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COLUMN SPACE DECOMPOSITION AND PARTIAL ORDER ON MATRICES*

N. EAGAMBARAM†, K. MANJUNATHA PRASAD‡, AND K.S. MOHANA§

Abstract. Motivated by the observation that there exists one-to-one correspondence between column space decompositions and row space decompositions of a matrix, the class of matrices dominated by this matrix under '$\leq$' is characterized in terms of characteristic of column space decompositions, where $\leq$ is a matrix partial order such as the star partial order, the sharp partial order, and the core partial order. The dominance property of the minus partial order over the other partial orders in the discussion resulted in providing a new definition of shorted matrix of a matrix with respect to column space decomposition. Also, extensions of a few results given in [O.M. Baksalary and G. Trenkler. Core inverse of matrices. *Linear Multilinear Algebra*, 58:681–697, 2010.] are presented in this paper.

Key words. Generalized inverse, Matrix partial order, Minus partial order, Star partial order, Core partial order, Space decomposition.

AMS subject classifications. 15A09, 15A24, 15B57.

1. Introduction. In the literature, we have several matrix partial orders defined on different subclasses of rectangular matrices of the same size, and the '*Minus Partial Order*' is a prominent among them [6, 23]. Many other matrix partial orders on the subclasses of rectangular matrices of the same size, defined earlier and also in sequel to these papers, are dominated by the minus partial order. The star [5], sharp [15] and core [3] partial orders are some examples of such matrix partial orders. It is well known that a pair of $m \times n$ matrices B and $A-B$ decompose the matrix A with reference to the minus partial order (i.e., $B,\ A-B \leq^{-} A$) if and only if the column spaces of B and $A-B$ decompose the column space of the matrix A (i.e., $\mathscr{C}(B) \oplus \mathscr{C}(A-B) = \mathscr{C}(A)$). The same is true with reference to the row spaces. In fact, given a matrix A, there is one-to-one correspondence between its matrix decompositions with reference to the minus partial order, its column space decompositions, and the row space decompositions. Some characterizations of certain matrix partial orders, such as the star and sharp partial orders, are given by the characteristics of both column space and row space decompositions. In fact, matrix B is less than A under the star partial order if and only if the column spaces of B and $A-B$ decompose the column space of A orthogonally, and row spaces of B and $A-B$ decompose the row space of A orthogonally. Even the shorted matrix of a matrix [1, 14, 18] is defined with respect to subspaces of column space and row

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space of the given matrix. Note that in [1], the choice of row subspace in the construction of a shorted matrix was obvious from the selection of column subspace, as the matrices in discussion were symmetric (in fact, positive semidefinite matrices), but the same is not true when the matrices of our interest are not symmetric. Now, in the light of the one-to-one correspondence between the column space decompositions and the row space decompositions of a given matrix A, we shall characterize all those matrices dominated by A with reference to star partial order (similarly, with reference to the sharp and core partial orders) in terms of characteristics of column space decomposition alone. In other words, with reference to a matrix partial order under consideration, we shall characterize the subspaces of column space of A corresponding to the matrix decompositions of A.

2. Preliminaries.

2.1. Matrices and generalized inverses. By notation $\mathbb{C}^n$, we denote the vector space of all n-tuples (usually written columnwise) with entries from the field of complex numbers $\mathbb{C}$. The symbols $\mathbb{C}^{m\times n}$ and $\mathrm{Mat}(\mathbb{C})$ denote the set of all $m\times n$ matrices and the class of all matrices, respectively, over $\mathbb{C}$. The dimension of a vector space $\mathscr{V}$ is denoted by $\mathscr{D}(\mathscr{V})$. The vector space spanned by the columns of A is called the *column space (column span)* of A and denoted by $\mathscr{C}(A)$. The *row space (row span)* of A denoted by $\mathscr{R}(A)$ is $\mathscr{C}(A^*)$, where A^* represents the conjugate transpose of A. The dimension of column space (which is incidentally the same as the dimension of row space) is called the *rank* of A and is denoted by $\rho(A)$. The *null space (kernel)* of a matrix A, denoted by $\mathscr{K}(A)$, is a subspace $\{x\in\mathbb{C}^n : Ax=0\}$. A matrix A is said to be a *Range Hermitian* or an *EP matrix* if $\mathscr{C}(A)=\mathscr{R}(A)$.

Given a matrix $A\in\mathbb{C}^{m\times n}$, we shall consider the following Moore–Penrose equations:

$$(1)\quad AXA=A,\quad (2)\quad XAX=X,\quad (3)\quad (AX)^*=AX,\quad (4)\quad (XA)^*=XA.$$

A matrix X satisfying (1) is called a *generalized inverse* or *1-inverse* or a *g-inverse* or sometimes an *inner inverse* of A. An arbitrary g-inverse is denoted by A^-. Similarly, a matrix X satisfying the matrix equation (2) is called a *2-inverse* or an *outer inverse* of A. An arbitrary outer inverse is denoted by $A^=$. A matrix X satisfying both (1) and (2) is called a *reflexive generalized inverse* or *(1,2)-inverse* of A. The *Moore–Penrose inverse* of A, denoted by A^+, is the matrix X satisfying (1)–(4) of the Moore–Penrose equations. For $A\in\mathrm{Mat}(\mathbb{C})$, the Moore–Penrose inverse A^+ always exists and is unique.

Whenever $m=n$, i.e., for a square matrix A, we shall consider the following matrix equations in addition to the Moore–Penrose equations:

$$(5)\quad AX=XA,\quad (1^k)\quad XA^{k+1}=A^k.$$

A square matrix X satisfying (1), (2) and (5) is called the *group inverse* (denoted by $A^{\#}$) of A, and X satisfying (2), (5) and (1^k), for some integer k, is called the *Drazin inverse* of

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A (denoted by A^D). The group inverse $A^\#$, when it exists, is unique with the property that $\mathscr{C}(A^\#)=\mathscr{C}(A)$ and $\mathscr{R}(A^\#)=\mathscr{R}(A)$. A matrix X satisfying (2), (3) and (1^k) for $k=1$ is called the *core–EP generalized inverse* of A (or simply, the *core inverse* as termed by Baksalary–Trenkler [3]). The core-EP generalized inverse of A, denoted by $A^{\textcircled{\#}}$ when it exists, is an outer inverse whose column space and row space are identical, and equal to the column space of A. For $A\in\mathbb{C}^{n\times n}$ with index one, both $A^\#$ and $A^{\textcircled{\#}}$ exist. For the details regarding the core inverse, readers are referred to [3, 11].

DEFINITION 2.1 (Space Decomposition). For any two subspaces $\mathscr{U}$ and $\mathscr{V}$, the sum $\mathscr{W}=\mathscr{U}+\mathscr{V}$ is said to be a *direct sum* and we write $\mathscr{W}=\mathscr{U}\oplus\mathscr{V}$ if $\mathscr{U}\cap\mathscr{V}=(0)$. In such a case, we say that $\mathscr{U}\oplus\mathscr{V}$ is a *space decomposition* of $\mathscr{W}$. By $\mathscr{U}\bigoplus\!\!\!\!\!\perp\mathscr{V}=\mathscr{W}$ we mean that $\mathscr{U}$ and $\mathscr{V}$ decomposes $\mathscr{W}$ orthogonally, i.e., $\mathscr{U}$ and $\mathscr{V}$ are orthogonal to each other and $\mathscr{U}+\mathscr{V}=\mathscr{W}$.

DEFINITION 2.2 (Disjoint Matrices). Given $B,C\in\mathbb{C}^{m\times n}$ are said to be *disjoint matrices* if $\mathscr{C}(B)\cap\mathscr{C}(C)=(0)$ and $\mathscr{R}(B)\cap\mathscr{R}(C)=(0)$.

The results in the following lemmas are quite elementary, but useful at several places in the present paper. The 'if part' of the Lemma 2.3 is given in [24, Lemma 2.2.4 (iii)], reader may refer to [10, Theorem 2.7] for a complete proof.

LEMMA 2.3. *Given nonnull matrices $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{p\times n}$, and $C\in\mathbb{C}^{m\times q}$, the following statements are equivalent.*

(i) *BA^-C is invariant for any choice of A^-.*
(ii) *$\mathscr{C}(C)\subseteq\mathscr{C}(A)$ and $\mathscr{R}(B)\subseteq\mathscr{R}(A)$.*

LEMMA 2.4. *Given a matrix $A\in\mathbb{C}^{n\times n}$ of index one, let B be a matrix such that $\mathscr{C}(B)=\mathbf{S}\subseteq\mathscr{C}(A)$. Then $\rho(AB)=\rho(B)=\mathscr{D}(\mathbf{S})$.*

Proof. Proof follows immediately from the fact that $A^\#$ exists, and therefore, $\mathscr{C}(B)=\mathbf{S}\subseteq\mathscr{C}(A)$ implies $B=AA^\#B=A^\#AB$. ☐

Readers are referred to [4, 24] for further reading on the fundamentals of matrices and generalized inverses of matrices.

