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REFINEMENTS OF THE OPERATOR JENSEN–MERCER INEQUALITY*

MOHSEN KIAN[†] AND MOHAMMAD SAL MOSLEHIAN[‡]

Abstract. A Hermite–Hadamard–Mercer type inequality is presented and then generalized to Hilbert space operators. It is shown that $f(M + m - \sum_{i=1}^n x_i A_i) \leq f(M) + f(m) - \sum_{i=1}^n f(x_i) A_i$, where f is a convex function on an interval $[m, M]$ containing 0, $x_i \in [m, M]$, $i = 1, \dots, n$, and A_i are positive operators acting on a finite dimensional Hilbert space whose sum is equal to the identity operator. A Jensen–Mercer operator type inequality for separately operator convex functions is also presented.

Key words. Jensen–Mercer inequality, Operator convex, Jensen inequality, Hermite–Hadamard inequality, Jointly operator convex.

AMS subject classifications. 47A63, 47A64.

1. Introduction. The well-known Jensen inequality for the convex functions states that if f is a convex function on an interval $[m, M]$, then

$$(1.1) \quad f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

for all $x_i \in [m, M]$ and all $\lambda_i \in [0, 1]$ ($i = 1, \dots, n$) with $\sum_{i=1}^n \lambda_i = 1$. The Hermite–Hadamard inequality asserts that if f is a convex function on $[m, M]$, then

$$f\left(\frac{m+M}{2}\right) \leq \frac{1}{M-m} \int_m^M f(x) dx \leq \frac{f(m)+f(M)}{2}.$$

For more information, see [5, 15] and references therein. Mercer [12] established a variant of the Jensen inequality (1.1) as follows.

THEOREM 1.1. *If f is a convex function on $[m, M]$, then*

$$(1.2) \quad f\left(M + m - \sum_{i=1}^n \lambda_i x_i\right) \leq f(M) + f(m) - \sum_{i=1}^n \lambda_i f(x_i)$$

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for all $x_i \in [m, M]$ and $\lambda_i \in [0, 1]$ ($i = 1, \dots, n$) with $\sum_{i=1}^n \lambda_i = 1$.

Inequality (1.2) is known as the Jensen–Mercer inequality. Recently, inequality (1.2) has been generalized; see [1, 2, 16].

Let $\mathbb{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a Hilbert space \mathcal{H} and I denote the identity operator. An operator A is positive (denoted by $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all vectors $x \in \mathcal{H}$. If, in addition, A is invertible, then it is strictly positive (denoted by $A > 0$). By $A \geq B$ we mean that $A - B$ is positive, while $A > B$ means that $A - B$ is strictly positive. An operator C is an isometry if $C^*C = I$, a contraction if $C^*C \leq I$ and an expansive operator if $C^*C \geq I$. A linear map Φ on $\mathbb{B}(\mathcal{H})$ is positive if $\Phi(A) \geq 0$ for each $A \geq 0$ and is strictly positive if $\Phi(A) > 0$ for each $A > 0$. A positive linear map Φ is strictly positive if and only if $\Phi(I) > 0$ [8].

A continuous function f defined on an interval J is said to be operator convex if

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$$

for all $\lambda \in [0, 1]$ and all self-adjoint operators A, B with spectra in J .

Hansen and Pedersen (see [6, Theorem 1.9]) presented an operator version of the Jensen inequality for an operator convex function by establishing that if f is operator convex on J , then

$$(1.3) \quad f(C^*AC) \leq C^*f(A)C$$

for any self-adjoint operator A with spectrum in J and any isometry C . Several versions of the Jensen operator inequality can be found in [6, 8]. Among them, we are interested in the following generalization of (1.3):

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \leq \sum_{i=1}^n \Phi_i(f(A_i)),$$

in which the operators A_i ($i = 1, \dots, n$) are self-adjoint with spectra in J and Φ_1, \dots, Φ_n are positive linear maps on $\mathbb{B}(\mathcal{H})$ with $\sum_{i=1}^n \Phi_i(I) = I$ [14]. Recently, J. Míćić, Z. Pavić and J. Pečarić [13] obtained a Jensen inequality for operators without the assumption of operator convexity.

