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## LINEAR SPACES AND PRESERVERS OF BOUNDED RANK-TWO PER-SYMMETRIC TRIANGULAR MATRICES\*

W.L. CHOOI<sup>†</sup>, K.H. KWA<sup>†</sup>, M.H. LIM<sup>†</sup>, AND Z.C. NG<sup>‡</sup>

**Abstract.** Let  $\mathbb{F}$  be a field and  $m, n$  be integers  $m, n \geq 3$ . Let  $\mathcal{SM}_n(\mathbb{F})$  and  $\mathcal{ST}_n(\mathbb{F})$  denote the linear space of  $n \times n$  per-symmetric matrices over  $\mathbb{F}$  and the linear space of  $n \times n$  per-symmetric triangular matrices over  $\mathbb{F}$ , respectively. In this note, the structure of spaces of bounded rank-two matrices of  $\mathcal{ST}_n(\mathbb{F})$  is determined. Using this structural result, a classification of bounded rank-two linear preservers  $\psi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{SM}_m(\mathbb{F})$ , with  $\mathbb{F}$  of characteristic not two, is obtained. As a corollary, a complete description of bounded rank-two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two is addressed.

**Key words.** Per-symmetric triangular matrices, Rank, Spaces of bounded rank-two matrices, Bounded rank-two linear preservers.

**AMS subject classifications.** 15A03, 15A04, 15A86.

**1. Introduction.** A linear mapping between matrix spaces is said to be rank- $k$  non-increasing (respectively, a rank- $k$  preserver) if it sends rank less than or equal to  $k$  matrices (respectively, if it sends rank  $k$  matrices) to matrices of the same type. Motivated by the studies of rank-one non-increasing linear mappings and rank-two non-increasing linear mappings on symmetric matrices [2, 5, 10, 11, 13] and rank-one non-increasing linear mappings on triangular matrices [3, 4], we investigate the structure of *bounded rank-two linear preservers*  $\psi$  on per-symmetric triangular matrices satisfying the condition

$$(1.1) \quad 1 \leq \text{rank } \psi(A) \leq 2 \quad \text{whenever} \quad 1 \leq \text{rank } A \leq 2,$$

where  $\text{rank } A$  denotes the rank of the matrix  $A$ .

It is a known fact that the structure of rank preservers is one of the basic results and useful in the study of linear preserver problems [9, 16]. Many linear preservers problems quite often depend on or can be solved with the help of such mappings. For instance, Minc [15] deduced from rank-one linear preservers the classical theorem

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of Frobenius [6] concerning determinant linear preservers. Watkins [17] classified commutativity linear preservers by using the structure of rank-one linear preservers. In [14], rank- $k$  non-increasing linear mappings were used by Loewy and Pierce to verify the John-Pierce conjecture [7] for certain balanced singular inertia classes. Beasley [1] showed that rank-additivity preserving linear mappings are rank- $k$  non-increasing. For works concerning rank preservers on various matrix spaces, we refer the reader to [16, Chapter 2] and [18, Chapter 2].

Let  $\mathbb{F}$  be a field and  $m, n$  be positive integers. Let  $\mathcal{M}_{m,n}(\mathbb{F})$  denote the linear space of  $m \times n$  matrices over  $\mathbb{F}$ . We abbreviate  $\mathcal{M}_{n,n}(\mathbb{F})$  to  $\mathcal{M}_n(\mathbb{F})$  and  $\mathcal{M}_{1,n}(\mathbb{F})$  to  $\mathbb{F}^n$ . Given  $A \in \mathcal{M}_{m,n}(\mathbb{F})$ , let  $A^+ := J_n A^T J_m \in \mathcal{M}_{n,m}(\mathbb{F})$ , where  $A^T$  stands for the transpose of  $A$  and  $J_n$  is the element of  $\mathcal{M}_n(\mathbb{F})$  with ones on the minor diagonal and zeros elsewhere. A matrix  $A \in \mathcal{M}_n(\mathbb{F})$  is called *per-symmetric* if it is symmetric around the minor diagonal, i.e.,  $A^+ = A$ . We denote by  $\mathcal{SM}_n(\mathbb{F})$  the linear subspace of  $\mathcal{M}_n(\mathbb{F})$  consisting of per-symmetric matrices, and  $\mathcal{ST}_n(\mathbb{F}) := \mathcal{SM}_n(\mathbb{F}) \cap \mathcal{T}_n(\mathbb{F})$ . Here  $\mathcal{T}_n(\mathbb{F})$  stands for the linear space of  $n \times n$  upper triangular matrices over  $\mathbb{F}$ . We shall call  $\mathcal{SM}_n(\mathbb{F})$  and  $\mathcal{ST}_n(\mathbb{F})$  the *per-symmetric matrix space* and the *per-symmetric triangular matrix space*, respectively.

