 0, 0, 2)\) or \((0, 0, 0, 4)\). Fix \( a = 1, b = 0.5, c = 2, d = 0.05, \) and \( f = 0.1 \). If \( e = 1.24 \), then \( \text{ri}(M) = (0, 4, 0, 0) \); if \( e = 1.23 \), then \( \text{ri}(M) = (2, 2, 0, 0) \). Therefore, by continuity, there is a value of \( e \) such that \( 1.23 < e < 1.24 \) with \( \text{ri}(M) = (0, 2, 0, 2) \). Hence, \( P_5 \) allows \( H_4 \) and since it does not allow \( (4, 0, 0, 0), (2, 0, 0, 2), \) or \( (0, 0, 0, 4) \), by Observation 2.2 it requires \( H_4 \).

2.2. Star sign patterns of order 4. Up to equivalence, the following are the only 4 \( \times \) 4 star sign patterns that are potentially stable, sign nonsingular, not sign stable, and for which its negative is not potentially stable (see [6]):

\[
S_1 = \begin{bmatrix}
- & + & + & + \\
- & 0 & 0 & 0 \\
+ & 0 & - & 0 \\
- & 0 & 0 & 0
\end{bmatrix}, \quad S_2 = \begin{bmatrix}
- & + & + & + \\
+ & 0 & 0 & 0 \\
+ & 0 & - & 0 \\
- & 0 & 0 & 0
\end{bmatrix}, \quad S_3 = \begin{bmatrix}
0 & + & + & + \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
S_4 = \begin{bmatrix}
- & + & + & + \\
+ & 0 & 0 & 0 \\
- & 0 & + & 0 \\
+ & 0 & 0 & 0
\end{bmatrix}, \quad S_5 = \begin{bmatrix}
+ & + & + & + \\
+ & 0 & 0 & 0 \\
+ & 0 & 0 & 0
\end{bmatrix}.
\]

Since each of the patterns \( S_1, S_2, \) and \( S_3 \) requires negative trace, the following result is immediate from [11, Theorem 2.3] and the table of realizations below.

**Theorem 2.5.** The star sign patterns \( S_1, S_2, \) and \( S_3 \) require \( H_4 \).
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<table>
<thead>
<tr>
<th>Realization of Sign Pattern</th>
<th>ri (0, 4, 0, 0)</th>
<th>ri (0, 2, 0, 2)</th>
<th>ri (2, 2, 0, 0)</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
-0.01 & 1 & 1 & 1 \\
-1 & -a & 0 & 0 \\
1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
\end{bmatrix}
\] \hspace{1em} & \in \mathbb{Q}(S_1) & a = 0.9 & \text{some } a \in (0.8, 0.9) & a = 0.8 |
| \[
\begin{bmatrix}
-1 & 1 & 1 & 1 \\
1 & -a & 0 & 0 \\
1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
\end{bmatrix}
\] \hspace{1em} & \in \mathbb{Q}(S_2) & a = 2.6 & a \in (2.5, 2.6) & a = 2.5 |
| \[
\begin{bmatrix}
0 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 \\
-\alpha & 0 & 0 & -1 \\
\end{bmatrix}
\] \hspace{1em} & \in \mathbb{Q}(S_3) & a = 2 & a = 1 & a = 0.5 |

By eliminating the other three refined inertias in Observation 2.2 and finding a realization for each refined inertia in \(H_4\), we now show that sign patterns \(S_4\) and \(S_5\) require \(H_4\).

**Theorem 2.6.** The star sign pattern \(S_4\) requires \(H_4\).

**Proof.** To consider refined inertia, any realization of \(S_4\) can be normalized to

\[
M = \begin{bmatrix}
-1 & 1 & 1 & 1 \\
\alpha & -a & 0 & 0 \\
-d & 0 & b & 0 \\
\epsilon & 0 & 0 & 0 \\
\end{bmatrix},
\]

where \(a, b, c, d, \epsilon \in \mathbb{R}^+\). The characteristic polynomial of \(M\) is \(c_M(x) = x^4 + p_1x^3 + p_2x^2 + p_3x + p_4\), where

\[
\begin{align*}
p_1 &= a - b + 1 \\
p_2 &= a - ab - b - c + d - \epsilon \\
p_3 &= ad - ab - ac + bc + be \\
p_4 &= abc.
\end{align*}
\]