Regarding the possible operator extensions of (1.2), there are some interesting works.

Let f be a continuous convex function on $[m, M]$, A_1, \dots, A_n be self-adjoint operators with spectra in $[m, M]$ and Φ_1, \dots, Φ_n be positive linear maps with $\sum_{i=1}^n \Phi_i(I) = I$. The following operator version of the Mercer inequality (1.2) was proved in [10]:

$$(1.4) \quad f\left(M + m - \sum_{i=1}^n \Phi_i(A_i)\right) \leq f(m) + f(M) - \sum_{i=1}^n \Phi_i(f(A_i)).$$

Furthermore, some refinements, applications and reformulations of inequality (1.4) for some other types of functions have been obtained in [3, 7, 9, 11].

The function $f : [m, M] \times [m', M'] \rightarrow \mathbb{R}$ is separately operator convex if the functions g_t and h_s defined by $g_t(s) = f(t, s) = h_s(t)$ are operator convex, respectively, on $[m', M']$ and $[m, M]$ for each $t \in [m, M]$ and $s \in [m', M']$.

In Section 2, we present a Hermite–Hadamard–Mercer type inequality and then generalize it for Hilbert space operators. In Section 3, we obtain another variant of inequality (1.4). Also we give a Jensen–Mercer operator type inequality for separately operator convex functions.

2. Hermite–Hadamard–Mercer type inequalities. In this section, we present a Hermite–Hadamard type inequality using the Mercer inequality (1.2) and then give its operator extension.

THEOREM 2.1. *Let f be a convex function on $[m, M]$. Then*

$$(2.1) \quad \begin{aligned} f\left(M + m - \frac{x+y}{2}\right) &\leq f(M) + f(m) - \int_0^1 f(tx + (1-t)y)dt \\ &\leq f(M) + f(m) - f\left(\frac{x+y}{2}\right), \end{aligned}$$

and

$$(2.2) \quad f\left(M + m - \frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y f(M+m-t)dt \leq f(M) + f(m) - \frac{f(x) + f(y)}{2}$$

for all $x, y \in [m, M]$.

Proof. It follows from the Jensen–Mercer inequality that

$$(2.3) \quad f\left(M + m - \frac{a+b}{2}\right) \leq f(M) + f(m) - \frac{f(a) + f(b)}{2}$$

for each $a, b \in [m, M]$. Let $t \in [0, 1]$ and $x, y \in [m, M]$. Replacing a and b respectively by $tx + (1-t)y$ and $(1-t)x + ty$ in (2.3), we obtain

$$(2.4) \quad f\left(M + m - \frac{x+y}{2}\right) \leq f(M) + f(m) - \frac{f(tx + (1-t)y) + f((1-t)x + ty)}{2}.$$

By integrating both sides of (2.4), we get

$$(2.5) \quad \begin{aligned} f\left(M + m - \frac{x+y}{2}\right) &\leq f(M) + f(m) \\ &\quad - \frac{1}{2} \int_0^1 (f(tx + (1-t)y) + f((1-t)x + ty))dt. \end{aligned}$$

Due to

$$(2.6) \quad \int_0^1 f(tx + (1-t)y)dt = \int_0^1 f((1-t)x + ty)dt = \frac{1}{y-x} \int_x^y f(t)dt,$$

inequality (2.5) gives rise to the first inequality of (2.1). The second inequality of (2.1) follows directly from the Hermite–Hadamard inequality.

Next we prove inequality (2.2). The Hermite–Hadamard inequality implies that

$$(2.7) \quad \begin{aligned} \int_0^1 f(M+m-(tx+(1-t)y))dt &= \int_0^1 f(t(M+m-x) + (1-t)(M+m-y))dt \\ &\geq f\left(\frac{M+m-x+M+m-y}{2}\right) \\ &= f\left(M+m-\frac{x+y}{2}\right). \end{aligned}$$

On the other hand, the Mercer inequality gives

$$(2.8) \quad f(M+m-(tx+(1-t)y)) \leq f(M) + f(m) - (tf(x) + (1-t)f(y)).$$

Integrating both sides of (2.8) we get

$$(2.9) \quad \begin{aligned} \int_0^1 f(M+m-(tx+(1-t)y))dt &\leq f(M) + f(m) - \int_0^1 (tf(x) + (1-t)f(y))dt \\ &= f(M) + f(m) - \frac{f(x) + f(y)}{2}. \end{aligned}$$

Inequality (2.2) now follows immediately from inequalities (2.6), (2.7) and (2.9). \square

The following operator version of inequality (2.2) holds true.