2.2. Partial order on a set. A relation on a set S is called a *preorder* on S if it satisfies reflexive and transitive properties. A preorder that is also an antisymmetric is called a *partial order.* A set S associated with a partial order (preorder; total order) $\leq$ is called a *partially ordered* (preordered; ordered) set. A partially ordered set is also known as a '*poset*' in short. A partial order $\leq_1$ on a set S_1 is said to be an *extended partial order* of a partial order $\leq_2$ on S_2, if $S_2\subseteq S_1$ and for $a,b\in S_2$,

$$b\leq_1 a\Leftrightarrow b\leq_2 a.$$

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A partial order $\leq_2$ on a set S_2 is said to be *dominated* by a partial order $\leq_1$ defined on S_1 if $S_2 \subseteq S_1$ and for $a,b \in S_2$,

$$b \leq_2 a \Rightarrow b \leq_1 a.$$

In such a case, we say that $\leq_1$ is a dominant partial order with reference to other partial order $\leq_2$.

Let $(S,\leq)$ be a partially ordered set and let $S_1 \subseteq S$. An element $x \in S_1$ is said to be a *maximal (minimal)* element in S_1 if from $x \leq y$ $(y \leq x)$ for any $y \in S_1$ it follows that $x = y$. An element $z \in S$ is said to be a *lower bound* (an *upper bound*) of S_1 if $z \leq x$ $(x \leq z)$ for every $x \in S_1$. The *greatest lower bound* (also known as *infimum*) of S_1, when it exists, is the unique maximal element in the set of lower bounds of S_1. The g.l.b., i.e., the greatest lower bound of S_1, when it exists, is denoted by $\wedge S_1$ or $\inf S_1$. Similarly, the *least upper bound* (also known as *supremum*) of S_1, when it exists, is the unique minimal element in the set of all upper bounds of S_1. The l.u.b., i.e., the least upper bound of S_1, when it exists, is denoted by $\vee S_1$ or $\sup S_1$. Whenever $S_1 = \{x_1,x_2\}$ is a two-elements set, $\wedge S_1$ and $\vee S_1$ are also denoted by $x_1 \wedge x_2$ and $x_1 \vee x_2$, respectively. A poset $(S,\leq)$ is said to be a *lattice* if $x_1 \wedge x_2$ and $x_1 \vee x_2$ are well defined for every pair of x_1,x_2 in S.

2.3. Minus partial order. Löwner [9] introduced a partial order on the class of nonnegative definite matrices defined by the relation $B \leq_L A$ whenever $A - B$ is nonnegative definite. The same has been extended to the class of all Hermitian matrices. This partial order was called the Löwner order. Subsequently, several other matrix partial orders on the different subclasses of rectangular matrices have been studied. The star, minus, sharp, and core partial orders are a few amongst such prominent matrix partial orders. Readers are referred to [17] for further reading on matrix partial orders.

Inspired by Drazin's work [5] on the star partial order defined on a semigroup with proper involution, Hartwig [6] introduced plus partial order (later renamed as minus partial order or simply minus order) on the set of regular elements in a semigroup. In the context of matrices, it was seen that $B \leq^* A$ (B related to A under the star relation) is equivalent to the conditions $B^+A = B^+B$ and $AB^+ = BB^+$. The question was whether the Moore–Penrose inverse was really necessary in the conditions to define a partial order on the set of regular elements in a semigroup. Hartwig in [6] replaced the Moore–Penrose inverse with a reflexive g-inverse in the conditions, and found that just a generalized inverse would suffice to obtain a partial order.

DEFINITION 2.5 (Minus Order. Hartwig [6], and Nambooripad [23]). The *minus order on the set* $\mathbb{C}^{m\times n}$ is a relation '$\leq^-$' defined by $B \leq^- A$ if

$$B^-A = B^-B,$$
$$AB^- = BB^-$$

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for some choice of B^-. In such a case, we say that the matrix B is less than the matrix A under the minus order.

The following lemma provides different characterizations of the minus partial order. For the proof, we refer to [14]. In fact, the equivalence of (i), (v) and (vi) in the context of left and right ideals in a regular ring is proved in [8].

LEMMA 2.6. *For $A, B \in \mathbb{C}^{m\times n}$, the following statements are equivalent.*

(i) *B and $A-B$ are disjoint matrices,*
i.e., $\mathscr{C}(B)\cap\mathscr{C}(A-B) = (0)$ and $\mathscr{R}(B)\cap\mathscr{R}(A-B) = (0)$.
(ii) $\rho(B)+\rho(A-B) = \rho(A)$.
(iii) $B \leq^- A$.
(iv) $\{A^-\} \subseteq \{B^-\}$.
(v) $\mathscr{C}(B)\oplus\mathscr{C}(A-B) = \mathscr{C}(A)$.
(vi) $\mathscr{R}(B)\oplus\mathscr{R}(A-B) = \mathscr{R}(A)$.

Let $\mathbf{S} = \mathscr{C}(E)$ and $\mathbf{T} = \mathscr{R}(F)$, for some $E \in \mathbb{C}^{m\times p}$ and $F \in \mathbb{C}^{q\times n}$. Define the set

$$\begin{aligned} \mathfrak{C} &= \{C \in \mathbb{C}^{m\times n} : \mathscr{C}(C) \subseteq \mathbf{S}, \mathscr{R}(C) \subseteq \mathbf{T}\} \\ &= \{C = EXF : X \in \mathbb{C}^{p\times q}\}. \end{aligned}$$

DEFINITION 2.7 (Shorted Matrix. Mitra–Puri [21, 22]). A matrix $B \in \mathfrak{C}$ is called a *shorted matrix of A relative to* $\mathbf{S}$ *and* $\mathbf{T}$, and denoted by $\mathbb{S}[A|\mathbf{S},\mathbf{T}]$ or $\mathbb{S}[A|E,F]$ if

$$\rho(A-B) = \min_{C\in\mathfrak{C}} \rho(A-C). \tag{2.1}$$

An equivalent maximality condition [18] for the minimality condition given in (2.1) is

$$B = \max_{C\in\mathfrak{C}} \{C \leq^- A\}. \tag{2.2}$$

A shorted matrix, as defined above, is known to be unique under certain regularity conditions. The shorted matrix could also be unique in some pathological situations, where the regularity conditions fail [16]. When regularity conditions hold, the unique shorted matrix has many attractive properties [13, 14, 22]. Some of these properties are lost when the shorted matrix is not unique.

The definition of shorted matrix as given in (2.2) extends the notion of shorted operator introduced by Anderson and Trapp. Initially, Anderson–Trapp [2] (also see [1]) introduced the concept of shorted operator as an operator satisfying certain maximal property. The study of shorted operator has significance in the context of electrical network theory. If A is the impedance matrix of a resistive n-port network, then A_s is the impedance matrix of the network obtained by shorting the last $n-s$ ports; thus, we call A_s a shorted operator. This shorted

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operator satisfies the maximal properties under the Löwner partial order. The shorted operator defined under the Löwner partial order and the shorted matrix under the minus partial order coincide when we confine to the class of positive semidefinite matrices. The notion of shorted operator has very interesting interpretation in the linear model [19, 20].

3. Minus order and space decomposition. By the definition of the minus partial order and its properties as stated in Lemma 2.6, we have the following lemma.

LEMMA 3.1. *Let $B \leq^- A$ and let C be any matrix such that $\mathscr{C}(C) = \mathscr{C}(B)$ and $\mathscr{R}(C) = \mathscr{R}(B)$ (i.e., B and C are space equivalent). Then $C \leq^- A$ implies $B = C$. In other words, B is the unique shorted matrix of A with respect to its column space and row space.*

Proof. By Lemma 2.6, we get that A^- is a g-inverse for both B and C. So, $\mathscr{C}(C) = \mathscr{C}(B)$ implies $(CA^-)B = B$, and $\mathscr{R}(C) = \mathscr{R}(B)$ implies $C(A^-B) = C$. □

For every column space decomposition $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$, the following statements are easily verified.

- (i) There exists a unique matrix $B \leq^- A$ such that $\mathscr{C}(B) = \mathbf{S}$ and $\mathscr{C}(A-B) = \mathbf{S}'$ (Proof is immediate from the observation that every $x \in \mathscr{C}(A)$ is uniquely written as $x = y + z$, where $y \in \mathbf{S}$ and $z \in \mathbf{S}'$).
- (ii) There exists a unique row space decomposition $\mathbf{T} \oplus \mathbf{T}'$ corresponding to the given column space decomposition, which is associated uniquely with $B \leq^- A$ satisfying (i) above.

So, we have the following corollary.