The normalized form \(M\) satisfies the conditions of Observation 2.2 by defining the map \(\chi : \mathbb{R}^5 \to \mathbb{R}^4\) as \(\chi(a, b, c, d, \epsilon) = (p_1, p_2, p_3, p_4)\). The Jacobian matrix of the map \(\chi\) is

\[
\text{Jac}_\chi = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-1-b & -1-a & -1 & 1 & -1 \\
d-b-\epsilon & c-a+\epsilon & b & a & b-a \\
be & ae & 0 & 0 & ab
\end{bmatrix}.
\]

Taking the \(4 \times 4\) submatrix formed by the first, third, fourth and fifth columns of \(\text{Jac}_\chi\), the determinant is \(-(a^2b + ab^2)\), which is nonzero since \(a, b > 0\). Therefore,
Jac₃ has rank 4. Since −S₄ does not appear in [6], it is not potentially stable, and thus S₄ does not allow refined inertia (4, 0, 0, 0). Consequently S₄ does not allow refined inertia (2, 0, 0, 2) or (0, 0, 0, 4) by Observation 1.1. Fix a = c = e = 1 and b = 0.1. If d = 2, then ri(M) = (0, 4, 0, 0); if d = 1.9, then ri(M) = (2, 2, 0, 0). By continuity and the sign nonsingularity of S₄, there exists a value of d such that 1.9 < d < 2 with ri(M) = (0, 2, 0, 2). Therefore, by Observation 2.2, the star sign pattern S₄ requires H₄.

LEMMA 2.7. The star sign pattern S₅ allows H₄.

Proof. Consider the matrix

\[
M = \begin{bmatrix}
  f & 1 & 1 & 1 \\
  -c & 0 & 0 & 0 \\
  -d & 0 & -a & 0 \\
  -e & 0 & 0 & -b
\end{bmatrix} \in Q(S₅),
\]

where a, b, c, d, e, f ∈ ℝ⁺. Fix a = b = c = d = e = 1. If f = 0.5, then ri(M) = (0, 4, 0, 0); if f = 0.6, then ri(M) = (2, 2, 0, 0). Therefore, by continuity, negativity of the trace and the sign nonsingularity of S₅, there exists a value of f such that 0.5 < f < 0.6 with ri(M) = (0, 2, 0, 2). Thus S₅ allows H₄.

THEOREM 2.8. The star sign pattern S₅ requires H₄.

Proof. To consider refined inertia, any realization of S₅ can be normalized to

\[
M = \begin{bmatrix}
  1 & 1 & 1 & 1 \\
  -c & 0 & 0 & 0 \\
  -d & 0 & -a & 0 \\
  -e & 0 & 0 & -b
\end{bmatrix}
\]

where a, b, c, d, e ∈ ℝ⁺. The characteristic polynomial of M is \(c_M(x) = x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4\), where

\[
p_1 = a + b - 1 \\
p_2 = ab - a - b + c + d + e \\
p_3 = ac - ab + ac + bc + bd \\
p_4 = abc.
\]

If S₅ allows refined inertia (0, 0, 0, 4), then there exists an M with characteristic polynomial \(x^4 + p_2 x^2 + p_4\), where \(p_2, p_4 > 0\). If \(p_1 = a + b - 1 = 0\) then \(a = 1 - b\). Thus since \(a > 0\) it follows that \(b < 1\). Now substituting \(a = 1 - b\) into \(p_3 = 0\) gives

\[
c = e(b - 1) + b(1 - b - d) = (e - b)(b - 1) - bd.
\]
Since $c > 0$ and $b < 1$, the first equality gives $d < 1 - b$ and the second equality gives $e < b$. Now substituting $a$ and $c$ into $p_2$ gives

$$p_2 = 2b - 2b^2 + d - 1 + bc - bd$$

$$= (1 - b)(d - 1 + b) + b(e - b).$$

Then $1 - b > 0$, $d - 1 + b < 0$ and $e - b < 0$ imply that $p_2 < 0$. Thus $S_5$ does not allow refined inertia $(0, 0, 0, 4)$. Since $-S_5$ is not potentially stable [6], it follows that $S_5$ does not allow refined inertia $(4, 0, 0, 0)$.

Finally, suppose that $S_5$ allows refined inertia $(2, 0, 0, 0)$. Then the coefficients of the characteristic polynomial satisfy

$$p_1 < 0, p_2 > 0, p_3 < 0, \text{ and } p_4 > 0.$$

Thus $p_1 = a + b - 1 < 0$. Now consider the following three cases.