THEOREM 2.2. *If f is convex on $[m, M]$, then*

$$(2.10) \quad \int_0^1 f(M+m-(t\Phi(A) + (1-t)\Phi(B)))dt \leq f(M) + f(m) - \frac{\Phi(f(A)) + \Phi(f(B))}{2},$$

$$(2.11) \quad f\left(m+M-\frac{\Phi(A)+\Phi(B)}{2}\right) \leq f(M) + f(m) - \int_0^1 f(t\Phi(A) + (1-t)\Phi(B))dt$$

for all self-adjoint operators A, B with spectra in $[m, M]$ and a unital positive linear map Φ . Furthermore, if f is operator convex, then

$$(2.12) \quad \begin{aligned} f\left(M+m-\frac{\Phi(A)+\Phi(B)}{2}\right) &\leq \int_0^1 f(M+m-(t\Phi(A) + (1-t)\Phi(B)))dt \\ &\leq f(M) + f(m) - \frac{\Phi(f(A)) + \Phi(f(B))}{2}. \end{aligned}$$

Proof. First note that since f is continuous, the vector valued integrals such as (2.10) exist for all self-adjoint operators A and B with spectra in $[m, M]$. We have

$$\begin{aligned} & \int_0^1 f(M + m - (t\Phi(A) + (1-t)\Phi(B)))dt \\ & \leq \int_0^1 [f(M) + f(m) - t\Phi(f(A)) - (1-t)\Phi(f(B))]dt \\ & \quad \text{(by the Jensen–Mercer operator inequality (1.4))} \\ & = f(M) + f(m) - \frac{\Phi(f(A)) + \Phi(f(B))}{2}, \end{aligned}$$

which is the desired inequality (2.10). Moreover, using the Jensen–Mercer operator inequality, we get

$$\begin{aligned} & f\left(m + M - \frac{\Phi(A) + \Phi(B)}{2}\right) \\ & = f\left(m + M - \frac{(t\Phi(A) + (1-t)\Phi(B)) + ((1-t)\Phi(A) + t\Phi(B))}{2}\right) \\ & \leq f(m) + f(M) - \frac{f(t\Phi(A) + (1-t)\Phi(B)) + f((1-t)\Phi(A) + t\Phi(B))}{2}. \end{aligned}$$

Integrating from both sides of the later inequality leads us to (2.11). If f is operator convex, then

$$\begin{aligned} & f\left(M + m - \frac{\Phi(A) + \Phi(B)}{2}\right) \\ & = f\left(\frac{(M + m - t\Phi(A) - (1-t)\Phi(B)) + (M + m - t\Phi(B) - (1-t)\Phi(A))}{2}\right) \\ (2.13) \quad & \leq \frac{1}{2} [f(M + m - t\Phi(A) - (1-t)\Phi(B)) + f(M + m - t\Phi(B) - (1-t)\Phi(A))]. \end{aligned}$$

Integrating from both sides of inequality (2.13) we get the first inequality of (2.12). The second one is clear. \square

EXAMPLE 2.3. If $f : J \rightarrow \mathbb{R}$ is a convex function, then the inequality

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA + (1-t)B)dt \leq \frac{f(A) + f(B)}{2}$$

may not hold in general [15]. To see this, consider the convex function $f(t) = t^4$ which appears in some counter-examples, starting with a work of M.-D. Choi [4], and Hermitian matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Nevertheless, inequalities (2.10) and (2.11) are valid for all $m, M \in J$ provided that the spectra of A and B are contained in $[m, M]$.