COROLLARY 3.2. *There is one-to-one correspondence between the set of all possible column space decompositions $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$ and the set of all possible row space decompositions $\mathbf{T} \oplus \mathbf{T}' = \mathscr{R}(A)$, where the correspondence is uniquely determined by the matrix B that is less than A under the minus order such that*

$$\mathscr{C}(B) = \mathbf{S} \quad \textit{and} \quad \mathscr{C}(A-B) = \mathbf{S}'$$

and

$$\mathscr{R}(B) = \mathbf{T} \quad \textit{and} \quad \mathscr{R}(A-B) = \mathbf{T}'.$$

For a nontrivial subspace $\mathbf{S}$ of $\mathscr{C}(A)$, there are infinitely many choices of $\mathbf{S}'$ such that $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$ and so there are infinitely many choices of the matrices B such that $B \leq^- A$ and $\mathscr{C}(B) = \mathbf{S}$. It would be very interesting to observe that for any fixed $\mathbf{S} \subseteq \mathscr{C}(A)$, we have infinitely many choices of B such that $\mathscr{C}(B) = \mathbf{S}$ and $B \leq^- A$, but $\mathscr{R}(A-B)$ is the same for all those choices of B. We shall prove this result in Corollary 3.5. For further development of discussion, it is necessary to recall the following concept of separability.

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DEFINITION 3.3 (Separability of a pair of subspaces **S** and **T**. Mitra [16]). Given $A \in \mathbb{C}^{m\times n}$, two subspaces $\mathbf{S} \subseteq \mathscr{C}(A)$ and $\mathbf{T} \subseteq \mathscr{R}(A)$ are said to be *separable with reference to the matrix A* if

$$FA^{-}E = 0 \quad \text{for some } A^{-}, \tag{3.1}$$

where $E \in \mathbb{C}^{m\times p}$ and $F \in \mathbb{C}^{q\times n}$ are such that $\mathscr{C}(E) = \mathbf{S}$ and $\mathscr{R}(F) = \mathbf{T}$. Equivalently,

$$y^{*}A^{-}x = 0 \quad \text{for some } A^{-} \tag{3.2}$$

for all $x \in \mathbf{S}$ and $y \in \mathbf{T}$.

Since x and y are in the column space and the row space of A respectively, by Lemma 2.3, (3.2) holds for every choice of A^{-}. The same is true in the case of (3.1) as $\mathscr{C}(E)$ and $\mathscr{R}(F)$ are subspaces of column and row spaces of A, respectively. By saying that **S** is separable with **T** or **T** is separable with **S**, without referring to the matrix A, we mean that **S** and **T** are separable with reference to A. In the following theorem, we prove that there exists a unique maximal separable subspace with reference to a given column subspace. The word 'maximal' in 'maximal separable subspace' is with reference to the partial order defined by '$\subseteq$' on the class of separable subspaces.

THEOREM 3.4. *Given $A \in \mathbb{C}^{m\times n}$ and a subspace $\mathbf{S} \subseteq \mathscr{C}(A)$, we have the following.*

(i) *There exists a unique maximal separable subspace $\mathbf{T}'$ with* **S**.
(ii) *For any oblique projector P onto* **S**, *the maximal separable space $\mathbf{T}'$ as in* (i) *is given by*

$$\mathbf{T}' = \mathscr{R}(QA),$$

where $Q = I - P$.
(iii) *For $\mathbf{T}'$ as in* (i),

$$\mathscr{D}(\mathbf{S}) + \mathscr{D}(\mathbf{T}') = \rho(A).$$

Proof. (i) Let F be a matrix such that $\mathscr{R}(F)$ is separable with **S**, but not maximal. If $y \notin \mathscr{R}(F)$ is separable with **S**, then for E such that $\mathbf{S} = \mathscr{C}(E)$ we have

$$y^{*}A^{-}E = 0 \quad \text{for all } A^{-}.$$

Now for F_1, a matrix obtained by augmenting the row y^{*} with F, it is obvious that $\mathscr{R}(F_1) \supsetneq \mathscr{R}(F)$ but $F_1A^{-}E = 0$. Since $\rho(F_1) > \rho(F)$ (as $y \notin \mathscr{R}(F)$), the existence of a maximal separable subspace with **S** is proved. To prove uniqueness of the maximal separable subspace with **S**, consider the matrices F_1 and F_2 such that $\mathscr{R}(F_1)$ and $\mathscr{R}(F_2)$ are two maximal separable subspaces with **S**. Further, $F_1A^{-}E = 0$ and $F_2A^{-}E = 0$ imply $FA^{-}E = 0$, where F is the matrix obtained by appending the rows of F_2 with rows of F_1. So, $\mathscr{R}(F)$ is separable with **S**. $\mathscr{R}(F_1) \neq \mathscr{R}(F_2)$ (certainly not comparable) implies that $\mathscr{R}(F)$ strictly contains each of

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$\mathscr{R}(F_1)$ and $\mathscr{R}(F_2)$, which contradicts the fact that $\mathscr{R}(F_1)$ and $\mathscr{R}(F_2)$ are maximal separable subspaces with $\mathbf{S}$.

(ii): For a projector P onto $\mathbf{S} \subseteq \mathscr{C}(A)$ and $Q = I - P$, we have $QAA^-P = QP = 0$, proving the separability of $\mathscr{R}(QA)$ with $\mathbf{S} = \mathscr{C}(P)$. Now for $y \in \mathscr{R}(A)$, we can write $y^* = z^*A$ for some z, and $y^* = z^*QA + z^*PA$. Further, $y^*A^-P = 0$, i.e., $y \in \mathbf{T}'$ implies $z^*PAA^-P = z^*P = 0$, and therefore, $y^* = z^*QA$. This proves that $\mathscr{R}(QA) = \mathbf{T}'$ leading to (ii).

(iii): For P and Q as defined in (ii), note that $PA(A^-)PA = PA$. By Lemma 2.6, we get $PA \leq^- A$, and therefore,

$$\rho(PA) + \rho(QA) = \rho(A).$$

Since P is an oblique projector onto $\mathbf{S} \subseteq \mathscr{C}(A)$, we have $\rho(PA) = \rho(P) = \mathscr{D}(\mathbf{S})$. From (ii) of the theorem, we have $\mathbf{T}' = \mathscr{R}(QA)$, and hence, $\mathscr{D}(\mathbf{S}) + \mathscr{D}(\mathbf{T}') = \rho(A)$. ∎

With the observation that the matrices in $\{B : B \leq^- A \text{ and } \mathscr{C}(B) = \mathbf{S}\}$ are obtained by PA for different oblique projectors P onto $\mathbf{S}$, we arrive at the following corollary.

COROLLARY 3.5. *For a fixed subspace $\mathbf{S}$ of $\mathscr{C}(A)$, the subspace $\mathscr{R}(A - B)$ is the same for all choices of B such that $B \leq^- A$ and $\mathscr{C}(B) = \mathbf{S}$. Similarly, for any fixed subspace $\mathbf{T}$ of $\mathscr{R}(A)$, the $\mathscr{C}(A - B)$ is the same for all choices of B such that $B \leq^- A$ with $\mathscr{R}(B) = \mathbf{T}$.*

In the following theorem, for a column space decomposition $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$, we give an explicit expression for the matrix B satisfying $B \leq^- A$, $\mathscr{C}(B) = \mathbf{S}$ and $\mathscr{C}(A - B) = \mathbf{S}'$.

THEOREM 3.6. *Given a matrix $A \in \mathbb{C}^{m\times n}$, let $E_1 \in \mathbb{C}^{m\times p}$ and $E_2 \in \mathbb{C}^{m\times q}$ be any two matrices such that $\mathscr{C}(E_1) = \mathbf{S}$ and $\mathscr{C}(E_2) = \mathbf{S}'$, where $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$. Also, let $\mathbf{T}$ and $\mathbf{T}'$ be the subspaces of $\mathscr{R}(A)$ such that $\mathbf{T}$ is maximal separable subspace with $\mathbf{S}'$ and $\mathbf{T}'$ is maximal separable subspace with $\mathbf{S}$. Then we have the following.*

(i) *If y is any vector from $\mathbf{T}$ separable with the entire $\mathbf{S}$, then y is the zero vector. Similarly, the vector from $\mathbf{T}'$ separable with the entire $\mathbf{S}'$ is the zero vector. Further, $\mathbf{T} \oplus \mathbf{T}' = \mathscr{R}(A)$.*

(ii) *The unique matrix B satisfying $B \leq^- A$, $\mathscr{C}(B) = \mathbf{S}$, and $\mathscr{C}(A - B) = \mathbf{S}'$ is given by*

$$B = E_1(F_1A^-E_1)^-F_1, \tag{3.3}$$

where A^- is an arbitrary g-inverse of A, $F_1 = (I - E_2E_2^-)A$, and E_2^- is an arbitrary g-inverse of E_2. In fact, for the matrix B as given in (3.3) we have

$$\mathscr{R}(B) = \mathbf{T} \quad \textit{and} \quad \mathscr{R}(A - B) = \mathbf{T}'.$$

Proof. (i): Suppose that y is any vector from $\mathbf{T}$ separable with the entire $\mathbf{S}$. Then from the definition of $\mathbf{T}'$, we conclude that y is also from $\mathbf{T}' \subset \mathscr{R}(A)$ which implies that y is separable with the entire $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$. In other words, $y^* = y^*A^-A = 0$, and hence, the zero vector

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is the only vector from $\mathbf{T}$ which is separable with the entire $\mathbf{S}$. In fact, we have proved that $\mathbf{T}\cap\mathbf{T}'=(0)$. Similarly, $\mathbf{T}'$ has no nonzero vector which is separable with $\mathbf{S}'$. Now referring to (iii) of Theorem 3.4, we see that $\mathscr{D}(\mathbf{S})+\mathscr{D}(\mathbf{T}')=\rho(A)=\mathscr{D}(\mathbf{S}')+\mathscr{D}(\mathbf{T})$. Therefore, $\mathbf{T},\mathbf{T}'\subseteq\mathscr{R}(A)$ and $\mathbf{T}\cap\mathbf{T}'=(0)$ imply that $\mathbf{T}\oplus\mathbf{T}'=\mathscr{R}(A)$.