**Case 1** $a = b$. Coefficients $p_1$, $p_2$, and $p_3$ become

$$p_1 = 2b - 1$$

$$p_2 = b^2 - 2b + c + d + e$$

$$p_3 = b(c - b + e + c + d).$$

First note that $2b - 1 < 0$ and so $b < 1$. Since $p_3 < 0$, it follows that $b > c + d + e + c$, and in particular $b > c + d + e$. But this implies that $p_2 = b(b - 2) + c + d + e < -b + c + d + e < 0$, which is a contradiction. Thus $b \neq a$ if $M$ has refined inertia $(2, 0, 0, 2)$.

**Case 2** $a > b$, i.e., $a = b + \epsilon$ for $\epsilon > 0$. As before consider the coefficients

$$p_1 = 2b + \epsilon - 1$$

$$p_2 = b(b + \epsilon) - b - \epsilon - b + c + d + e$$

$$p_3 = bc + \epsilon c - b^2 - bc + be + \epsilon e + be + bd.$$  

Since $p_1 < 0$, $2b + \epsilon - 1 < 0$ and so $2b + \epsilon < 1$. Since $p_3 < 0$, it follows that $-(b + \epsilon)b + (b + \epsilon)c + (b + \epsilon)e + bc + bd < 0$ and so $b + \epsilon > c + d + e + c + \frac{c(e + e)}{b} + \frac{c(c + e)}{b}$, and in particular $b + \epsilon > c + d + e$. Finally, this gives $p_2 = -b(2 - \epsilon - b) - \epsilon - c + d + e < -b - \epsilon + c + d + e < 0$, which is a contradiction. Therefore, if $a > b$, then $M$ does not have refined inertia $(2, 0, 0, 2)$.

**Case 3** $a < b$. The proof of this case follows from that of Case 2 by interchanging $e$ and $d$ in the expression for $p_3$, and by replacing $b$ with $a$ throughout.
Therefore, these three cases imply that \( S_5 \) does not allow refined inertia \((2, 0, 0, 2)\).
By Observation 2.2 and Lemma 2.7, \( S_5 \) requires \( H_4 \).

In order to obtain a characterization of the \( 4 \times 4 \) tree sign patterns that require \( H_4 \), we started with the list of potentially stable path and star sign patterns in [6, 8]. Elimination of those sign patterns that are sign stable or not sign nonsingular resulted in 21 tree sign patterns \( P_1, P_2, P_3, P_4, P_5 \) and \( S_1, S_2, S_3, S_4, S_5 \) above together with \( P_6, P_7, P_8, P_9, P_{10}, P_{11} \) and \( S_6, S_7, S_8, S_9, S_{10} \) in the Appendix. Each of the patterns in the Appendix is shown to allow refined inertia \((4, 0, 0, 0)\), i.e., its negative is potentially stable, leading to the following characterization.

**Theorem 2.9.** A \( 4 \times 4 \) tree sign pattern requires \( H_4 \) if and only if it is potentially stable, sign nonsingular, not sign stable, and its negative is not potentially stable.

### 3. Star sign patterns of order \( n \).

**3.1. Extending a star sign pattern.** Using the star sign patterns \( S_1 \) and \( S_2 \), we construct \( n \times n \) star sign patterns that allow \( H_n \). Note that these sign patterns are potentially stable by [5, Theorems 4.3 and 3.5].

**Theorem 3.1.** With \( \pm \) taken to be either + or −, the \( n \times n \) star sign patterns

\[
S = \begin{bmatrix}
- & + & + & + & + & \ldots & + \\
+ & - & 0 & 0 & 0 & \ldots & 0 \\
- & 0 & 0 & 0 & 0 & \ldots & 0 \\
\pm & 0 & 0 & - & 0 & \ldots & 0 \\
\pm & 0 & 0 & 0 & - & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\pm & 0 & 0 & 0 & \ldots & 0 & - \\
\pm & 0 & 0 & 0 & \ldots & 0 & 0 & - \\
\end{bmatrix}
\]

require \( H_n \) for \( n = 3 \) and \( 4 \), and allow \( H_n \) for \( n \geq 5 \).