3. A variant of the Jensen–Mercer inequality for operators. We use an idea from [17] to obtain one of our main results.

THEOREM 3.1. *Let A_i ($i = 1, \dots, n$) be positive operators acting on a finite dimensional Hilbert space \mathcal{H} with $\sum_{i=1}^n A_i = I$. If f is convex on an interval $[m, M]$ containing 0, then*

$$(3.1) \quad f\left(M + m - \sum_{i=1}^n x_i A_i\right) \leq f(M) + f(m) - \sum_{i=1}^n f(x_i) A_i$$

for all $x_1, \dots, x_n \in [m, M]$.

Proof. Clearly, the spectrum of $M + m - \sum_{i=1}^n x_i A_i$ is contained in $[m, M]$. Without loss of generality we may assume that $f(0) = 0$ (if $f(0) \neq 0$ we may consider the convex function $g(x) = f(x) - f(0)$ instead of f). There is a Hilbert space \mathfrak{H} containing \mathcal{H} and a family of mutually orthogonal projections P_i ($i = 1, \dots, n$) on \mathfrak{H} such that $\sum_{i=1}^n P_i = I_{\mathfrak{H}}$ and $A_i = P P_i P|_{\mathcal{H}}$ for each $i = 1, \dots, n$, where P is the projection from \mathfrak{H} onto \mathcal{H} [17]. Therefore,

$$\begin{aligned} f\left(M + m - \sum_{i=1}^n x_i A_i\right) &= f\left(M + m - \sum_{i=1}^n x_i P P_i P|_{\mathcal{H}}\right) \\ &= f\left(M + m - P\left(\sum_{i=1}^n x_i P_i\right)P|_{\mathcal{H}} - (I - P)0(I - P)|_{\mathcal{H}}\right) \\ &\leq f(M) + f(m) - P f\left(\sum_{i=1}^n x_i P_i\right)P|_{\mathcal{H}} \\ &\quad - (I - P)f(0)(I - P)|_{\mathcal{H}} \\ &\quad \text{(by Jensen–Mercer operator inequality (1.4) with} \\ &\quad \Phi_1(A) = PAP \text{ and } \Phi_2(A) = (1 - P)A(1 - P)) \\ &\leq f(M) + f(m) - P\left(\sum_{i=1}^n f(x_i)P_i\right)P|_{\mathcal{H}} \\ &\quad \text{(by } f(0) = 0 \text{ and the spectral theorem)} \\ &= f(M) + f(m) - \sum_{i=1}^n f(x_i)A_i. \quad \square \end{aligned}$$

COROLLARY 3.2. *Let f and x_i ($i = 1, \dots, n$) be as in Theorem 3.1 and $f(0) = 0$. If $\sum_{i=1}^n A_i \leq I$, then inequality (3.1) remains true.*

Proof. Put $B = I - \sum_{i=1}^n A_i$. Then $B + \sum_{i=1}^n A_i = I$. Hence,

$$f\left(M + m - \sum_{i=1}^n x_i A_i\right) = f\left(M + m - \sum_{i=1}^n x_i A_i - 0B\right)$$

$$\begin{aligned} &\leq f(M) + f(m) - \sum_{i=1}^n f(x_i)A_i - f(0)B \quad (\text{by (3.1)}) \\ &\leq f(M) + f(m) - \sum_{i=1}^n f(x_i)A_i \quad (\text{by } f(0) = 0). \quad \square \end{aligned}$$

The following particular case of (3.1) is of special interest.

COROLLARY 3.3. *Let A_i ($i = 1, \dots, n$) be positive operators acting on a finite dimensional Hilbert space \mathcal{H} with $\sum_{i=1}^n A_i = I$. If f is convex on the interval $[m, M]$ containing -1 and 1 , then*

$$f\left(M + m + \sum_{i=1}^k A_i - \sum_{i=k+1}^n A_i\right) \leq f(M) + f(m) - f(-1) \sum_{i=1}^k A_i - f(1) \sum_{i=k+1}^n A_i.$$