The study of rank- $k$  non-increasing linear mappings led naturally to the investigation of *linear spaces of bounded rank  $k$*  (i.e., linear subspaces consisting of matrices of rank at most  $k$ ) and  *$k$ -spaces* (i.e., linear subspaces consisting of the zero matrix and matrices of rank  $k$ ). In this note, we first give a classification of linear spaces of bounded rank-two per-symmetric matrices of  $\mathcal{ST}_n(\mathbb{F})$  over an arbitrary field  $\mathbb{F}$ . As a corollary, a description of 2-spaces of  $\mathcal{ST}_n(\mathbb{F})$  is obtained. We next deduce from the structural result of spaces of bounded rank-two per-symmetric triangular matrices a characterization of bounded rank-two linear preservers from  $\mathcal{ST}_n(\mathbb{F})$  into  $\mathcal{SM}_m(\mathbb{F})$ , with  $m, n \geq 3$  and  $\mathbb{F}$  of characteristic not two. As an immediate consequence, a complete description of bounded rank-two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two is addressed.

As a side remark, the structure of rank-one non-increasing linear mappings on triangular matrices is much more complicated than the one of those on symmetric matrices. Some examples of rank-one non-increasing linear mappings and rank-two non-increasing linear mappings on per-symmetric triangular matrices are given at the end of this note to indicate the aptness of condition (1.1) in arriving at our results.

In the sequel, we write  $\{f_1, \dots, f_m\}$  and  $\{e_1, \dots, e_n\}$  for the standard bases of  $\mathcal{M}_{m,1}(\mathbb{F})$  and  $\mathcal{M}_{n,1}(\mathbb{F})$ , respectively, and let  $E_{ij} := f_i \cdot e_j^T$  be the matrix unit in  $\mathcal{M}_{m,n}(\mathbb{F})$  with one as the  $(i, j)$  entry and zero elsewhere. We use  $\langle u_1, \dots, u_r \rangle$  designate the linear span of the vectors  $u_1, \dots, u_r$ .

**2. Preliminaries.** Let  $\mathbb{F}$  be a field and  $n$  be an integer such that  $n \geq 2$ . For each  $\alpha \in \mathbb{F}$  and each pair of integers  $i, j$  satisfying  $1 \leq i, j \leq n$  and  $j \neq n + 1 - i$ , we set

$$(2.1) \quad Z_{ij}^\alpha := E_{ij} + E_{ij}^+ + \alpha E_{i, n+1-i} \in \mathcal{M}_n(\mathbb{F})$$

and write  $Z_{ij} = Z_{ij}^0$  for short. It is obvious that  $Z_{ij}^\alpha$  is a per-symmetric triangular matrix for every  $1 \leq i \leq j \leq n + 1 - i$  and  $j \neq n + 1 - i$ .

We begin with a result on the decomposition of per-symmetric triangular matrices.

**LEMMA 2.1.** *Let  $\mathbb{F}$  be a field and  $n$  be an integer such that  $n \geq 2$ . A nonzero matrix  $A \in \mathcal{ST}_n(\mathbb{F})$  is of rank  $k$  if and only if there exist an integer  $0 \leq h \leq \frac{k}{2}$ , scalars  $\alpha_1, \dots, \alpha_h \in \mathbb{F}$ , nonzero scalars  $\beta_{2h+1}, \dots, \beta_k \in \mathbb{F}$ , and an invertible matrix  $P \in \mathcal{T}_n(\mathbb{F})$  such that*

$$A = P \left( \sum_{i=1}^h Z_{s_i t_i}^{\alpha_i} + \sum_{i=2h+1}^k \beta_i E_{p_i, n+1-p_i} \right) P^+,$$

where  $\{s_1, \dots, s_h, n + 1 - t_1, \dots, n + 1 - t_h, p_{2h+1}, \dots, p_k\}$  and  $\{t_1, \dots, t_h, n + 1 - s_1, \dots, n + 1 - s_h, n + 1 - p_{2h+1}, \dots, n + 1 - p_k\}$  are two sets of  $k$  distinct positive integers such that  $1 \leq s_i \leq t_i \leq n + 1 - s_i$  and  $t_i \neq n + 1 - s_i$  for  $i = 1, \dots, h$ , and  $1 \leq p_i \leq \frac{n+1}{2}$  for  $i = 2h + 1, \dots, k$ ; and  $(\alpha_1, \dots, \alpha_h) \neq 0$  only if  $\mathbb{F}$  has characteristic two.

*Proof.* The proof of sufficiency is immediate. We now consider necessity.