**Proof.** For \( n = 3 \), the sign pattern \( S \) is equivalent to a pattern in the Appendix of [1], and thus requires \( H_3 \). For \( n = 4 \), sign pattern \( S \) with the \((4, 1)\) entry equal to − is equivalent to \( S_1 \) listed above. Taking the \((4, 1)\) entry to be +, sign pattern \( S \) is equivalent to \( S_2 \) listed above. Thus for \( n = 4 \), \( S \) requires \( H_4 \).

For \( n \geq 5 \), we show that sign patterns \( S \) allow each refined inertia in \( H_n \). Consider sign pattern \( \tilde{S} \) obtained from \( S \) by replacing the \((j, 1)\) and \((1, j)\) entries with zero for \( j = 3, 4, \ldots, n \). The eigenvalues of \( \tilde{S} \) are the eigenvalues of the leading principal \( 2 \times 2 \)
submatrix, one 0, and \(n - 3\) negative real numbers. Consider

\[
\tilde{S} = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 \\
 a & -1 & 0 & 0 & 0 & 0 & \ldots & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
 0 & 0 & 0 & -10 & 0 & 0 & \ldots & 0 \\
 0 & 0 & 0 & 0 & -10 & 0 & \ldots & 0 \\
 \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 0 & 0 & 0 & \ldots & 0 & -10 & 0 \\
 0 & 0 & 0 & \ldots & 0 & 0 & -10 \\
\end{bmatrix} \in Q(\tilde{S})
\]

and

\[
S(a, \epsilon) = \begin{bmatrix}
-1 & 1 & \frac{\epsilon}{n} & \frac{\epsilon}{n} & \frac{\epsilon}{n} & \frac{\epsilon}{n} & \ldots & \frac{\epsilon}{n} \\
 a & -1 & 0 & 0 & 0 & 0 & \ldots & 0 \\
 -\epsilon & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
 \pm \epsilon & 0 & -10 & 0 & 0 & \ldots & 0 \\
 \pm \epsilon & 0 & 0 & 0 & -10 & 0 & \ldots & 0 \\
 \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 \pm \epsilon & 0 & 0 & 0 & \ldots & 0 & -10 & 0 \\
 \pm \epsilon & 0 & 0 & 0 & \ldots & 0 & 0 & -10 \\
\end{bmatrix} \in Q(S)
\]

with \(a, \epsilon > 0\).

If \(a > 1\), then the determinant of the leading \(2 \times 2\) principal submatrix of \(\tilde{S}\) is negative, and thus the eigenvalues of \(\tilde{S}\) are one negative real number, one positive real number, 0, and \(-10\) with multiplicity \(n - 3\). For \(\epsilon > 0\) sufficiently small, the refined inertia of \(S(a, \epsilon)\) is \((2, n - 2, 0, 0)\), since the sign of the determinant of \(S(a, \epsilon)\) is \((-1)^n\) and the eigenvalues of \(S(a, \epsilon)\) are small perturbations of those of \(\tilde{S}\).

Now if \(a < 1\), then \(n - 1\) of the eigenvalues of \(\tilde{S}\) are negative and one is zero. From the properties above, for sufficiently small \(\epsilon > 0\), the refined inertia of \(S(a, \epsilon)\) is \((0, n, 0, 0)\).

Fix \(a_1\) such that \(1 < a_1 < 8\) and \(\epsilon_1 > 0\) sufficiently small so that \(S(a_1, \epsilon_1)\) has refined inertia \((2, n - 2, 0, 0)\), and fix \(a_2\) such that \(0 < a_2 < 1\) and \(\epsilon_2 > 0\) sufficiently small so that \(S(a_2, \epsilon_2)\) has refined inertia \((0, n, 0, 0)\). Let \(\epsilon_3 = \min\{\epsilon_1, \epsilon_2, \frac{1}{2}\}\) and note that \(S(a_1, \epsilon_3)\) has refined inertia \((2, n - 2, 0, 0)\) and \(S(a_2, \epsilon_3)\) has refined inertia \((0, n, 0, 0)\). Now consider the matrices \(S(a, \epsilon_3)\) for \(a_2 < a < a_1\). By Geršgorin’s disc theorem [9, p. 14-5], each of these matrices has \(n - 3\) eigenvalues that lie within a closed disc of radius \(\epsilon_3 \leq \frac{1}{2}\) centered at \(-10\) in the complex plane. Furthermore, using Geršgorin’s theorem, it follows that the other three eigenvalues lie within a disjoint closed disc of radius 9 centered at the origin. Since the sign of the determinant of
$S(a, \epsilon_3)$ is $(-1)^n$, as $a$ decreases continuously from $a_1$ to $a_2$ there must be a value $\hat{a}$ in the interval $(a_2, a_1)$ for which the refined inertia of $S(\hat{a}, \epsilon_3)$ is $(0, n - 2, 0, 2)$. Therefore, the $n \times n$ star sign patterns $S$ allow $H_n$ for $n \geq 5$ and require $H_n$ for $n = 3$ and 4.