REMARK 3.4. It should be mentioned that (3.1) implies a weaker version of the Jensen–Mercer operator inequality. Assume that X_i ($i = 1, \dots, n$) are self-adjoint operators on a finite dimensional Hilbert space \mathcal{H} with spectra in $[m, M]$ and $w_i \in [0, 1]$ with $\sum_{i=1}^n w_i = 1$. Let $X_i = \sum_{j=1}^k \lambda_{ij} P_{ij}$ ($i = 1, \dots, n$) be the spectral decomposition of X_i so that $\lambda_{ij} \in [m, M]$. It follows from $\sum_{i=1}^n \sum_{j=1}^k w_i P_{ij} = I$ that

$$\begin{aligned} f\left(M + m - \sum_{i=1}^n w_i X_i\right) &= f\left(M + m - \sum_{i=1}^n \sum_{j=1}^k w_i \lambda_{ij} P_{ij}\right) \\ &\leq f(M) + f(m) - \sum_{i=1}^n \sum_{j=1}^k f(\lambda_{ij}) w_i P_{ij} \quad (\text{by (3.1)}) \\ &= f(M) + f(m) - \sum_{i=1}^n w_i f\left(\sum_{j=1}^k \lambda_{ij} P_{ij}\right) \\ &= f(M) + f(m) - \sum_{i=1}^n w_i f(X_i). \end{aligned}$$

Let f be an operator convex function with $f(0) \leq 0$. It follows from the Jensen operator inequality (1.3) that

$$(3.2) \quad C^* f(A) C \leq f(C^* A C)$$

for any invertible expansive operator C and any self-adjoint operator A .

THEOREM 3.5. *Let $m < M$ and let Φ be a positive linear map on $\mathbb{B}(\mathcal{H})$ with $0 < \Phi(I) \leq I$. Let*

$$m' = \min\{m \langle \Phi(I)x, x \rangle; \|x\| = 1\} \quad \text{and} \quad M' = \max\{M \langle \Phi(I)x, x \rangle; \|x\| = 1\}.$$

If $f : J \rightarrow \mathbb{R}$ is an operator convex function with $f(0) \leq 0$ and $[m, M] \cup [m', M'] \subseteq J$, then

$$f((m + M)\Phi(I) - \Phi(A)) \leq f(m) + f(M) - \Phi(f(A))$$

for any self-adjoint operator A with spectrum contained in $[m, M]$.

Proof. Define the positive linear map Ψ on $\mathbb{B}(\mathcal{H})$ by

$$\Psi(X) = \Phi(I)^{-\frac{1}{2}}\Phi(X)\Phi(I)^{-\frac{1}{2}}, \quad (X \in \mathbb{B}(\mathcal{H})).$$

Then Ψ is unital, and it follows from (1.4) that

$$f(m + M - \Psi(A)) \leq f(m) + f(M) - \Psi(f(A))$$

for each self-adjoint operator A with spectrum in $[m, M]$. Therefore,

$$f(m + M - \Phi(I)^{-\frac{1}{2}}\Phi(A)\Phi(I)^{-\frac{1}{2}}) \leq f(m) + f(M) - \Phi(I)^{-\frac{1}{2}}\Phi(f(A))\Phi(I)^{-\frac{1}{2}}.$$

Hence,

$$(3.3) \quad \begin{aligned} & f(\Phi(I)^{-\frac{1}{2}}((m + M)\Phi(I) - \Phi(A))\Phi(I)^{-\frac{1}{2}}) \\ & \leq \Phi(I)^{-\frac{1}{2}}((f(m) + f(M))\Phi(I) - \Phi(f(A)))\Phi(I)^{-\frac{1}{2}}. \end{aligned}$$

On the other hand, $\Phi(I)^{-\frac{1}{2}}$ is an expansive operator. Using (3.2) we obtain

$$(3.4) \quad \begin{aligned} & \Phi(I)^{-\frac{1}{2}}(f((m + M)\Phi(I) - \Phi(A))\Phi(I)^{-\frac{1}{2}}) \\ & \leq f(\Phi(I)^{-\frac{1}{2}}((m + M)\Phi(I) - \Phi(A))\Phi(I)^{-\frac{1}{2}}). \end{aligned}$$

Now, the result follows from inequalities (3.3) and (3.4). \square

COROLLARY 3.6. Let f and \mathcal{H} be as in Theorem 3.5. Let Φ_i ($i = 1, \dots, n$) be positive linear maps on $\mathbb{B}(\mathcal{H})$ with $0 < \Phi(I) = \sum_{i=1}^n \Phi_i(I) \leq I$. Then

$$f\left((m + M)\Phi(I) - \sum_{i=1}^n \Phi_i(A_i)\right) \leq f(m) + f(M) - \sum_{i=1}^n \Phi_i(f(A_i))$$

for all self-adjoint operators A_i with spectra in $[m, M]$.

