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A NOTE ON INVERSE-ORTHOGONAL TOEPLITZ MATRICES*

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Abstract. In this note, inverse-orthogonal Toeplitz matrices are investigated, and it is proved that every such a matrix is equivalent to a circulant one. As a corollary, it is showed that a real Hadamard matrix of order $n > 2$ with Toeplitz structure is necessarily circulant.

1. Introduction. A Toeplitz matrix is an $n \times n$ matrix T, where $t_{i,j} = t_{j-i}$ for every $1 \leq i, j \leq n$, i.e., a matrix of the form

$$T = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & t_{n-1} \\ t_{-1} & t_0 & t_1 & & \\ t_{-2} & t_{-1} & t_0 & & \vdots \\ \vdots & & & \ddots & \\ t_{-n+1} & & \cdots & & t_0 \end{bmatrix}. \tag{1.1}$$

Such matrices arise in many applications [7, 9]. We mention here two special kind of Toeplitz matrices as follows. One important class is formed by circulant matrices, where each row vector is rotated one element to the right relative to the preceding row vector. In particular, with the notations of (1.1) we have $t_i = t_{-n+i}$, for each $i = 1, 2, \ldots, n-1$ within circulant matrices. Another class of Toeplitz matrices are negacyclic matrices, in which $t_i = -t_{-n+i}$ holds for each $i = 1, 2, \ldots, n-1$. For the general treatment of Toeplitz matrices we refer the reader to [6, 7].

In this note, we investigate the existence of inverse-orthogonal Toeplitz matrices. A matrix $[M]_{i,j} = m_{i,j}$ of order n is called inverse-orthogonal (also called as type II, [14]; or "jacket", within the engineering literature [13]), if its inverse satisfies $[M^{-1}]_{i,j} = \frac{1}{nm_{j,i}}$, i.e., the inverse matrix can be obtained by taking element-wise inverse and transposition, up to a negligible constant factor. Equivalently, these matrices satisfy the following relations:

$$\sum_{k=1}^{n} \frac{m_{i,k}}{m_{j,k}} = n\delta_{ij}, \qquad i, j = 1, 2, \ldots, n. \tag{1.2}$$

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An inverse-orthogonal matrix in which all entries are of modulus 1 is called a complex Hadamard matrix [10, 16]. Hadamard matrices have both theoretical applications ranging from harmonic analysis [12] to quantum information theory [4]; as well as applications in signal processing [13, 17].

It is easy to see that if K is an inverse-orthogonal matrix, then for every permutation matrices P_1, P_2 and for every invertible diagonal matrices D_1, D_2 so is the matrix $H = P_1D_1KD_2P_2$. Matrices related in this fashion are called equivalent. Finding all inverse-orthogonal matrices up to equivalence turns out to be a challenging problem, and has been solved only up to orders $n \leq 5$ [14]. For partial results regarding the case $n = 6$, see [16].

The special case of circulant inverse-orthogonal matrices has been heavily investigated during the early 1990s by Björck and coauthors, who were motivated by finding all solutions to the so-called cyclic n-roots problem [2]. Currently there is a complete characterization of cyclic n-roots up to $n = 10$ [5], while numerical results are available for $n = 11$ [3]. Starting from $n = 12$ only partial results are known [15]. It is worthwhile noting that the number of $n \times n$ circulant Hadamard matrices, up to equivalence, is not finite when n has a square divisor [1].

By relaxing the condition "circulant" to "Toeplitz", we were hoping to find new examples of complex Hadamard matrices. Surprisingly, however, equation (1.2) puts heavy restrictions onto the structure of inverse-orthogonal Toeplitz matrices, and such an envisioned generalization is not possible. In particular, we prove the following rather unexpected result in Section 2.

Theorem 1.1. *Every inverse-orthogonal Toeplitz matrix is equivalent to a circulant matrix.*

Thus, the two concepts are exactly the same within the class of inverse-orthogonal matrices.