The next example gives one instance of the above sign patterns $S$ that does not require $H_n$ for $n \geq 5$.

**Example 3.2.** The sign pattern

$$S = \begin{bmatrix}
- & + & + & + \\
+ & 0 & 0 & 0 \\
- & 0 & 0 & 0 \\
- & 0 & 0 & - \\
+ & 0 & 0 & - \\
\end{bmatrix}$$

allows $H_5$ by Theorem 3.1. Consider the following realization

$$S = \begin{bmatrix}
-0.1 & 1 & 1 & 1 & 1 \\
1000 & -100 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
-100 & 0 & 0 & -1 & 0 \\
50 & 0 & 0 & 0 & -0.1 \\
\end{bmatrix} \in Q(S).$$

Since the eigenvalues of $S$ are approximately $-109.1318, 3.3189 \pm 4.1492i, 0.0025$, and 1.2914, the refined inertia of $S$ is $(4, 1, 0, 0)$. Hence, the sign pattern $S$ does not require $H_5$.

It follows by continuity that any $n \times n$ sign pattern with $S$ as a $5 \times 5$ principal subpattern allows at least four eigenvalues with positive real part, and thus does not require $H_n$.

**3.2. A necessary condition for requiring $H_n$.** If a star sign pattern requires $H_n$, then its digraph has some additional structure, namely that exactly one leaf vertex does not have a loop. The next result follows immediately from [5, Theorems 3.5 and 4.2].

**Lemma 3.3.** [5] Let $S = [\sigma_{ij}]$ be an $n \times n$ potentially stable star sign pattern with 1 as the center vertex in $D(S)$ and without loss of generality $\sigma_{11} = +$ for $i = 2, \ldots, n$. Then

(i) if $\sigma_{11} \in \{+, 0\}$ then there exists $i$ such that $\sigma_{ii} = -$ and $\sigma_{ii} = -$, and

(ii) for $i = 2, \ldots, n$,

$$|\{i|\sigma_{ii} = + \text{ and } \sigma_{ii} = +\}| = \left\lfloor \frac{|\{\sigma_{ii} = +\}|}{2} \right\rfloor.$$
Theorem 3.4. For \( n \geq 3 \), let \( S = [\sigma_{ij}] \) be an \( n \times n \) star sign pattern with 1 as the center vertex in \( D(S) \). If \( S \) is sign nonsingular, potentially stable and not sign stable, then there exists a unique \( i \) such that \( 2 \leq i \leq n \) and \( \sigma_{ii} = 0 \).

Proof. Since all \( S \in Q(S) \) are nonsingular, at most one \( \sigma_{ii} \) can be zero. We now show by contradiction that at least one \( \sigma_{ii} \) must be zero. Recall that without loss of generality \( \sigma_{ii} = + \) for \( i = 2, \ldots, n \). If \( \sigma_{ii} \neq 0 \) for all \( i = 2, \ldots, n \), then one of the following cases must occur.

Case 1. Let \( \sigma_{i1} = - \) for \( i = 2, \ldots, n \). Then \( \sigma_{i1} \) has the same sign for \( i = 2, \ldots, n \); otherwise if say \( \sigma_{i1} \sigma_{kk} = - \), then any \( S = [s_{ij}] \in Q(S) \) has terms in \( \det(S) \)
\[-s_{kk}s_{i1}s_{i1} \prod_{j \neq 1,i,k} s_{jj} \quad \text{and} \quad -s_{ii}s_{1k}s_{kk} \prod_{j \neq 1,i,k} s_{jj}\]
of opposite sign, violating the sign nonsingularity of \( S \). If \( \sigma_{i1} = + \) for all \( i = 2, \ldots, n \), then \( S \in Q(S) \) is symmetrizable by a positive diagonal similarity. Thus since \( S \) is potentially stable, sign nonsingular and symmetrizable, it must also be sign stable (since for any \( S \in Q(S) \), all eigenvalues of \( S \) are real and thus negative). On the other hand, if \( \sigma_{i1} = - \) for \( i = 2, \ldots, n \), then \( S \) is sign stable if \( \sigma_{i1} = - \) or 0 [3, Corollary 10.2.3], and \( S \) is not sign nonsingular if \( \sigma_{i1} = + \). Thus each case gives a contradiction.