Proof. Assume that A_1, \dots, A_n are self-adjoint operators on \mathcal{H} with spectra in $[m, M]$ and Φ_1, \dots, Φ_n are positive linear maps on $\mathbb{B}(\mathcal{H})$ with $0 < \sum_{i=1}^n \Phi_i(I) \leq I$. For $A, B \in \mathbb{B}(\mathcal{H})$ assume that $A \oplus B$ is the operator defined on $\mathbb{B}(\mathcal{H} \oplus \mathcal{H})$ by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. Now apply Theorem 3.5 to the self-adjoint operator A on the Hilbert space $\mathcal{H} \oplus \dots \oplus \mathcal{H}$ defined by $A = A_1 \oplus \dots \oplus A_n$ and the positive linear map Φ defined on $\mathbb{B}(\mathcal{H} \oplus \dots \oplus \mathcal{H})$ by $\Phi(A) = \Phi_1(A_1) \oplus \dots \oplus \Phi_n(A_n)$. \square

The next theorem yields a Jensen–Mercer operator type inequality for separately operator convex functions.

THEOREM 3.7. *Let $f : [m, M] \times [m', M'] \rightarrow \mathbb{R}$ be a separately operator convex function. Let Φ_i, Ψ_j , ($1 \leq i \leq r, 1 \leq j \leq k$) be positive linear maps on $\mathbb{B}(\mathcal{H})$ with $\sum_{i=1}^r \Phi_i(I) = I = \sum_{j=1}^k \Psi_j(I)$. Then*

$$\begin{aligned} & f \left(M + m - \sum_{i=1}^r \Phi_i(A_i), M' + m' - \sum_{j=1}^k \Psi_j(B_j) \right) \\ & \leq f(m, m') + f(m, M') + f(M, m') + f(M, M') - 2f \left(\sum_{i=1}^r \Phi_i(A_i), \frac{M' + m'}{2} \right) \\ & \quad - 2f \left(\frac{m + M}{2}, \sum_{j=1}^k \Psi_j(B_j) \right) + f \left(\sum_{i=1}^r \Phi_i(A_i), \sum_{j=1}^k \Psi_j(B_j) \right) \end{aligned}$$

for all self-adjoint operators A_i with spectra in $[m, M]$ and B_j with spectra in $[m', M']$.

Proof. Since f is separately convex, we have

$$(3.5) \quad f(m + M - t, s) \leq f(m, s) + f(M, s) - f(t, s),$$

and

$$(3.6) \quad f(t, M' + m' - s) \leq f(t, m') + f(t, M') - f(t, s)$$

for all $t \in [m, M]$ and $s \in [m', M']$. Adding inequalities (3.5) and (3.6) we obtain

$$(3.7) \quad \begin{aligned} f(t, s) & \leq \frac{1}{2} [f(m, s) + f(M, s) + f(t, m') + f(t, M') \\ & \quad - f(t, M' + m' - s) - f(m + M - t, s)], \end{aligned}$$

for all $t \in [m, M]$ and $s \in [m', M']$. Using functional calculus for inequality (3.7) we get

$$(3.8) \quad \begin{aligned} & f \left(M + m - \sum_{i=1}^r \Phi_i(A_i), M' + m' - \sum_{j=1}^k \Psi_j(B_j) \right) \\ & \leq \frac{1}{2} \left[f \left(m, M' + m' - \sum_{j=1}^k \Psi_j(B_j) \right) + f \left(M, M' + m' - \sum_{j=1}^k \Psi_j(B_j) \right) \right. \\ & \quad + f \left(M + m - \sum_{i=1}^r \Phi_i(A_i), m' \right) + f \left(M + m - \sum_{i=1}^r \Phi_i(A_i), M' \right) \\ & \quad \left. - f \left(M + m - \sum_{i=1}^r \Phi_i(A_i), \sum_{j=1}^k \Psi_j(B_j) \right) - f \left(\sum_{i=1}^r \Phi_i(A_i), M' + m' - \sum_{j=1}^k \Psi_j(B_j) \right) \right]. \end{aligned}$$