(ii): For an arbitrary E_2^-, the matrix $E_2E_2^-$ is an oblique projector onto $\mathbf{S}'$. Define $F_1=(I-E_2E_2^-)A$. Now, from the part (ii) of Theorem 3.4, we obtain $\mathbf{T}=\mathscr{R}(F_1)$. Again, referring to the part (iii) of Theorem 3.4, we see that

$$\rho(A)=\mathscr{D}(\mathbf{S}')+\mathscr{D}(\mathbf{T}),$$

and this implies $\mathscr{D}(\mathbf{T})=\mathscr{D}(\mathbf{S})$. In other words, $\rho(E_1)=\rho(F_1)$. Since $\mathscr{C}(E_1)=\mathbf{S}\subseteq\mathscr{C}(A)$, we have $AA^-E_1=E_1$ and

$$F_1A^-E_1=(I-E_2E_2^-)AA^-E_1=(I-E_2E_2^-)E_1.$$

Also note that the invariance of $F_1A^-E_1$ follows from Lemma 2.3, and the definitions of the matrices E_1 and F_1. Since $\mathbf{S}$ and $\mathbf{S}'=\mathscr{K}(I-E_2E_2^-)$ are disjoint, we get

$$\rho(F_1A^-E_1)=\rho((I-E_2E_2^-)E_1)=\rho(E_1)=\rho(F_1).$$

So, $\mathscr{C}(F_1A^-E_1)=\mathscr{C}(F_1)$ and $\mathscr{R}(F_1A^-E_1)=\mathscr{R}(E_1)$. Now referring to Lemma 2.3, we get that the matrix $B=E_1(F_1A^-E_1)^-F_1$ is uniquely determined. From this expression for B, it is obvious that $\rho(B)\le\rho(F_1),\rho(E_1)$. By postmultiplying both sides of $B=E_1(F_1A^-E_1)^-F_1$ by A^-E_1, we get $BA^-E_1=E_1(F_1A^-E_1)^-F_1(A^-E)=E_1(F_1A^-E_1)^-(F_1A^-E)=E_1$, and hence, $\rho(E_1)\le\rho(B)$. Therefore $\rho(B)=\rho(E_1)=\rho(F_1)$. Now consider an arbitrary A^- and observe that

$$BA^-B=E_1(F_1A^-E_1)^-F_1A^-E_1(F_1A^-E_1)^-F_1=E_1(F_1A^-E_1)^-F_1=B.$$

Therefore, B is a matrix such that $B\le^- A$ (by Lemma 2.6), $\mathscr{C}(B)=\mathbf{S}$, and $\mathscr{R}(B)=\mathbf{T}$. Since $\mathscr{C}(E_2)$ is the maximal separable subspace with $\mathbf{T}=\mathscr{R}(B)$, by Corollary 3.5 we get that $\mathscr{C}(A-B)=\mathbf{S}'$, and similarly, $\mathscr{R}(A-B)=\mathbf{T}'$. $\square$

REMARK 3.7. Note that the expression we obtained for B in (3.3) is exactly the same as the unique shorted matrix $\mathbb{S}[A|\mathbf{S},\mathbf{T}]$ with the regularity condition $\rho(FA^-E)=\rho(F)=\rho(E)$, where $\mathscr{C}(E)=\mathbf{S}$ and $\mathscr{R}(F)=\mathbf{T}$ as in [18]. This prompts us to define a regular shorted matrix with respect to a column space decomposition. The use of the word 'regular' is due to the condition, under which we discuss the shorted matrix, equivalent to the regularity conditions discussed in the literature with reference to uniqueness of shorted matrix.

We shall consider any partial order $\le$ dominated by $\le^-$ on a subclass $\mathbb{P}$ of $\mathbb{C}^{m\times n}$. For a matrix A and its column space decomposition $\mathbf{S}\oplus\mathbf{S}'=\mathscr{C}(A)$, define

$$\mathfrak{C}=\{C\in\mathbb{C}^{m\times n}:\mathscr{C}(C)\subseteq\mathbf{S}\ \text{ and }\ \mathscr{C}(A-C)\supseteq\mathbf{S}'\}.$$

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DEFINITION 3.8 (Shorted matrix with respect to a column space decomposition). Given a matrix partial order $\leq$ on $\mathbb{P}$, $A \in \mathbb{P}$ and subspace $\mathbf{S}$ of $\mathscr{C}(A)$, *a shorted matrix of A with respect to a column space decomposition* $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$ is a maximal element

$$\mathfrak{S}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}'] = \max_{C \in \mathfrak{C}} \{C \leq A\}.$$

A *regular shorted matrix* of A with respect to a column space decomposition $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$, denoted by $\mathfrak{R}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ when it exists, is the matrix $B = \mathfrak{S}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ with $\mathscr{C}(B) = \mathbf{S}$ and $\mathscr{C}(A-B) = \mathbf{S}'$.

Though we prefer to denote the above shorted matrix by $\mathfrak{S}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ whenever the matrix partial order under discussion is clear by context, sometimes we may denote the same also by $\mathfrak{S}[A, \leq |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$, particularly when we involve more than one partial order in the discussion. Similarly, for a given $\mathbf{S}$ if the choice of possible decomposition $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$ is unique due to the property of $\leq$, we write $\mathfrak{S}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ simply as $\mathfrak{S}[A|\mathbf{S}]$ or $\mathfrak{S}[A, \leq |\mathbf{S}]$. Also, in the case of regular shorted matrix $\mathfrak{R}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$, we use the notations $\mathfrak{R}[A, \leq |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$, $\mathfrak{R}[A|\mathbf{S}]$ and $\mathfrak{R}[A, \leq |\mathbf{S}]$ for convenience.

REMARK 3.9. If the partial order $\leq$ is the same as the minus partial order $\leq^-$, then by Theorem 3.6, $\mathfrak{S}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ uniquely exists and is given by the expression (3.3). In fact, $\mathfrak{S}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ is the matrix B such that $B \leq^- A$, $\mathscr{C}(B) = \mathbf{S}$, and $\mathscr{C}(A-B) = \mathbf{S}'$. In other words, $\mathfrak{S}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}'] = \mathfrak{R}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ for every choice of decomposition $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$. Also in this case, $\mathfrak{R}[A|\mathbf{S}', \mathbf{S} \oplus \mathbf{S}'] = A - B$.

EXAMPLE 3.10. Consider $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, a nonsingular matrix of size 2×2. Let $E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Now for $\mathscr{C}(E_1) = \mathbf{S}$ and $\mathscr{C}(E_2) = \mathbf{S}'$, we have $S \oplus S' = \mathscr{C}(A) = \mathbb{C}^2$. To find a matrix $B = \mathfrak{S}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$, we shall first compute the following matrices:

$$E_2 E_2^- = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad F_1 = (I - E_2 E_2^-)A = \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix},$$

where $E_2^- = \begin{bmatrix} 0 & 1 \end{bmatrix}$ is a choice of a g-inverse for E_2. Now, $F_1 A^- E_1 = F_1 A^{-1} E_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and by taking $(F_1 A^- E_1)^- = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}$, we obtain

$$B = E_1 (F_1 A^- E_1)^- F_1 = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

It is easily verified that $BA^{-1}B = B$, and therefore, the matrix B so obtained is below A under minus partial order. From the structure of B and $A - B$ it is clear that $\mathscr{C}(B) = \mathbf{S}$ and $\mathscr{C}(A-B) = \mathbf{S}'$.

Now, the natural problem is as follows.

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Problem. If a matrix partial order $\leq$ is dominated by $\leq^-$, characterize all the subspaces $\mathbf{S}$ of $\mathscr{C}(A)$, (and $\mathbf{S}'$) for which there exists a matrix $B \leq A$ such that $\mathscr{C}(B) = \mathbf{S}$ (and $\mathscr{C}(A-B) = \mathbf{S}'$).

In the following sections, we shall address this problem for the cases of the star, sharp, and core orders.

4. Star partial order and column space decomposition. Drazin [5] was the first to notice that the relation $\leq^*$, as defined in (4.1), is a partial order on a semigroup with proper involution and, Hartwig-Drazin [7] were the first to call this the star-order.

DEFINITION 4.1 ($*$ - relation on $\mathbb{C}^{m\times n}$). Given matrices $A, B \in \mathbb{C}^{m\times n}$, we write $B \leq^* A$ if

$$B^*B = B^*A \quad \text{and} \quad BB^* = AB^*. \tag{4.1}$$

$\leq^*$ defined above is a partial order on $\mathbb{C}^{m\times n}$ and is called *the star partial order* or simply *the star-order*.