Case 2. Let \( \sigma_{i1} = - \) and \( \sigma_{ii} = + \) for some \( i \) such that \( 2 \leq i \leq n \). Since \( S \) is potentially stable and sign nonsingular with \( \text{sgn}(\det(S)) = \text{sgn}((-1)^n) \) for all \( S \in Q(S) \), there exists \( k \neq i \) such that \( \sigma_{ik} = + \). Therefore, by Lemma 3.3 (ii), since the right hand side of (3.1) is at least one, the equality in (3.1) implies that \( i \) and \( k \) can be chosen, without loss of generality, so that \( \sigma_{i1} = - \) and \( \sigma_{kk} = + \). Therefore, as in Case 1, any \( S \in Q(S) \) has two terms in \( \det(S) \) of opposite sign, violating the sign nonsingularity of \( S \).

Case 3. Let \( \sigma_{i1} \in \{+,0\} \) and \( \sigma_{ii} = + \) for some \( i \) such that \( 2 \leq i \leq n \). By Lemma 3.3 (ii), \( i \) can be chosen such that \( \sigma_{i1} = - \). By Lemma 3.3 (i), there exists a \( k \) such that \( \sigma_{kk} = - \) and \( \sigma_{kk} = + \). Thus, as in Case 1, the sign nonsingularity of \( S \) is violated. \( \square \)

The next result follows immediately from Theorem 3.4 since if \( S \) requires \( \mathbb{H}_n \) then it is potentially stable, sign nonsingular and not sign stable.

Corollary 3.5. For \( n \geq 3 \), let \( S = [\sigma_{ij}] \) be an \( n \times n \) star sign pattern with 1 as the center vertex in \( D(S) \). If \( S \) requires \( \mathbb{H}_n \), then there exists a unique \( i \) such that \( 2 \leq i \leq n \) and \( \sigma_{ii} = 0 \).
4. Reducible sign patterns. Reducible sign patterns that either require or allow $\mathbb{H}_n$ are considered in [1]. The following result is an extension of [1, Observation 1.5] for the requires problem.

**Theorem 4.1.** Suppose $A = \begin{bmatrix} A_1 & \# \\ O & A_2 \end{bmatrix}$, where $A_1$ is a sign pattern of order $n_1$, $A_2$ is a sign pattern of order $n_2$ and $\#$ denotes an arbitrary $n_1 \times n_2$ sign pattern. Then $A$ requires $\mathbb{H}_{n_1+n_2}$ if and only if exactly one sign pattern $A_i$ requires $\mathbb{H}_{n_i}$ and the other sign pattern $A_j$ is sign stable.

**Proof.** Suppose first without loss of generality that $A_1$ requires $\mathbb{H}_{n_1}$ and $A_2$ is sign stable. Therefore, any realization of $A$ necessarily has refined inertia in $\mathbb{H}_{n_1+n_2}$ and $A$ requires $\mathbb{H}_{n_1+n_2}$.

Conversely, if $A$ requires $\mathbb{H}_{n_1+n_2}$, then exactly one sign pattern $A_i$ requires $\mathbb{H}_{n_i}$, in which case the other sign pattern $A_j$ must be sign stable. $\square$

Now consider the allows problem for the reducible sign pattern $A$ in Theorem 4.1. From [1, Observation 1.5], if $A_i$ allows $\mathbb{H}_{n_i}$ and $A_j$ is potentially stable with distinct $i, j \in \{1, 2\}$, then $A = A_i \oplus A_j$ allows $\mathbb{H}_{n_i+n_j}$. However the following proposition and example show that the converse is false.

**Proposition 4.2.** Let

$$P = \begin{bmatrix} 0 & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix}.$$

The path sign pattern $P$ allows only two refined inertiases, namely $(0, 5, 0, 0)$ and $(0, 3, 0, 2)$.