Since f is separately convex, the functions g_s and h_t defined by $g_s(t) = f(t, s) = h_t(s)$ are convex on $[m, M]$ and $[m', M']$ respectively. It follows from (1.4) that

$$(3.9) \quad f \left(m, M' + m' - \sum_{j=1}^k \Psi_j(B_j) \right) \leq f(m, m') + f(m, M') - \sum_{j=1}^k \Psi_j(f(m, B_j)),$$

$$(3.10) \quad f \left(M, M' + m' - \sum_{j=1}^k \Psi_j(B_j) \right) \leq f(M, m') + f(M, M') - \sum_{j=1}^k \Psi_j(f(M, B_j)),$$

$$(3.11) \quad f \left(M + m - \sum_{i=1}^r \Phi_i(A_i), m' \right) \leq f(m, m') + f(M, m') - \sum_{i=1}^r \Phi_i(f(A_i, m')),$$

$$(3.12) \quad f \left(M + m - \sum_{i=1}^r \Phi_i(A_i), M' \right) \leq f(m, M') + f(M, M') - \sum_{i=1}^r \Phi_i(f(A_i, M')).$$

Summing inequalities (3.9), (3.10), (3.11) and (3.12) we obtain

$$\begin{aligned} & f \left(m, M' + m' - \sum_{j=1}^k \Psi_j(B_j) \right) + f \left(M, M' + m' - \sum_{j=1}^k \Psi_j(B_j) \right) \\ & + f \left(M + m - \sum_{i=1}^r \Phi_i(A_i), m' \right) + f \left(M + m - \sum_{i=1}^r \Phi_i(A_i), M' \right) \\ & \leq 2 [f(m, m') + f(m, M') + f(M, m') + f(M, M')] \\ & \quad - \sum_{j=1}^k \Psi_j(f(m, B_j) + f(M, B_j)) - \sum_{i=1}^r \Phi_i(f(A_i, M') + f(A_i, m')) \\ & \leq 2 [f(m, m') + f(m, M') + f(M, m') + f(M, M')] \quad (\text{by the convexity}) \\ & \quad - 2 \left[\sum_{j=1}^k \Psi_j \left(f \left(\frac{m+M}{2}, B_j \right) \right) + \sum_{i=1}^r \Phi_i \left(f \left(A_i, \frac{M'+m'}{2} \right) \right) \right] \\ & \leq 2 [f(m, m') + f(m, M') + f(M, m') + f(M, M')] \quad (\text{by the operator convexity}) \\ (3.13) \quad & - 2 \left[f \left(\frac{m+M}{2}, \sum_{j=1}^k \Psi_j(B_j) \right) + f \left(\sum_{i=1}^r \Phi_i(A_i), \frac{M'+m'}{2} \right) \right]. \end{aligned}$$

Also, since f is separately operator convex, we have

$$\frac{1}{2} f \left(M + m - \sum_{i=1}^r \Phi_i(A_i), \sum_{j=1}^k \Psi_j(B_j) \right) + \frac{1}{2} f \left(\sum_{i=1}^r \Phi_i(A_i), \sum_{j=1}^k \Psi_j(B_j) \right)$$

$$(3.14) \quad \geq f \left(\frac{M+m}{2}, \sum_{j=1}^k \Psi_j(B_j) \right),$$

and

$$(3.15) \quad \frac{1}{2} f \left(\sum_{i=1}^r \Phi_i(A_i), M' + m' - \sum_{j=1}^k \Psi_j(B_j) \right) + \frac{1}{2} f \left(\sum_{i=1}^r \Phi_i(A_i), \sum_{j=1}^k \Psi_j(B_j) \right) \\ \geq f \left(\sum_{i=1}^r \Phi_i(A_i), \frac{M' + m'}{2} \right).$$

Combining inequalities (3.8), (3.13), (3.14) and (3.15), one can easily conclude the desired result. \square

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