2. Inverse-orthogonal Toeplitz matrices.

First we provide the reader with some explicit examples. The following is a well-known example of circulant complex Hadamard matrices, see e.g. [8].

Example 1. Let $n \geq 2$ be an integer, and let $\mathbb{Z}_n^*$ denote the set of invertible elements in the ring $\mathbb{Z}_n$. For $(\alpha, \beta) \in \mathbb{Z}_n^* \times \mathbb{Z}_n$ consider the row vector $x(\alpha, \beta)$ of length n with entries

$$[x(\alpha,\beta)]_i = \begin{cases} \exp\left(\frac{2\pi\mathbf{i}}{n}\left(\frac{\alpha i^2}{2} + \beta i\right)\right), & i \in \mathbb{Z}_n, \quad n \text{ even}, \\ \exp\left(\frac{2\pi\mathbf{i}}{n}\left(\frac{\alpha i(i-1)}{2} + \beta i\right)\right), & i \in \mathbb{Z}_n, \quad n \text{ odd}, \end{cases}$$

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where $\mathbf{i}$ is the complex imaginary unit. Then the circulant complex Hadamard matrix whose first row is $x(\alpha, \beta)$ is equivalent to the discrete Fourier transform (DFT) matrix.

The “Potts model”, introduced by Jones [11], constitutes another important class of circulant inverse-orthogonal matrices.

Example 2. Let us denote by I_n and J_n the identity matrix and the matrix of all 1s of order n, respectively. Further, let α_n be any root of the quadratic equation $\alpha^2 + (n-2)\alpha + 1 = 0$. Then, the matrix $P = (\alpha_n - 1)I_n + J_n$ is a circulant inverse-orthogonal matrix. Note that for $n > 4$ the matrix arising from P does not have unimodular entries, and hence, it is not a complex Hadamard matrix.

Finally, we provide examples of order $n = 4$ as follows.

Example 3. Let a, b, c be nonzero complex numbers. Then the matrices, arising of the form below are all inverse-orthogonal Toeplitz matrices of order 4:

$$T_4^{(3)}(a,b,c) = \begin{bmatrix} a & b & c & -\frac{bc}{a} \\ -\frac{ab}{c} & a & b & c \\ \frac{a^2}{c} & -\frac{ab}{c} & a & b \\ \frac{a^2 b}{c^2} & \frac{a^2}{c} & -\frac{ab}{c} & a \end{bmatrix}.$$

Observe that the matrix above has 3 independent parameters which is one more that of circulant matrices of the same size. This extra parameter, however, arises due to the following non-trivial symmetry of Toeplitz matrices. We use the notation $\mathrm{Diag}(a_1, a_2, \ldots, a_n)$ to refer to the $n \times n$ diagonal matrix with diagonal entries a_i, $i = 1, 2, \ldots, n$.

Lemma 2.1. *Let T be a Toeplitz matrix of order n, and let a be an arbitrary, while b be a nonzero complex number. Then the following is a Toeplitz matrix as well*:

$$T^{(2)}(a,b) = a\mathrm{Diag}(1, b^{-1}, b^{-2}, \ldots, b^{-n+1})T\mathrm{Diag}(1, b, b^2, \ldots, b^{n-1}).$$

Proof. It is easy to see that the (i,j)th entry of $T^{(2)}(a,b)$ reads $t_{i,j}ab^{j-i}$. ◻

Note that for all a and nonzero b the matrices $T^{(2)}(a,b)$ are all equivalent to the starting-point matrix T. Nevertheless, one can transform a circulant matrix into a Toeplitz matrix with two prescribed entries via Lemma 2.1 which might be useful in practical applications. For more involved examples, we refer the reader to [15].

Now we state a structural result concerning inverse-orthogonal Toeplitz matrices.