**Proof.** First notice that $P$ satisfies the hypotheses of [3, Theorem 10.2.1] and so it is sign semi-stable, i.e., does not allow any eigenvalues with positive real part. Thus since $P$ is sign nonsingular with negative trace and sign semi-stable, the only possible refined inertiases are $(0, 1, 0, 4)$, $(0, 3, 0, 2)$ and $(0, 5, 0, 0)$. To eliminate refined inertia $(0, 1, 0, 4)$ consider

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -b & 0 & 1 & 0 & 0 \\ 0 & -c & -a & 1 & 0 \\ 0 & 0 & -d & 0 & 1 \\ 0 & 0 & 0 & -e & 0 \end{bmatrix} \in Q(P)$$

where $a, b, c, d, e \in \mathbb{R}^+$, which has characteristic polynomial $c_A(x) = x^5 + ax^4 +$
Refined Inertias of Tree Sign Patterns

(b + c + d + e)x^3 + a(b + e)x^2 + (bd + be + ce)x + ab. If A has refined inertia $(0, 1, 0, 4)$, then the characteristic polynomial of A is $(x + \alpha)(x^2 + \beta)(x^2 + \gamma) = x^5 + \alpha x^4 + (\beta + \gamma)x^3 + \alpha(\beta + \gamma)x^2 + \beta \gamma x + \alpha \beta \gamma$ with $\alpha, \beta, \gamma \in \mathbb{R}^+$. Equating these polynomials gives

$$a = \alpha$$

$$a(b + e) = \alpha(\beta + \gamma) \Rightarrow b + e = \beta + \gamma$$

$$b + c + d + e = \beta + \gamma \Rightarrow c + d = 0.$$  

This is a contradiction and so $P$ does not allow refined inertia $(0, 1, 0, 4)$. If $P$ is a realization of $\mathcal{P}$ with all nonzero entries having magnitude 1, then $ri(P) = (0, 3, 0, 2)$. If $\tilde{P}$ is obtained from $P$ by changing the $(2, 1)$ entry to $-2$, then $ri(\tilde{P}) = (0, 5, 0, 0)$. Therefore, the only refined inertias allowed by $\mathcal{P}$ are $(0, 3, 0, 2)$ and $(0, 5, 0, 0)$. □

**Example 4.3.** The path sign patterns

$$\mathcal{P}_1 = \begin{bmatrix} 0 & + & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 \\ 0 & - & - & + & 0 \\ 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & - & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{P}_2 = \begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix}$$

do not allow $\mathbb{H}_5$ and $\mathbb{H}_4$, respectively, however $\mathcal{A} = \mathcal{P}_1 \oplus \mathcal{P}_2$ allows, but does not require, $\mathbb{H}_9$. To see this, first note that $\mathcal{P}_1$ is sign semi-stable by Proposition 4.2. Hence, $\mathcal{P}_1$ does not allow refined inertia $(2, 3, 0, 0)$ and consequently does not allow $\mathbb{H}_5$. Next notice that since any realization of $\mathcal{P}_2$ is symmetrizable, $\mathcal{P}_2$ does not allow refined inertia $(0, 2, 0, 2)$. Therefore, $\mathcal{P}_2$ does not allow $\mathbb{H}_4$. However, $\mathcal{P}_2$ is potentially stable and so it allows refined inertia $(0, 4, 0, 0)$. Using a realization of $\mathcal{P}_1$ that is stable and a realization of $\mathcal{P}_2$ that is stable, $\mathcal{A}$ allows refined inertia $(0, 9, 0, 0)$. By Observation 4.2 there is a realization $P_1 \in Q(\mathcal{P}_1)$ that has refined inertia $(0, 3, 0, 2)$. Using a realization of $\mathcal{P}_2$ that is stable, $\mathcal{A}$ allows refined inertia $(0, 7, 0, 2)$. Finally if $\tilde{P}_2$ is obtained from $P_2$ by replacing the diagonal entries with zero, then the refined inertia of any realization $\tilde{P}_2 \in Q(\tilde{P}_2)$ is $(2, 2, 0, 0)$, since all eigenvalues of $\tilde{P}_2$ are real, nonzero and (from the characteristic polynomial) $-\alpha$ is an eigenvalue if and only if $\alpha$ is an eigenvalue. Thus there exists an $\epsilon > 0$ sufficiently small so that $\tilde{P}_2 - \epsilon I$ has refined inertia $(2, 2, 0, 0)$. Using this realization of $\mathcal{P}_2$ and a realization of $\mathcal{P}_1$ that is stable gives a realization of $\mathcal{A}$ with refined inertia $(2, 7, 0, 0)$. Therefore, $\mathcal{A}$ allows $\mathbb{H}_9$. Finally, using the realizations $P_1$ and $\tilde{P}_2 - \epsilon I$ above gives a realization of $\mathcal{A}$ that has refined inertia $(2, 5, 0, 2)$ and so $\mathcal{A}$ does not require $\mathbb{H}_9$.