Let  $A = (a_{ij}) \in \mathcal{ST}_n(\mathbb{F})$  be a nonzero rank  $k$  matrix. We denote by  $A_{(i)}$  and  $A^{(j)}$  the  $i$ -th row and the  $j$ -th column of the matrix  $A$ , respectively. Let  $A^{(j_0)}$  be the first nonzero column from the left of  $A$ , and let  $a_{i_0 j_0}$  be the first nonzero entry from the bottom of the column  $A^{(j_0)}$ . Then  $a_{i_0 j_0} = 0$  for every  $i_0 + 1 \leq i \leq n$ , and  $a_{ij} = 0$  for every  $1 \leq i \leq n$  and  $1 \leq j \leq j_0 - 1$ , and also  $1 \leq i_0 \leq j_0 \leq n + 1 - i_0$  since  $A \in \mathcal{ST}_n(\mathbb{F})$ . We divide our proof into the following two cases:

*Case I:*  $j_0 = n + 1 - i_0$ . For each  $1 \leq s \leq i_0 - 1$ , we apply the following elementary row and column operations on  $A$ :

$$(2.2) \quad A_{(s)} \rightarrow A_{(s)} - a_{s j_0} a_{i_0 j_0}^{-1} A_{(i_0)} \quad \text{and} \quad A^{(n+1-s)} \rightarrow A^{(n+1-s)} - a_{i_0, n+1-s} a_{i_0 j_0}^{-1} A^{(j_0)}.$$

For each  $1 \leq s \leq i_0 - 1$ , there exists the elementary matrix  $I_n - c_s E_{s i_0} \in \mathcal{T}_n(\mathbb{F})$  corresponding to the row operation  $A_{(s)} \rightarrow A_{(s)} - c_s A_{(i_0)}$ , where  $c_s = a_{s j_0} a_{i_0 j_0}^{-1} \in \mathbb{F}$ . Since  $A^+ = A$ , we have  $a_{i_0, n+1-s} = a_{s j_0}$  for every  $1 \leq s \leq i_0 - 1$ , and so there exists an invertible matrix  $P_1 \in \mathcal{T}_n(\mathbb{F})$  such that

$$(2.3) \quad P_1 A P_1^+ = a_{i_0 j_0} E_{i_0 j_0} + B$$

for some matrix  $B = (b_{ij}) \in \mathcal{ST}_n(\mathbb{F})$  such that  $b_{ij_0} = 0$  for every  $1 \leq i \leq n$ ,  $b_{i_0j} = 0$  for every  $1 \leq j \leq n$ , and  $b_{ij} = 0$  for every  $1 \leq i \leq n$  and  $1 \leq j \leq j_0 - 1$ .

*Case II:*  $j_0 \neq n + 1 - i_0$ . Without loss of generality, we may assume  $a_{i_0j_0} = 1 = a_{n+1-j_0, n+1-i_0}$ . For each  $1 \leq s \leq i_0 - 1$ , we apply the following elementary row and column operations on  $A$ :

$$A_{(s)} \rightarrow A_{(s)} - a_{sj_0}A_{(i_0)} \quad \text{and} \quad A^{(n+1-s)} \rightarrow A^{(n+1-s)} - a_{n+1-j_0, n+1-s}A^{(n+1-i_0)},$$

and it is followed by the elementary row and column operations on  $A$ :

$$A_{(t)} \rightarrow A_{(t)} - a_{t, n+1-i_0}A_{(n+1-j_0)} \quad \text{and} \quad A^{(n+1-t)} \rightarrow A^{(n+1-t)} - a_{i_0, n+1-t}A^{(j_0)}$$

for every  $1 \leq t \leq n - j_0$ . We note that, for each  $1 \leq s \leq i_0 - 1$  (respectively, for each  $1 \leq t \leq n - j_0$ ), there exists the elementary matrix  $I_n - a_{sj_0}E_{s i_0}$  (respectively,  $I_n - a_{t, n+1-i_0}E_{t, n+1-j_0}$ ) in  $\mathcal{T}_n(\mathbb{F})$  corresponding to the row operation  $A_{(s)} \rightarrow A_{(s)} - a_{sj_0}A_{(i_0)}$  (respectively,  $A_{(t)} \rightarrow A_{(t)} - a_{t, n+1-i_0}A_{(n+1-j_0)}$ ). Since  $a_{n+1-j_0, n+1-s} = a_{sj_0}$  for every  $1 \leq s \leq i_0 - 1$ , and  $a_{i_0, n+1-t} = a_{t, n+1-i_0}$  for every  $1 \leq t \leq n - j_0$ , there exists an invertible matrix  $P_1 \in \mathcal{T}_n(\mathbb{F})$  such that

$$(2.4) \quad P_1AP_1^+ = Z_{i_0j_0}^{\alpha_1} + B$$

for some scalar  $\alpha_1 \in \mathbb{F}$  and matrix  $B = (b_{ij}) \in \mathcal{ST}_n(\mathbb{F})$  such that  $b_{ij_0} = 0$  for every  $1 \leq i \leq n$ ,  $b_{i_0j} = 0$  for  $1 \leq j \leq n$ , and  $b_{ij} = 0$  for every  $1 \leq i \leq n$  and  $1 \leq j \leq j_0 - 1$ .