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PROPOSITION 2.2. *Suppose that T is an inverse-orthogonal Toeplitz matrix. Then, with the notations from* (1.1)*, we have*

$$t_{-\ell} = \frac{t_{-1}}{t_{n-1}} t_{n-\ell}, \qquad \ell = 1, 2, \ldots, n-1. \tag{2.1}$$

Proof. Note that formula (2.1) holds trivially for $\ell = 1$. We begin the proof by showing that it holds for $\ell = 2$ as well. We assume that $n \geq 3$.

Consider the condition (1.2) within the first two rows of T, i.e., for $(i, j) = (2, 1)$. We have

$$\frac{t_{-1}}{t_0} + \frac{t_0}{t_1} + \cdots + \frac{t_{n-2}}{t_{n-1}} = 0. \tag{2.2}$$

Next consider (1.2) for the pair of rows $(i, j) = (3, 2)$, and derive

$$\frac{t_{-2}}{t_{-1}} + \frac{t_{-1}}{t_0} + \frac{t_0}{t_1} + \cdots + \frac{t_{n-3}}{t_{n-2}} = 0.$$

Add $\frac{t_{n-2}}{t_{n-1}}$ to both sides and use (2.2) to conclude that (2.1) holds for $\ell = 2$ as well. In particular, we arrived at case $\ell = 1$ of the following:

$$\frac{t_{-\ell-1}}{t_{-\ell}} = \frac{t_{n-\ell-1}}{t_{n-\ell}}, \qquad \ell = 1, 2, \ldots, n-2. \tag{2.3}$$

Now we use mathematical induction to prove that (2.3) holds for all $1 \leq \ell \leq n-2$. We can assume that $n \geq 4$ and (2.3) already holds for some $\ell \leq n-3$. Then, we consider the condition (1.2) for the pair of rows $(\ell + 3, \ell + 2)$:

$$\sum_{k=1}^{n} \frac{t_{-\ell-2+k-1}}{t_{-\ell-1+k-1}} = 0,$$

and rewrite it, using (2.3), to

$$\frac{t_{-\ell-2}}{t_{-\ell-1}} + \sum_{\substack{k=1 \\ k \neq n-\ell}}^{n} \frac{t_{k-2}}{t_{k-1}} = 0.$$

Adding the missing term $t_{n-\ell-2}/t_{n-\ell-1}$ to both sides, and reducing via (2.2) concludes the validity of (2.3) for $\ell + 1$.

We finish the proof by observing that (2.3) connects the consecutive terms $t_{-\ell}$ and $t_{-\ell-1}$, and hence,

$$t_{-\ell-1} = t_{-\ell} \frac{t_{n-\ell-1}}{t_{n-\ell}} = \frac{t_{-1}}{t_{n-1}} t_{n-\ell} \frac{t_{n-\ell-1}}{t_{n-\ell}} = \frac{t_{-1}}{t_{n-1}} t_{n-\ell-1}. \qquad \square$$

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Now we are ready to prove Theorem 1.1 rigorously.

Proof of Theorem 1.1. Let T be an inverse-orthogonal Toeplitz matrix as in (1.1), and let $x = \sqrt[n]{t_{-1}/t_{n-1}}$, where the operator $\sqrt[n]{\cdot}$ denotes the principal nth root. By Lemma 2.1 the matrix

$$C = \mathrm{Diag}\left(1, x^{-1}, x^{-2}, \dots, x^{-n+1}\right) T \mathrm{Diag}\left(1, x, x^2, \dots, x^{n-1}\right)$$

is a Toeplitz matrix, which is equivalent to T. We claim that C is a circulant matrix. To see this, it is enough to show that $c_{1,j} = c_{n-j+2,1}$ for every $j = 2, \dots, n$. It is clear that $c_{1,j} = t_{j-1}x^{j-1}$. On the other hand, by using Proposition 2.2, we have

$$c_{n-j+2,1} = t_{-n+j-1}x^{-n+j-1} = \frac{t_{-1}}{t_{n-1}} t_{j-1}x^{-n+j-1} = t_{j-1}x^{j-1}. \qquad \square$$