5. **Concluding remarks.** Each path sign pattern with $n = 3$ (listed in Appendix) and $n = 4$ (listed in Section 2.4 above) that requires $\mathbb{H}_9$ has a zero in the $(1, 1)$ entry, the $(n, n)$ entry or both, i.e., at least one leaf in its digraph. By
Corollary 3.5, each star sign pattern of order \( n \geq 3 \) that requires \( H_n \) has a zero at a unique leaf vertex in its digraph, but the question of whether or not a path sign pattern \( P \) of order \( n \geq 5 \) that requires \( H_n \) must have a zero at a leaf vertex in \( D(P) \) remains open. The question also remains open as to whether or not this is true for every tree sign pattern \( A \) with order \( n \geq 5 \) that is potentially stable, sign nonsingular and not sign stable.

Necessary and sufficient conditions for a tree sign pattern to require \( H_3 \) are given by [1, Theorem 2.1] and to require \( H_4 \) by Theorem 2.9. The requires problem for \( H_n \) with \( n \geq 5 \) remains open.

6. Appendix. In addition to \( P_1, \ldots, P_5 \) and \( S_1, \ldots, S_5 \), up to equivalence there are eleven \( 4 \times 4 \) tree sign patterns from [6] and [8] that are sign nonsingular, potentially stable and not sign stable. We now list these sign patterns and show below that they allow refined inertia \((4, 0, 0, 0)\), and thus do not require \( H_4 \).

Let

\[
P_6 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & 0 & 0 \\ 0 & 0 & 0 & + \\ 0 & 0 & 0 & - \end{bmatrix}, \quad P_7 = \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & 0 & 0 \\ 0 & 0 & - & + \\ 0 & 0 & 0 & - \end{bmatrix}, \quad P_8 = \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
P_9 = \begin{bmatrix} + & + & 0 & 0 \\ - & + & 0 & 0 \\ 0 & 0 & 0 & + \\ 0 & 0 & 0 & - \end{bmatrix}, \quad P_{10} = \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & 0 & 0 \\ 0 & 0 & - & + \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_{11} = \begin{bmatrix} 0 & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
S_6 = \begin{bmatrix} - & + & + & + \\ - & 0 & 0 & 0 \\ - & 0 & + & 0 \\ + & 0 & 0 & 0 \end{bmatrix}, \quad S_7 = \begin{bmatrix} - & + & + & + \\ + & 0 & 0 & 0 \\ - & 0 & + & 0 \\ - & 0 & 0 & 0 \end{bmatrix}, \quad S_8 = \begin{bmatrix} 0 & + & + & + \\ + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
S_9 = \begin{bmatrix} - & + & + & + \\ + & 0 & 0 & 0 \\ + & 0 & 0 & 0 \\ - & 0 & 0 & - \end{bmatrix}, \quad S_{10} = \begin{bmatrix} + & + & + & + \\ - & 0 & 0 & 0 \\ - & 0 & 0 & 0 \\ + & 0 & 0 & 0 \end{bmatrix}.
\]

Each of these sign patterns is equivalent to the negative of one of these sign patterns, as the following table specifies.
Considering for example $S_6$ and $S_{10}$, if all entries in $S_6$ are negated, then a sign pattern that is equivalent to $S_{10}$ is obtained and vice versa. Since these two sign patterns are potentially stable, taking a stable realization of $S_6$ and negating it gives a matrix that has four eigenvalues with positive real part and a sign pattern equivalent to $S_{10}$. Therefore, $S_{10}$ does not require $\mathbb{H}_4$. Similarly, $S_6$ does not require $\mathbb{H}_4$. By a similar argument, these 11 sign patterns all allow refined inertia $(4, 0, 0, 0)$ and hence do not require $\mathbb{H}_4$. However, it can be shown with numerical examples that each sign pattern allows $\mathbb{H}_4$.

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