In view of (2.3) and (2.4), if  $B = 0$ , then we are done. Suppose that  $B \neq 0$ . Let  $b_{i_1j_1}$  be the first nonzero entry from the bottom of the first nonzero column of  $B$  counting from the left of the matrix  $B$ . Evidently,  $j_1 > j_0$ ,  $i_1 \neq i_0$  and  $1 \leq i_1 \leq j_1 \leq n + 1 - i_1$ . Since  $b_{ij_0} = 0$  for all  $1 \leq i \leq n$ ,  $b_{i_0j} = 0$  for every  $1 \leq j \leq n$ , and  $b_{ij} = 0$  for every  $1 \leq i \leq n$  and  $1 \leq j \leq j_0 - 1$ , by applying suitable elementary row and column operations similar to (2.2) when  $j_1 = n + 1 - i_1$  (respectively, similar to (2.4) when  $j_1 \neq n + 1 - i_1$ ), there exists an invertible matrix  $P_2 \in \mathcal{T}_n(\mathbb{F})$  such that

$$P_2BP_2^+ = a_{i_1j_1}E_{i_1j_1} + C$$

for some matrix  $C \in \mathcal{ST}_n(\mathbb{F})$ , and  $P_2E_{i_0j_0}P_2^+ = E_{i_0j_0}$  (respectively,

$$P_2BP_2^+ = Z_{i_1j_1}^{\alpha_2} + C$$

for some scalar  $\alpha_2 \in \mathbb{F}$  and matrix  $C \in \mathcal{ST}_n(\mathbb{F})$ , and  $P_2Z_{i_0j_0}^{\alpha_1}P_2^+ = Z_{i_0j_0}^{\alpha_1}$ ). If  $C = 0$ , then we are done. Suppose that  $C \neq 0$ . Since  $A$  is of rank  $k$ , by repeating a similar argument on  $C$ , there exist an integer  $0 \leq h \leq \frac{k}{2}$ , scalars  $\alpha_1, \dots, \alpha_h, \beta_{2h+1}, \dots, \beta_k \in \mathbb{F}$ , and an invertible matrix  $Q \in \mathcal{T}_n(\mathbb{F})$  such that

$$(2.5) \quad QAQ^+ = \sum_{i=1}^h Z_{s_i t_i}^{\alpha_i} + \sum_{i=2h+1}^k \beta_i E_{p_i, n+1-p_i},$$

where  $\{s_1, \dots, s_h, n+1-t_1, \dots, n+1-t_h, p_{2h+1}, \dots, p_k\}$  and  $\{t_1, \dots, t_h, n+1-s_1, \dots, n+1-s_h, n+1-p_{2h+1}, \dots, n+1-p_k\}$  are two sets of  $k$  distinct positive integers such that  $1 \leq s_i \leq t_i \leq n+1-s_i$  and  $t_i \neq n+1-s_i$  for  $i = 1, \dots, h$ , and  $1 \leq p_i \leq \frac{n+1}{2}$  for  $i = 2h+1, \dots, k$

We denote  $D = QAQ^+$ . If  $\mathbb{F}$  is of characteristic not two, then, for each  $1 \leq i \leq h$ , we further perform the following elementary row and column operations on  $D$ :

$$D_{(s_i)} \rightarrow D_{(s_i)} - \frac{\alpha_i}{2} D_{(n+1-t_i)} \quad \text{and} \quad D^{(n+1-s_i)} \rightarrow D^{(n+1-s_i)} - \frac{\alpha_i}{2} D^{(t_i)}$$

to annihilate  $\alpha_i$  in  $Z_{s_i t_i}^{\alpha_i}$  as described in (2.5). Since  $s_i < n+1-t_i$  for every  $1 \leq i \leq h$ , there exists an invertible  $P \in \mathcal{T}_n(\mathbb{F})$  such that

$$PAP^+ = \sum_{i=1}^h Z_{s_i t_i} + \sum_{i=2h+1}^k \beta_i E_{p_i, n+1-p_i} \cdot \square$$

As a corollary of Lemma 2.1, we notice that if  $A \in \mathcal{ST}_n(\mathbb{F})$  is of rank bounded by two, then there exists an invertible matrix  $P \in \mathcal{T}_n(\mathbb{F})$  such that either

$$A = P(\alpha E_{s, n+1-s} + \beta E_{t, n+1-t})P^+$$

for some  $\alpha, \beta \in \mathbb{F}$  and some integers  $1 \leq s < t \leq \frac{n+1}{2}$ , or

$$A = PZ_{st}^\lambda P^+$$

for some integers  $1 \leq s \leq t \leq n+1-s$  with  $t \neq n+1-s$ , and some scalar  $\lambda \in \mathbb{F}$  with  $\lambda \neq 0$  only if  $\text{char } \mathbb{F} = 2$ .