The star partial order has some interesting lattice properties and we refer to [7, 12, 17] for further reading.

The star partial order is dominated by the minus partial order ($B \leq^* A \Rightarrow B \leq^- A$) and the proof follows from Definitions 2.5 and 4.1, and the equivalence of (iii) and (iv) of Lemma 4.2. Proof of the following lemma is immediate from the space equivalence of B^* and B^+.

LEMMA 4.2. *For $A, B \in \mathbb{C}^{m\times n}$, the following statements are equivalent.*

(i) $\mathscr{C}(B) \perp \mathscr{C}(A-B)$ *and* $\mathscr{R}(B) \perp \mathscr{R}(A-B)$.
(ii) $B^*(A-B) = (A-B)B^* = 0$.
(iii) $B^*A = B^*B$ *and* $AB^* = BB^*$.
(iv) $B^+A = B^+B$ *and* $AB^+ = BB^+$.

The above lemma presents different characterizations of B for which $B \leq^* A$. In fact, (i) of Lemma 4.2 characterizes B with respect to the column space and row space decompositions of the matrix A. From the one-to-one correspondence between matrix decomposition under the minus partial order, column space decompositions, and row space decompositions, as we have seen in Corollary 3.2, we would like to replace (i) of the above Lemma 4.2 by an appropriate condition on the column space decomposition alone. With reference to the star partial order, unlike in the case of the minus partial order, a matrix decomposition of A corresponding to an arbitrary column space decomposition may not exist. Lemma 4.2 (i) suggests that having chosen $\mathbf{S}$ for $\mathscr{C}(B)$, where $B \leq^* A$, then $\mathbf{S}'$ in the corresponding space decomposition is essentially orthogonal to $\mathbf{S}$. In fact, with reference to $\mathbf{S} \subseteq \mathscr{C}(A)$ there exists a unique orthogonal decomposition of $\mathscr{C}(A)$. So, whenever $\mathbf{S} \bigoplus\!\!\!\!\perp \mathbf{S}' = \mathscr{C}(A)$, we may conveniently write $\mathfrak{S}[A|\mathbf{S}, \mathbf{S} \bigoplus\!\!\!\!\perp \mathbf{S}']$ simply as $\mathfrak{S}[A|\mathbf{S}]$, when it exists. The uniqueness of $\mathfrak{S}[A|\mathbf{S}]$ follows from the subsequent Theorem 4.3, and for the proof of this theorem we refer

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to [12, Theorem 14].

THEOREM 4.3. *Let A be an $m\times n$ complex matrix and $P,Q\in[0,\,A]$, where $[0,\,A]$ represents the set of all $m\times n$ matrices B such that $B\leq^* A$. Then $P\vee Q$ and $P\wedge Q$ are well defined in $[0,\,A]$ with $\mathscr{C}(P\vee Q)=\mathscr{C}(P)+\mathscr{C}(Q)$ and $\mathscr{C}(P\wedge Q)=\mathscr{C}(P)\cap\mathscr{C}(Q)$. In other words, $[0,\,A]$ is a lattice.*

Now we have the following theorem.

THEOREM 4.4. *Given an $m\times n$ matrix A, $\mathfrak{S}[A|\mathbf{S}]$ is unique.*

Proof. For $B,C\leq^* A$, by Theorem 4.3 we get $B\vee C\leq^* A$. Also, $\mathscr{C}(B),\mathscr{C}(C)\subseteq\mathbf{S}$ implies $\mathscr{C}(B\vee C)\subseteq\mathbf{S}$. Hence, the uniqueness of $\mathfrak{S}[A|\mathbf{S}]$ follows. $\square$

Considering a subspace $\mathbf{S}$ and the orthogonal decomposition $\mathbf{S}\bigoplus\mathbf{S}'$ of $\mathscr{C}(A)$, we may not have a corresponding matrix decomposition with reference to the star partial order. In fact, $\mathfrak{S}[A|\mathbf{S}]$ could be the zero matrix. In the following example, we find that for a matrix A and subspace $\mathbf{S}$ of $\mathscr{C}(A)$, there is no matrix B such that $B\leq^* A$ and $\mathscr{C}(B)=\mathbf{S}$. In fact, in this case $\mathfrak{S}[A|\mathbf{S},\mathbf{S}\bigoplus\mathbf{S}']$ is the null matrix.

EXAMPLE 4.5. Consider $A=\begin{bmatrix}1&-2\\-2&1\end{bmatrix}$ and $\mathbf{S}=\mathscr{C}(E)$, where $E=\begin{bmatrix}1\\2\end{bmatrix}$. Note that A is a nonsingular matrix and $\mathscr{C}(E)\subseteq\mathscr{C}(A)$. If B is a nonzero 2×2 matrix of rank one such that $\mathscr{C}(B)=\mathscr{C}(E)$, then $B=\begin{bmatrix}\lambda_1&\lambda_2\\2\lambda_1&2\lambda_2\end{bmatrix}$ for some λ_1,λ_2, not both of them equal to zero. Now, $BB^*=\begin{bmatrix}\lambda_1^2+\lambda_2^2&2(\lambda_1^2+\lambda_2^2)\\2(\lambda_1^2+\lambda_2^2)&4(\lambda_1^2+\lambda_2^2)\end{bmatrix}$, $AB^*=\begin{bmatrix}\lambda_1-2\lambda_2&2\lambda_1-4\lambda_2\\-2\lambda_1+\lambda_2&-4\lambda_1+2\lambda_2\end{bmatrix}$, $B^*B=\begin{bmatrix}5\lambda_1^2&5\lambda_1\lambda_2\\5\lambda_1\lambda_2&5\lambda_2^2\end{bmatrix}$ and $B^*A=\begin{bmatrix}-3\lambda_1&0\\-3\lambda_2&0\end{bmatrix}$. Suppose that $B^*B=B^*A$, in which case $\lambda_2=0$ and $\lambda_1=0$ or $-3/5$. Since B is a nonzero matrix, we are left with the only possibility that $\lambda_2=0$ and $\lambda_1=-3/5$. But, for the matrix B obtained by substituting $\lambda_2=0$ and $\lambda_1=-3/5$, we find that $BB^*\neq AB^*$. Hence, there is no matrix B such that $B\leq^* A$ and $\mathscr{C}(B)=\mathbf{S}$, and in fact, $\mathfrak{S}[A|\mathbf{S}]=0$ in this case.

Now, we shall consider a column space decomposition $\mathbf{S}\oplus\mathbf{S}'=\mathscr{C}(A)$. We know that there exists a unique B such that $B\leq^- A$ and $\mathscr{C}(B)=\mathbf{S}$ corresponding to any column space decomposition considered. The condition $\mathscr{C}(B)\perp\mathscr{C}(A-B)$ holds if and only if (from the unique correspondence of column space decomposition with matrix decomposition) $B=PA$ and $A-B=(I-P)A$ for the orthogonal projection P onto $\mathbf{S}$. Now considering the orthogonality between row spaces of B and $A-B$, we see that $B\leq^* A$ if and only if $PAA^*(I-P)=0$. So, we have the following theorem.

THEOREM 4.6. *For $A\in\mathbb{C}^{m\times n}$, $\mathbf{S}=\mathscr{C}(E_1)\subseteq\mathscr{C}(A)$, and $P=E_1E_1^+$, the orthogonal projector onto $\mathbf{S}$, the following statements are equivalent.*

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(i) *There exists a matrix $B \leq^* A$ such that $\mathscr{C}(B) = \mathbf{S}$.*
(ii) $PAA^*(I-P) = 0$.
(iii) *P commutes with AA^*, i.e., $PAA^* = AA^*P$.*
(iv) *PAA^* is Hermitian.*

With $\mathbf{S}$ and P satisfying any one of the above equivalent conditions, the matrix $B \leq^* A$ such that $\mathscr{C}(B) = \mathbf{S}$ is given by PA.

The following corollary provides further characterizations of $\mathbf{S}$ satisfying the properties stated in the above theorem.

COROLLARY 4.7. *For $A \in \mathbb{C}^{m\times n}$, $\mathbf{S} = \mathscr{C}(E_1) \subseteq \mathscr{C}(A)$, the following statements are equivalent.*

(i) *There exists a matrix $B \leq^* A$ such that $\mathscr{C}(B) = \mathbf{S}$.*
(ii) *$\mathbf{S}$ is an invariant space under AA^* (i.e., $AA^*(\mathbf{S}) \subseteq \mathbf{S}$).*
(iii) *$QAA^*Q = AA^*Q$ for every oblique projector Q onto $\mathbf{S}$.*

Proof. (i) $\Rightarrow$ (ii) follows from (i) $\Rightarrow$ (iii) of Theorem 4.6. (ii) $\Rightarrow$ (iii) follows from the invariance property of $\mathbf{S}$ under AA^* and the fact that Q is an oblique projector onto $\mathbf{S}$.