3. Real Hadamard matrices with Toeplitz structure. Next we briefly discuss Toeplitz real Hadamard matrices.

COROLLARY 3.1. *Let H be a Toeplitz real Hadamard matrix. Then H is either circulant or negacyclic.*

Proof. The result follows from Proposition 2.2: if the quotient $t_{-1}/t_{n-1} = 1$, then by formula (2.1) we have $t_{-\ell} = t_{n-\ell}$ for $\ell = 1, 2, \dots, n-1$, i.e., the matrix is circulant. Otherwise $t_{-1}/t_{n-1} = -1$, and hence, $t_{-\ell} = -t_{n-\ell}$, i.e., the matrix is negacyclic. $\square$

EXAMPLE 4. The following are the only known Toeplitz real Hadamard matrices, up to equivalence:

$$H_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad H_4 = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Here, we prove that the negacyclic case is vacuous for orders $n > 2$.

LEMMA 3.2. *There exists a negacyclic real Hadamard matrix of order n if and only if $n = 1, 2$.*

Proof. The first two matrices of size $n = 1, 2$, displayed in Example 4, are negacyclic. We thus assume that $n > 2$, and hence, n is necessarily doubly even. Now suppose, to the contrary, that there exists a negacyclic real Hadamard matrix H of order n with first row $[x_1, x_2, \dots, x_n]$. We can normalize this row by multiplying H from the right by the diagonal matrix $\mathrm{Diag}(x_1, x_2, \dots, x_n)$ to yield a Hadamard matrix with first two rows being $[1, 1, \dots, 1]$ and $[-x_1x_n, x_1x_2, x_2x_3, \dots, x_{n-1}x_n]$, respectively. Due to the orthogonality of these two rows, we find that there exist

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exactly $n/2$ entries of 1 and -1 in the second row. Therefore, as n is doubly even, the product of the entries in the second row is exactly 1, which is a contradiction. $\square$

It is conjectured that no circulant Hadamard matrices exists of orders $n > 4$ [10]. The proof of this, however, remains elusive despite continuous efforts.

4. Inverse-orthogonal matrices with Toeplitz core. As a final remark, we note that one can ask about the existence of inverse-orthogonal matrices of order n whose first row and column are all normalized to 1 and have a lower right $n-1 \times n-1$ Toeplitz submatrix. These conditions, however, easily imply that this particular lower right submatrix is circulant. We conclude the paper with the following non-trivial example from [16].

EXAMPLE 5. Let $\alpha < 0$ be the unique negative root of the cubic polynomial $x^3 - 40169x^2 + 122486812x + 124134308$, and further define

$$\begin{aligned} h_\alpha(u) = \ & 2054570000u^6 + 4109140000u^5 + (16\alpha^2 + 9768064\alpha - 5227993936)u^4 \\ & + (1956\alpha^2 + 64132324\alpha - 12170223176)u^3 + (11393\alpha^2 + 427075897\alpha \\ & + 1676016222)u^2 + (4644\alpha^2 + 2280446676\alpha + 8524444776)u - 17074\alpha^2 \\ & + 3269963754\alpha + 2727593304. \end{aligned}$$

The six real roots of $h_\alpha(u)$, $r_1 < r_2 < r_3 < r_4 < r_5 < r_6$, describe twice the real part of six complex unimodular numbers $z_i = r_i/2 + \mathbf{i}\sqrt{1 - r_i^2/4}$, $i = 1, 2, \ldots, 6$. In particular, the circulant matrix whose first row is $[z_1, z_3, 1/z_2, z_4, 1/z_5, 1/z_6]$, bordered by a row and column of numbers 1 is a complex Hadamard matrix of order 7. We note that these six roots can be described by radicals [16].

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