Inspired by this observation, we define

$$(2.6) \quad u \circ v := u \cdot v^+ + v \cdot u^+ \quad \text{and} \quad u^2 := u \cdot u^+$$

for every  $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$ , where  $u \cdot v^+$  denotes the usual matrix product of  $u \in \mathcal{M}_{n,1}(\mathbb{F})$  and  $v^+ \in \mathbb{F}^n$ . It can easily be verified that  $(u, v) \mapsto u \circ v$  is a symmetric bilinear map from  $\mathcal{M}_{n,1}(\mathbb{F}) \times \mathcal{M}_{n,1}(\mathbb{F})$  into  $\mathcal{M}_n(\mathbb{F})$ . We also see that

$$e_i \circ e_j = E_{i, n+1-j} + E_{i, n+1-j}^+ \quad \text{and} \quad e_i^2 = E_{i, n+1-i}$$

for all integers  $1 \leq i, j \leq n$ . In view of (2.1), we have

$$Z_{ij}^\alpha = e_i \circ e_{n+1-j} + \alpha e_i^2$$

for every  $\alpha \in \mathbb{F}$  and  $1 \leq i, j \leq n$  with  $j \neq n+1-i$ . Note that  $\{e_i \circ e_j \mid 1 \leq i < j \leq n\} \cup \{e_i^2 \mid 1 \leq i \leq n\}$  and  $\{e_i \circ e_j \mid 1 \leq i < j \leq n+1-i\} \cup \{e_i^2 \mid 1 \leq i \leq \frac{n+1}{2}\}$  are the standard bases of  $\mathcal{SM}_n(\mathbb{F})$  and  $\mathcal{ST}_n(\mathbb{F})$ , respectively.

It follows immediately from (2.6) that the following elementary properties hold and their straightforward proofs are omitted. Let  $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$ ,  $a, b, c \in \mathbb{F}$  and  $P \in \mathcal{M}_n(\mathbb{F})$ . We have

- (P1)  $(u \otimes v)^+ = u \otimes v$  and  $(u^2)^+ = u^2$ ,
- (P2)  $u^2 = 0 \Leftrightarrow u = 0$ ,
- (P3)  $u \otimes v = 0 \Leftrightarrow$  either  $u = 0$  or  $v = 0$  when  $\text{char } \mathbb{F} \neq 2$ ; and  $u \otimes v = 0 \Leftrightarrow u, v$  are linearly dependent when  $\text{char } \mathbb{F} = 2$ ,
- (P4)  $P(u \otimes v)P^+ = (Pu) \otimes (Pv)$  and  $P(u^2)P^+ = (Pu)^2$ , and
- (P5)  $\text{rank}(a(u \otimes v) + bu^2 + cv^2) \leq 2$ ; and  $\text{rank}(a(u \otimes v) + bu^2 + cv^2) = 2 \Leftrightarrow u, v$  are linearly independent and  $a^2 \neq bc$ .

LEMMA 2.2. Let  $u, v, x, y \in \mathcal{M}_{n,1}(\mathbb{F})$  and  $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{F}$ .

- (a) If  $a_1u \otimes v + b_1u^2 + c_1v^2 = a_2x \otimes y + b_2x^2 + c_2y^2 \neq 0$  with  $a_i^2 \neq b_i c_i$  for  $i = 1, 2$ , then  $\langle u, v \rangle = \langle x, y \rangle$ .
- (b) If  $\mathbb{F}$  has characteristic not two, then  $u \otimes v = x \otimes y \neq 0$  if and only if there exists a nonzero  $a \in \mathbb{F}$  such that either  $u = ax$  and  $v = a^{-1}y$ , or  $u = ay$  and  $v = a^{-1}x$ .

*Proof.* (a) By our hypothesis, together with (2.6), we obtain

$$(2.7) \quad u \cdot (a_1v^+ + b_1u^+) + v \cdot (a_1u^+ + c_1v^+) = x \cdot (a_2y^+ + b_2x^+) + y \cdot (a_2x^+ + c_2y^+).$$

Since  $a_i^2 \neq b_i c_i$  for  $i = 1, 2$ , we have  $u, v$  are linearly independent if and only if  $x, y$  are linearly independent. Thus,  $\langle u, v \rangle = \langle x, y \rangle$  when  $u, v$  are linearly independent. If  $u, v$  are linearly dependent, assuming  $u, x \neq 0$ , then  $v = \lambda_1 u$  and  $y = \lambda_2 x$  for some scalars  $\lambda_1, \lambda_2 \in \mathbb{F}$ . By (2.7), we obtain  $(2a_1\lambda_1 + b_1 + \lambda_1^2 c_1)u^2 = (2a_2\lambda_2 + b_2 + \lambda_2^2 c_2)x^2 \neq 0$ , and so  $\langle u \rangle = \langle x \rangle$ . We are done.