(iii) $\Rightarrow$ (i): Let Q be the orthogonal projector onto $\mathbf{S}$, in particular, such that $QAA^*Q = AA^*Q$. Clearly, QAA^*Q is Hermitian, and hence, Q commutes with AA^*. Now (i) follows from Theorem 4.6. □

COROLLARY 4.8. *For $A \in \mathbb{C}^{m\times n}$, if $\mathbf{S},\mathbf{T}$ are any two subspaces of $\mathscr{C}(A)$ such that*

(i) *there exist matrices B and C satisfying $B,C \leq^* A$, $\mathscr{C}(B) = \mathbf{S}$ and $\mathscr{C}(C) = \mathbf{T}$,*
(ii) *$\mathbf{S} \perp \mathbf{T}$,*

then $B+C \leq^* A$ and $\rho(B+C) = \rho(B)+\rho(C)$.

Proof. If P and Q are the orthogonal projectors onto $\mathbf{S}$ and $\mathbf{T}$, respectively, from (ii) we get $PQ = QP = 0$, and therefore, $P+Q$ is the orthogonal projector onto the subspace $\mathbf{S}\oplus\mathbf{T}$ of $\mathscr{C}(A)$. Clearly, $\rho(P+Q) = \rho(P)+\rho(Q)$. Now referring to the conditions (i) of the corollary and (iii) of Theorem 4.6, we see that both P and Q commute with AA^*. This in turn means that $P+Q$ also commutes with AA^*, and therefore, $B+C = (P+Q)A \leq^* A$. Additionally, note that $\rho(B+C) = \rho(P+Q) = \rho(P)+\rho(Q) = \rho(B)+\rho(C)$. □

The following corollary follows from Theorem 4.6.

COROLLARY 4.9. *If $E_1 = y$ is a nonzero column vector from $\mathscr{C}(A)$, in other words, $\mathbf{S} = \mathscr{C}(E_1)$ is a one-dimensional subspace, then there exists a matrix B such that $B \leq^* A$ and $\mathscr{C}(B) = \mathbf{S}$ if and only if y is an eigenvector of AA^*.*

Proof. Note that the orthogonal projector P onto $\mathscr{C}(E_1)$, as in Theorem 4.6, is given by

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$\frac{1}{y^*y}yy^*$. Now from (ii) of Theorem 4.6, we get

$$AA^*\frac{1}{y^*y}yy^*(=AA^*P=PAA^*P)=\frac{1}{y^*y}yy^*AA^*\frac{1}{y^*y}yy^*=\frac{1}{(y^*y)^2}yy^*AA^*yy^*.$$

Hence, by postmultiplying by y on both sides, we get that y is an eigenvector of AA^*.

Conversely, if y is an eigenvector of AA^*, it is easily verified that the orthogonal projector $P=\frac{1}{y^*y}yy^*$ commutes with AA^*. Now, by Theorem 4.6 we conclude that $B=PA=\frac{1}{y^*y}yy^*A$ is the matrix of rank one such that $\mathscr{C}(B)=\mathscr{C}(E_1)$ and $B\leq^* A$. $\square$

REMARK 4.10. For $A\in\mathbb{C}^{m\times n}$, we have the following.

(i) Start with an arbitrary eigenvector of AA^* corresponding to a singular value of A, obtain $B_1(\leq^* A)$ of rank one as in Corollary 4.9. Now obtain a matrix B_2 of rank one such that $B_2\leq^*(A-B_1)$ in a similar way. Clearly, $B_1^*B_2=0=B_2^*B_1$. So, by Corollary 4.8 we get $B_1+B_2\leq^* A$. Continue to obtain $B_i\leq^* A-(B_1+\cdots+B_{i-1})$; $i=2,3,\ldots r$, where r is the rank of the matrix A. In fact, A can be written as a sum of matrices B_i $(\leq^* A)$ of rank one such that $B_i^*B_j=0=B_j^*B_i$ $(i\neq j)$. For the benefit of readers, we give the following quick proof.
Note that $B_1^*B_2=0$ and $B_1B_2^*=0$ as $B_2\leq^*(A-B_1)$. By Corollary 4.8, we get that $B_1+B_2\leq^* A$ and $\rho(B_1+B_2)=2$. Having $B_i\leq^* A-(B_1+\cdots+B_{i-1})$ such that $B_k^*B_l=0$ for $k\neq l<i$, we see that $(B_1+\cdots+B_{i-1})B_i^*=0$ implies $B_k^*B_kB_i^*=0$, equivalently, $B_kB_i^*=0$ for all $k<i$. Similarly, $B_kB_l^*=0$ for $k\neq l<i$ together with $B_i^*(B_1+\cdots+B_{i-1})=0$ imply $B_i^*B_l=0$ for all $l<i$. So, the decomposition of matrix A of rank r as a sum of rank one matrices with the said properties follows by induction.

(ii) For the star order, $\mathfrak{S}[A|\mathbf{S}]$ is nonzero if and only if there exists at least one eigenvector of AA^* in $\mathbf{S}$. In fact, $\mathfrak{S}[A|\mathbf{S}]$ is the sum of rank one matrices B_i, $i=1,\ldots,r$, as in Corollary 4.9, corresponding to distinct and orthogonal eigenvectors of AA^* from $\mathbf{S}$.

(iii) The interval of matrices $[0,\ A]$ with reference to the star partial order is finite if and only if there exists a unique choice for the set of orthonormal eigenvectors of AA^*. In other words, nonzero singular values of A are distinct. If the nonzero singular values are not distinct, then the interval $[0,\ A]$ is not a finite set. For example, the interval $[0,\ I]$, where I is the identity matrix, is not a finite set and every orthogonal projector is a member of this interval.

EXAMPLE 4.11. Consider the same matrix $A=\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix}$ as in Example 4.5. Now take $\mathbf{S}=\mathscr{C}(E)$, where $E=\begin{bmatrix}1\\ 1\end{bmatrix}$ and is an eigenvector of AA^*. Further, the Moore–Penrose inverse of E is $E^+=\dfrac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}$ and $P=EE^+=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$. Now, write $B_1=PA=\begin{bmatrix}-\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}$. From the structure of B_1, it is clear that $\mathscr{C}(B_1)=\mathbf{S}$. By direct computa-

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tion, it can be verified that B_1 satisfies $B_1^*B_1 = B_1^*A$ and $B_1B_1^* = AB_1^*$, and hence, B_1 is the matrix with $\mathscr{C}(B_1) = \mathbf{S}$ and $B_1 \leq^* A$.

Now, write $B_2 = A - B_1 = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix}$ and observe that $A = B_1 + B_2$ is a sum of rank one matrices such that $B_1, B_2 \leq^* A$ as in (i) of Remark 4.10.

Observe that A, in this example, has distinct singular values (i.e., 1 and 3). For any matrix B of rank one such that $B \leq^* A$, referring to the construction of B given in the proof of Corollary 4.9, it is one of the matrices B_1 and B_2 we have above. So, the interval $[0, \quad A]$ has exactly four elements (finite), i.e., B_1 and B_2 given above and the trivial elements 0 and A. So, we have demonstrated (iii) of Remark 4.10.

5. Sharp order and core order. Inspired by the role of the Moore–Penrose inverse in providing an alternate definition for the star partial order (see (iv) of Lemma 4.2), Mitra [15] used the group inverse to define the sharp order and later Baksalary–Trenkler [3] used the core-EP generalized inverse to define the core partial order on the class of square matrices of index one. The star, sharp, and core partial orders are defined on the different classes of matrices and are characterized by considering additional conditions on the choice of B^- in the Definition 2.5 of the minus partial order. The star order is specified by taking $B^- = B^+$ – the Moore–Penrose inverse of B in the Definition 2.5, we take $B^\#$ – the group inverse of B in the case of the sharp order, and in the case of the core order it is $B^{\textcircled{\#}}$ – the core–EP generalized inverse of B. So, the minus partial order dominates each of the star, sharp, and core orders in their respective class of matrices over which they are defined.

DEFINITION 5.1. For the matrices A and B of size $n \times n$ and of index one, we say that B is less than A with reference to a matrix relation called *the sharp relation* if

$$B^\# B = B^\# A \quad \text{and} \quad AB^\# = BB^\#$$

in which case, we write $B \leq^\# A$.

DEFINITION 5.2. For the matrices A and B of size $n \times n$ and of index one, B is said to be less than A with reference to a matrix relation called *the core relation* if

$$B^{\textcircled{\#}} B = B^{\textcircled{\#}} A \quad \text{and} \quad AB^{\textcircled{\#}} = BB^{\textcircled{\#}}$$

in which case, we write $B \leq^{\textcircled{\#}} A$.

It is well established in the literature [3, 15] that both the sharp and core relations define partial orders on the class of matrices of index one. In this section, we are interested in studying $\mathfrak{S}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ with reference to each of these partial orders. In particular, our main interest is to characterize the decompositions $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$ for which we have $B = \mathfrak{R}[A|\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ with $\mathscr{C}(B) = \mathbf{S}$. A few properties of the sharp and core relations are listed in the following

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lemma, some of which are required for the further discussion and the other are of academic interest.