(b) The proof of sufficiency is straightforward. We consider necessity. First note that  $u, v, x, y$  are nonzero and  $\langle u, v \rangle = \langle x, y \rangle$  by (a). If  $u, v$  are linearly dependent, then  $\langle u \rangle = \langle v \rangle = \langle x \rangle = \langle y \rangle$ . Let  $u = ax$  and  $v = by$  for some nonzero scalars  $a, b \in \mathbb{F}$ . Then  $x \otimes y = u \otimes v = ab(x \otimes y)$  implies that  $b = a^{-1}$ , as desired. Suppose now that  $u, v$  are linearly independent. Then  $x, y$  are linearly independent and either  $\langle x \rangle \neq \langle v \rangle$  or  $\langle x \rangle \neq \langle u \rangle$ . We consider  $\langle x \rangle \neq \langle v \rangle$  as the second case can be verified similarly. Then  $x = au + bv$  and  $y = cu + dv$  for some  $a, b, c, d \in \mathbb{F}$  with  $a \neq 0$ . Then  $u \otimes v = x \otimes y = (ad + bc)u \otimes v + 2acu^2 + 2bdv^2$  leads to  $(ad + bc - 1)u \otimes v + 2acu^2 + 2bdv^2 = 0$ . Since  $u \otimes v, u^2$  and  $v^2$  are linearly independent, we get  $ad + bc = 1$  and  $ac = 0 = bd$ . Since  $a \neq 0$ , we have  $c = 0$  implies that  $ad = 1$  and  $b = 0$ . So  $x = au$  and  $y = a^{-1}v$ .  $\square$

For each integer  $1 \leq i \leq n$ , we denote

$$\mathcal{U}_{i,n} := \{ (u_1, \dots, u_i, 0, \dots, 0)^T \in \mathcal{M}_{n,1}(\mathbb{F}) \mid u_1, \dots, u_i \in \mathbb{F} \}$$

and  $\mathcal{U}_{0,n} := \{0\} \subset \mathcal{M}_{n,1}(\mathbb{F})$ . When  $n$  is clear from the context,  $\mathcal{U}_{i,n}$  is abbreviated to  $\mathcal{U}_i$ .

LEMMA 2.3. Let  $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$ . Then the following assertions hold.

- (a)  $u^2 \in \mathcal{ST}_n(\mathbb{F}) \setminus \{0\}$  if and only if  $u \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$  for some  $1 \leq p \leq \frac{n+1}{2}$ .
- (b)  $u \circ v \in \mathcal{ST}_n(\mathbb{F}) \setminus \{0\}$  if and only if either
- (i) there exist integers  $1 \leq p \leq n+1-q$  such that  $u \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$  and  $v \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$ , or
  - (ii) there exists an integer  $\frac{n+1}{2} < q \leq n$  such that  $u, v \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$  in which  $v = \alpha u + z$  for some  $\alpha \in \mathbb{F} \setminus \{0\}$  and  $z \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$  with  $1 \leq p \leq n+1-q$ , and this case holds only if  $\mathbb{F}$  has characteristic two.

*Proof.* (a) This is an immediate consequence of (2.6).

(b) Sufficiency is clear. We consider necessity. Since  $u \circ v \neq 0$ , we argue in two cases:

*Case A:* If  $u \circ v$  is of rank one, then, by Lemma 2.1,  $u \circ v = \alpha x^2$  for some  $\alpha \in \mathbb{F} \setminus \{0\}$  and  $x \in \mathcal{U}_p$  with  $1 \leq p \leq \frac{n+1}{2}$ . Then  $\text{char } \mathbb{F} \neq 2$  and  $u, v$  are nonzero linearly dependent vectors such that  $\langle u \rangle = \langle x \rangle = \langle v \rangle$ . So  $u, v \in \mathcal{U}_p$  and statement (i) holds true.

*Case B:* If  $u \circ v$  is of rank two, then, by Lemma 2.1, we consider two subcases:

*Case B-1:*  $u \circ v = \alpha x^2 + \beta y^2$  for some  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$  and linearly independent vectors  $x, y \in \mathcal{U}_p$  with  $1 \leq p \leq \frac{n+1}{2}$ . By Lemma 2.2 (a), we have  $\langle u, v \rangle = \langle x, y \rangle$ . Then  $u, v \in \mathcal{U}_p$  and statement (i) holds true.

*Case B-2:*  $u \circ v = x \circ y + \lambda x^2$  for some  $\lambda \in \mathbb{F}$  and linearly independent vectors  $x \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$ ,  $y \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$  with  $1 \leq p \leq n+1-q \leq n+1-p$  and  $p \neq q$ . By Lemma 2.2 (a), we have  $\langle u, v \rangle = \langle x, y \rangle$ . Then

$$(2.8) \quad u = ax + by \quad \text{and} \quad v = cx + dy$$

for some  $a, b, c, d \in \mathbb{F}$ . We thus have  $u \circ v = (2ac)x^2 + (2bd)y^2 + (ad + bc)x \circ y$ , and hence,

$$(2ac - \lambda^2)x^2 + (2bd)y^2 + (ad + bc - 1)x \circ y = 0.$$

Since  $x^2, y^2, x \circ y$  are linearly independent, we have  $2bd = 0$ . We first consider  $\text{char } \mathbb{F} \neq 2$ . Then  $bd = 0$  implies that either  $b = 0$  or  $d = 0$ . It follows from (2.8) that either  $u \in \mathcal{U}_p$  or  $v \in \mathcal{U}_p$  with  $1 \leq p \leq \frac{n+1}{2}$ , and so statement (i) holds true. Next, if  $\text{char } \mathbb{F} = 2$ , then  $u \circ v = (ad + bc)x \circ y$ . If  $q \leq \frac{n+1}{2}$ ,  $b = 0$ , or  $d = 0$ , then, by (2.8), statement (i) holds. If  $q > \frac{n+1}{2}$  and  $b, d \neq 0$ , then  $1 \leq p < \frac{n+1}{2}$ , and by (2.8), we have  $u, v \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$  and  $y = b^{-1}(u - ax)$ . So

$$v = cx + dy = cx + b^{-1}d(u - ax) = \alpha u + z,$$

where  $\alpha = b^{-1}d \in \mathbb{F}$  and  $z = b^{-1}(ad + bc)x \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$ . It is clear that  $\alpha \neq 0$  and  $u, z$  are linearly independent vectors. Thus, statement (ii) holds.  $\square$



Let  $u \in \mathcal{M}_{n,1}(\mathbb{F})$  and  $\mathcal{V}$  be a subset of  $\mathcal{M}_{n,1}(\mathbb{F})$ . We denote

$$u \circ \mathcal{V} := \{u \circ v : v \in \mathcal{V}\}.$$

It is immediate that  $u \circ \mathcal{V}$  is a linear subspace of  $\mathcal{S}\mathcal{M}_n(\mathbb{F})$  when  $\mathcal{V}$  is a linear subspace.

LEMMA 2.4. *Let  $x \in \mathcal{M}_{n,1}(\mathbb{F})$  be nonzero. If  $x \circ \mathcal{M}_{n,1}(\mathbb{F})$  contains two linearly independent elements  $u \circ v + \alpha u^2$ ,  $u \circ w + \beta u^2$  for some  $u, v, w \in \mathcal{M}_{n,1}(\mathbb{F})$  and  $\alpha, \beta \in \mathbb{F}$ , then  $\langle u \rangle = \langle x \rangle$ .*

*Proof.* Denote  $A = u \circ v + \alpha u^2$  and  $B = u \circ w + \beta u^2$ . Clearly,  $u, x$  are nonzero since  $A, B$  are linearly independent. It follows from Lemma 2.2 (a) that  $x \in \langle u, v \rangle$  and  $x \in \langle u, w \rangle$ . The result follows immediately when  $u, w$  are linearly dependent. Consider now  $u, w$  are linearly independent. Suppose that  $v \notin \langle u, w \rangle$ . Then  $x \in \langle u, v \rangle \cap \langle u, w \rangle = \langle u \rangle$  because  $u, v, w$  are linearly independent. We next consider  $v \in \langle u, w \rangle$ . Then  $A = a(u \circ w) + bu^2$  for some scalars  $a, b \in \mathbb{F}$ . Since  $A, B$  are linearly independent, it follows that  $0 \neq A - aB \in x \circ \mathcal{M}_{n,1}(\mathbb{F})$ , and thus,  $u^2 \in x \circ \mathcal{M}_{n,1}(\mathbb{F})$ . Then  $u^2 = x \circ y$  for some  $y \in \mathcal{M}_{n,1}(\mathbb{F})$ . Since  $u^2$  is of rank one, we have  $x, y$  are linearly dependent. If  $\text{char } \mathbb{F} = 2$ , then  $x \circ y = 0$  by (P3), and so  $u^2 = 0$ , an impossibility. We thus have  $\text{char } \mathbb{F} \neq 2$  and  $u^2 = \lambda x^2$  for some nonzero  $\lambda \in \mathbb{F}$ . Therefore,  $\langle u \rangle = \langle x \rangle$ , as required.  $\square$

Let  $u, v, w \in \mathcal{M}_{n,1}(\mathbb{F})$ . One sees immediately that  $u, v, w$  are linearly independent implies  $u \circ v, v \circ w, w \circ u$  are linearly independent. The converse is true if the characteristic of  $\mathbb{F}$  is two. It can also be checked that if  $u, v, w$  are linearly independent and  $\mathbb{F}$  has characteristic two, then each nonzero element in  $\langle u \circ v, v \circ w, w \circ u \rangle$  has rank two. By this observation, we next obtain a result that describes the uniqueness of  $\langle u \circ v, v \circ w, w \circ u \rangle$ .