LEMMA 5.3. *Let $A, B \in \mathbb{C}^{n\times n}$ be matrices of index one and let $C = A - B$. Then the following hold:*

(i) $B \leq^{\#} A$ *if and only if*

$$BA = B^2 = AB, \quad \textit{equivalently}, \quad BC = CB = 0. \tag{5.1}$$

(ii) $B \leq^{\#} A$ *if and only if C is also of index one and $C \leq^{\#} A$.*

(iii) $B \leq^{\textcircled{\#}} A$ *if and only if*

$$B^*B = B^*A \quad \textit{and} \quad B^2 = AB, \quad \textit{equivalently}, \quad B^*C = CB = 0. \tag{5.2}$$

(iv) *The relation defined by $\leq^{\textcircled{\#}}$ on the class of matrices of index one is a partial order.*

(v) *If the matrices A, B are from the class of Hermitian matrices (or to be more general, EP matrices), a subset of matrices of index one, then*

$$B \leq^{*} A \Leftrightarrow B \leq^{\#} A \Leftrightarrow B \leq^{\textcircled{\#}} A.$$

Proof. Here, we provide a quick proof of the lemma. Readers may refer to [3, 15] for the detailed and original proofs.

Proofs of the parts (i) and (iii) are immediate from the fact that

$$\mathscr{C}(B) = \mathscr{R}(B^{+}) = \mathscr{C}(B^{\#}) = \mathscr{C}(B^{\textcircled{\#}}) = \mathscr{R}(B^{\textcircled{\#}}) \quad \text{and} \quad \mathscr{R}(B) = \mathscr{C}(B^{+}) = \mathscr{R}(B^{\#}).$$

The part (v), in which case the matrices A and B are EP (range Hermitian), follows from (i), (iii) and the definition of the star relation. In fact, in this case $B^{+} = B^{\#} = B^{\textcircled{\#}}$.

From the part (i), $B \leq^{\#} A$ implies $A^2 = B^2 + C^2$. Since A and B are of index one, $A^2 = B^2 + C^2$ implies $\rho(C^2) = \rho(C)$. So, B and C simultaneously satisfy the conditions discussed in the part (i), and hence, (ii) follows immediately.

The reflexive property of $\leq^{\textcircled{\#}}$ is trivial from the definition. Antisymmetry of $\leq^{\textcircled{\#}}$ follows from the fact that $\leq^{\textcircled{\#}}$ is dominated by $\leq^{-}$. If $D \leq^{\textcircled{\#}} B$ and $B \leq^{\textcircled{\#}} A$, the transitive property of $\leq^{\textcircled{\#}}$ follows immediately from (iii), $\mathscr{C}(D) \subseteq \mathscr{C}(B)$ and $\mathscr{R}(D) \subseteq \mathscr{R}(B)$. Hence, $\leq^{\textcircled{\#}}$ is a partial order on the class of the matrices of index one. ☐

Now, we shall characterize all subspaces $\mathbf{S}$ of $\mathscr{C}(A)$ for which we have a subspace decomposition $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$ such that regular shorted matrix $B = \mathfrak{R}[A, \leq^{\textcircled{\#}} | \mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ exists.

THEOREM 5.4. *For a matrix A of index one and the space decomposition $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$, there exists a matrix $B = \mathfrak{R}[A, \leq^{\textcircled{\#}} | \mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ if and only if the following hold:*

(i) $\mathbf{S}$ *is an invariant space under A (equivalently, $QAQ = AQ$ for every oblique projector Q onto $\mathbf{S}$).*

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(ii) $\mathbf{S} \perp \mathbf{S}'$; *in other words* $\mathbf{S} \oplus\!\!\!\!\bigcirc \mathbf{S}' = \mathscr{C}(A)$.

In the case of a space decomposition $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$ *satisfying the above conditions, the regular shorted matrix of A with respect to this space decomposition under the core order is given by*

$$\mathfrak{R}[A, \leq^{\textcircled{\#}} |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}'] = PA = EE^{+}A,$$

where $\mathscr{C}(E) = \mathbf{S}$ *and* $P = EE^{+}$ *is the orthogonal projector onto* $\mathbf{S}$.

Proof. Suppose B is a regular shorted matrix of A such that $B \leq^{\textcircled{\#}} A$, $\mathscr{C}(B) = \mathbf{S}$ and $\mathscr{C}(A-B) = \mathbf{S}'$. By (iii) of Lemma 5.3, $B \leq^{\textcircled{\#}} A$ implies $B^2 = AB$. So, $\mathscr{C}(B) = \mathbf{S}$ is an invariant space under A and (i) is proved. Again referring to (5.2), we have $B^*(A-B) = 0$, and therefore, $\mathscr{C}(A-B) = \mathbf{S}'$ implies (ii).

Conversely, let $\mathbf{S} \oplus \mathbf{S}'$ be a space decomposition of $\mathscr{C}(A)$ such that $\mathbf{S}$ is an invariant space under A and $\mathbf{S} \perp \mathbf{S}'$. From the invariance of $\mathbf{S}$ under A, we have $QAQ = AQ$ for every oblique projector Q onto $\mathbf{S}$. Let E be a matrix such that $\mathscr{C}(E) = \mathbf{S}$ and write $Q = P = EE^{+}$, the orthogonal projector onto $\mathbf{S}$. Now for $B = PA$, we shall prove that B is the regular shorted matrix with the required properties. From the definition of B, we have $B^*A = A^*P^*A = A^*PA = A^*P^*PA = B^*B$ and $AB = APA = PAPA = B^2$. A is of index one and $\mathscr{C}(B) = \mathbf{S} \subseteq \mathscr{C}(A)$ implies $\rho(B) = \rho(AB) = \rho(B^2)$ (by Lemma 2.4), and therefore, B is of index one. Since P is the orthogonal projector onto $\mathbf{S}$ ($\subseteq \mathscr{C}(A)$), we have $\mathscr{C}(PA) \oplus\!\!\!\!\bigcirc \mathscr{C}((I-P)A) = \mathscr{C}(A)$ and $\mathscr{C}(PA) = \mathbf{S}$. So, from the uniqueness of the orthogonal complement $\mathbf{S}'$ of $\mathbf{S}$ in $\mathscr{C}(A)$, we have $\mathscr{C}(A-B) = \mathbf{S}'$. Therefore, $PA = \mathfrak{R}[A, \leq^{\textcircled{\#}} |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$. $\square$

The following corollaries are immediate from the above theorem.

COROLLARY 5.5. *For a matrix A of index one and* $\mathbf{S} \subseteq \mathscr{C}(A)$, *let E be any matrix such that* $\mathscr{C}(E) = \mathbf{S}$ *and* $P = EE^{+}$. *Then* $PAP = AP$ *if and only if there exists the unique* $\mathfrak{R}[A, \leq^{\textcircled{\#}} |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ *for some* $\mathbf{S}'$. *In fact, in this case* $\mathbf{S}' = \mathscr{C}((I-P)A)$.

COROLLARY 5.6. *For a matrix A of index one and* $\mathbf{S} \subseteq \mathscr{C}(A)$, *a shorted matrix* $\mathfrak{S}[A, \leq^{\textcircled{\#}} |\mathbf{S}]$ *with respect to* $\mathbf{S}$ *uniquely exists and is given by* $\mathfrak{R}[A, \leq^{\textcircled{\#}} |\mathbf{S}_1]$, *where* $\mathbf{S}_1$ *is the maximal invariant space under A in* $\mathbf{S}$.

REMARK 5.7. In the Corollary 5.5, we have observed that for an invariant space $\mathbf{S}(\subseteq \mathscr{C}(A))$ under A, there is a unique $\mathbf{S}'$ for which the regular shorted matrix $\mathfrak{R}[A, \leq^{\textcircled{\#}} |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ exists. Also, we have obtained a necessary and sufficient condition on a subspace $\mathbf{S}$ of $\mathscr{C}(A)$ to have a regular shorted matrix B of A under the core order with $\mathscr{C}(B) = \mathbf{S}$. The same characterization, given in Corollary 5.5, also provides a reason due to which $B \leq^{\textcircled{\#}} A$ need not imply $A - B \leq^{\textcircled{\#}} A$. Baksalary–Trenkler in [3] observed that for $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, the matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is such that $B \leq^{\textcircled{\#}} A$ but $A - B \not\leq^{\textcircled{\#}} A$. In this case, $A - B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

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is a matrix whose column space equals $\mathbf{S}' = \mathscr{C}(F)$, where $F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{S}'$ is not an invariant space under A. So, we cannot expect $A - B \leq^{\textcircled{\#}} A$ in this case.

Now, we shall proceed to obtain a characterization of the decomposition $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$ for which $\mathfrak{R}[A, \leq^{\#} |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ exists. In the following example, we observe that the invariance property of space $\mathbf{S}$ is not sufficient for a matrix A of index one to have a regular shorted matrix with reference to the sharp order.

EXAMPLE 5.8. Consider a matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, an invertible matrix in the *Jordan form*. Suppose there exists $B = xy^*$, a matrix of rank one such that $B \leq^{\#} A$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $y^* = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$. Since B is of index one, (5.1) implies $0 \neq B^2 = xy^*xy^* = xy^*A = Axy^*$, and therefore, $x(y^*x) = Ax$. So, x is an eigenvector of A corresponding to the eigenvalue $\lambda = y^*x$. Similarly, y is an eigenvector of A^* corresponding to the eigenvalue $\overline{\lambda} = x^*y$. Comparing $Ax = \lambda x$, we get $\lambda = 1$ and $x_2 = 0$. Similarly, we get $y_1 = 0$. This would imply that $1 = \lambda = y^*x = 0$, a contradiction. So, the above matrix A of index one in the Jordan form has no matrix B of rank one such that $B \leq^{\#} A$. From this example, we can even interpret that for some invariant space $\mathbf{S}$, spanned by an eigenvector, we may not have any choice of $\mathbf{S}'$ such that $\mathfrak{R}[A, \leq^{\#} |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ exists.

THEOREM 5.9. *Let A be an $n \times n$ matrix of index one and $\mathbf{S}$ be a subspace of $\mathscr{C}(A)$. Then the following are equivalent.*

(i) *There exists a matrix B such that $B \leq^{\#} A$ and $\mathscr{C}(B) = \mathbf{S}$.*
(ii) *For some $\mathbf{S}'$ such that $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$, the regular shorted matrix $\mathfrak{R}[A, \leq^{\#} |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ exists.*
(iii) *$\mathbf{S}$ is invariant space under A, for which there exists an invariant space $\mathbf{S}'$ under A such that $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$.*
(iv) *There exists a space decomposition $\mathbf{S} \oplus \mathbf{S}' = \mathscr{C}(A)$ such that for every oblique projectors P and Q onto $\mathbf{S}$ and $\mathbf{S}'$, respectively,*

$$PAP = AP \quad \textit{and} \quad QAQ = AQ.$$

Proof. (i) $\Rightarrow$ (ii) is trivial, where $\mathbf{S}'$ in (ii) is $\mathscr{C}(A - B)$.

(ii) $\Rightarrow$ (iii): For $B = \mathfrak{R}[A, \leq^{\#} |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$, from the definition of the regular shorted matrix, we conclude that B is a matrix such that $B \leq^{\#} A$, $\mathscr{C}(B) = \mathbf{S}$ and $\mathscr{C}(A - B) = \mathbf{S}'$. Now from (5.1), we get $B^2 = AB$, and therefore, $\mathbf{S} = \mathscr{C}(B)$ is an invariant space under A. Again, from (5.1), we see that the matrix $C = A - B$ satisfies $C^2 = CA = AC$. Since A is matrix of index one and $\mathscr{C}(C) \subseteq \mathscr{C}(A)$, by Lemma 2.4 we get $\rho(C) = \rho(AC)$. So, $\rho(C^2) = \rho(C)$, and therefore, $C^2 = CA = AC$ implies $C \leq^{\#} A$. Hence, $\mathbf{S}' = \mathscr{C}(C)$ is an invariant space under A (as in the case of B) and this proves (iii).

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(iii) $\Rightarrow$ (iv) is trivial from the invariance property of subspaces $\mathbf{S}$ and $\mathbf{S}'$ with reference to A.

(iv) $\Rightarrow$ (i): For $E_1 = P$ and $E_2 = Q$, we have $\mathscr{C}(E_1) = \mathbf{S}$ and $\mathscr{C}(E_2) = \mathbf{S}'$. From (3.3) in Theorem 3.6, we have

$$B = E_1(F_1A^-E_1)^-F_1,$$

where $F_1 = (I-Q)A$, satisfies $B \leq^- A$ with $\mathscr{C}(B) = \mathbf{S}$ and $\mathscr{C}(A-B) = \mathbf{S}'$. Now by substitution, we get $B = P[(I-Q)AA^-P]^-(I-Q)A = P[(I-Q)P]^-(I-Q)A$ and similarly $C = A-B = Q[P(I-Q)]^-(I-P)A$. From (iv), we have that $PAP = AP$ and $QAQ = AQ$. Therefore,

$$\begin{aligned} BC &= P[(I-Q)P]^-(I-Q)AQ[P(I-Q)]^-PA \\ &= P[(I-Q)P]^-(I-Q)QAQ[P(I-Q)]^-PA = 0, \end{aligned}$$

and similarly

$$\begin{aligned} CB &= Q[P(I-Q)]^-(I-P)AP[(I-Q)P]^-(I-Q)A \\ &= Q[P(I-Q)]^-(I-P)PAP[(I-Q)P]^-(I-Q)A = 0. \end{aligned}$$

So, $B^2 = BA = AB$. By Lemma 2.4, $\mathbf{S} = \mathscr{C}(B) \subseteq \mathscr{C}(A)$ and since A is of index one, we have $\rho(B) = \rho(AB)$. Therefore, $B^2 = AB = BA$ implies that B is of index one and $B \leq^\# A$. $\square$

REMARK 5.10. It is quite interesting to note that a corollary analogue to Corollary 5.5 for the sharp order is not possible. In fact, for $\mathbf{S} \subseteq \mathscr{C}(A)$, even if we have B such that $B \leq^\# A$ with $\mathscr{C}(A) = \mathbf{S}$, the choice of $\mathbf{S}'$ need not be unique to have $\mathfrak{R}[A, \leq^{\textcircled{\#}} |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$. For example, consider $A = I$, the identity matrix of size $n \times n$, and a subspace $\mathbf{S} \subseteq \mathbb{C}^n$. Note that every oblique projector Q onto $\mathbf{S}$ satisfies $Q \leq^\# I$. So, in the case of the sharp order, we do not take liberty of writing $\mathfrak{R}[A, \leq^\# |\mathbf{S}, \mathbf{S} \oplus \mathbf{S}']$ as $\mathfrak{R}[A, \leq^\# |\mathbf{S}]$.

Referring to (5.2), we have $B \leq^{\textcircled{\#}} A \Rightarrow B^2 = AB$ and in such a case $AB = BA \Leftrightarrow B(A-B) = 0$. The following theorem extends Theorem 9 of [3].

THEOREM 5.11. *Let A, B and C be matrices such that $B + C = A$ and the index of A is one. Then the following statements are equivalent.*

(i) $B \leq^\# A$ *and* $B \leq^{\textcircled{\#}} A$.
(ii) $B \leq^{\textcircled{\#}} A$ *and* $BC = 0$.
(iii) $B \leq^{\textcircled{\#}} A$ *and* $C \leq^{\textcircled{\#}} A$.
(iv) $B \leq^\# A$ *and* $B^*C = C^*B = 0$.
(v) $C \leq^\# A$ *and* $B^*C = C^*B = 0$.
(vi) $C \leq^{\textcircled{\#}} A$ *and* $CB = 0$.
(vii) $B \leq^{\textcircled{\#}} A$ *and* $B^2 \leq^{\textcircled{\#}} A^2$.
(viii) $C \leq^{\textcircled{\#}} A$ *and* $C^2 \leq^{\textcircled{\#}} A^2$.

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Proof. From (i) and (iii) of Lemma 5.3, it follows that each of (i), (ii), (iii), (iv), (v), and (vi) of the theorem is equivalent to $B^*C = C^*B = BC = CB = 0$.

If any of (i)–(vi) holds, then $B \leq^{\#} A$ and further $BC = CB = 0$ yields $A^2 = B^2 + C^2$. Since B and C are of index one, we have that the matrices B^2 and C^2 are also of index one. Note that $CB = 0$ implies $C^2B^2 = 0$, and $B^*C = 0$ entails $(B^2)^*C^2 = 0$. Therefore, $B^2 \leq^{\#} A^2$, thus proving (vii). Conversely, suppose (vii) holds. Then $B \leq^{\#} A$ yields that B is of index one and $B^*C = C^*B = 0 = CB$. Now, $B^2 \leq^{\#} A^2$ implies

$$(B^2)^*B^2 = (B^2)^*A^2 = (B^2)^*(B^2 + BC + C^2) \quad \text{(noting that } CB = 0\text{)},$$

from which we get

$$(B^2)^*BC + (B^2)^*C^2 = 0. \tag{5.3}$$

Noting that $B^*C = 0$ implies $(B^2)^*C^2 = 0$, and thus, the equation (5.3) reduces to $(B^2)^*BC = 0$. Since B is of index one, so is B^*, and therefore, $(B^2)^*BC = 0 \Rightarrow B^*BC = 0 \Rightarrow BC = 0$. Thus, (vii) is also equivalent to $B^*C = C^*B = BC = CB = 0$.

Similarly, the equivalence of (viii) with (i)–(vi) is proved. □

Now the following corollary is immediate.

COROLLARY 5.12. *If A, B and C are matrices of index one and $A = B + C$ then any two of the following conditions imply the third one.*

(i) $B \leq^{\#} A$.
(ii) $BC = 0 = CB$.
(iii) $B^2 \leq^{\#} A^2$.

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