LEMMA 2.5. *Let  $\mathbb{F}$  be a field of characteristic two and  $u, v, w, x, y, z \in \mathcal{M}_{n,1}(\mathbb{F})$  be vectors such that  $u, v, w$  are linearly independent. Then  $\langle u \circ v, v \circ w, w \circ u \rangle = \langle x \circ y, y \circ z, z \circ x \rangle$  if and only if  $\langle u, v, w \rangle = \langle x, y, z \rangle$ .*

*Proof.* We first claim that if  $a, b \in \mathcal{M}_{n,1}(\mathbb{F})$  are linearly independent vectors, then

$$(2.9) \quad a \circ b \in \langle u \circ v, v \circ w, w \circ u \rangle \quad \Rightarrow \quad a, b \in \langle u, v, w \rangle.$$

Note that  $a \circ b = \alpha u \circ v + \beta v \circ w + \gamma w \circ u$  for some  $\alpha, \beta, \gamma \in \mathbb{F}$  with  $(\alpha, \beta, \gamma) \neq 0$ . We consider only for the case  $\alpha \neq 0$  as the other cases can be proved similarly. Then  $a \circ b = (u + \beta\alpha^{-1}w) \circ (\alpha v + \gamma w)$  implies that  $\langle a, b \rangle = \langle u + \beta\alpha^{-1}w, \alpha v + \gamma w \rangle$  by Lemma 2.2 (a). We thus have  $a, b \in \langle u + \beta\alpha^{-1}w, \alpha v + \gamma w \rangle \subseteq \langle u, v, w \rangle$ , as claimed.

If  $\langle x \circ y, y \circ z, z \circ x \rangle = \langle u \circ v, v \circ w, w \circ u \rangle$ , then  $x, y, z$  are linearly independent. By (2.9), we have  $x, y, z \in \langle u, v, w \rangle$ , and so  $\langle z, y, z \rangle = \langle u, v, w \rangle$ . Conversely, if  $\langle x, y, z \rangle = \langle u, v, w \rangle$ , then  $x \circ y, y \circ z, z \circ x$  are linearly independent vectors contained

in  $\langle u \otimes v, v \otimes w, w \otimes u \rangle$ .  $\square$

LEMMA 2.6. Let  $\mathbb{F}$  be a field,  $m$  and  $n$  be integers such that  $m \geq n \geq 2$ , and  $P \in \mathcal{M}_{m,n}(\mathbb{F})$  be a full rank matrix. Then  $PAP^+ \in \mathcal{ST}_m(\mathbb{F})$  for every  $A \in \mathcal{ST}_n(\mathbb{F})$  if and only if  $Pe_i \in \mathcal{U}_{p_i,m} \setminus \mathcal{U}_{p_i-1,m}$  for  $i = 1, \dots, n$  such that  $1 \leq p_i \leq \frac{m+1}{2}$  for every  $1 \leq i \leq \frac{n+1}{2}$ , and  $p_i \leq m+1-p_j$  for every  $1 \leq i < j \leq n+1-i$ . In particular,  $P \in \mathcal{T}_n(\mathbb{F})$  when  $m = n$ .

*Proof.* Denote  $u_i = Pe_i$  for  $i = 1, \dots, n$ . So  $u_1, \dots, u_n$  are linearly independent. Let  $Pe_i \in \mathcal{U}_{p_i,m} \setminus \mathcal{U}_{p_i-1,m}$  for  $i = 1, \dots, n$ . Recall that  $\{e_i^2 \mid 1 \leq i \leq \frac{n+1}{2}\} \cup \{e_i \otimes e_j \mid 1 \leq i < j \leq n+1-i\}$  is a basis of  $\mathcal{ST}_n(\mathbb{F})$ . For each  $1 \leq i \leq \frac{n+1}{2}$ , by (P4) and Lemma 2.3 (a), we have  $P(e_i^2)P^+ = u_i^2 \in \mathcal{ST}_m(\mathbb{F})$  since  $p_i \leq \frac{m+1}{2}$ . Again, by (P4) and Lemma 2.3 (b),  $P(e_i \otimes e_j)P^+ = u_i \otimes u_j \in \mathcal{ST}_m(\mathbb{F})$  for every  $1 \leq i < j \leq n+1-i$ . This proves sufficiency. For necessity, we argue in the following two cases.

*Case I:  $m > n$ .* In view of Lemma 2.3 (a),  $u_i^2 = P(e_i^2)P^+ \in \mathcal{ST}_m(\mathbb{F})$  for  $1 \leq i \leq \frac{n+1}{2}$  implies that  $1 \leq p_i \leq \frac{m+1}{2}$  for every  $1 \leq i \leq \frac{n+1}{2}$ . On the other hand, by Lemma 2.3 (b),  $u_i \otimes u_j = P(e_i \otimes e_j)P^+ \in \mathcal{ST}_m(\mathbb{F})$  and  $p_i \leq \frac{m+1}{2}$  for  $1 \leq i < j \leq n+1-i$  leads to  $p_i \leq m+1-p_j$  for every  $1 \leq i < j \leq n+1-i$ . This establishes the desired conclusion.

*Case II:  $m = n$ .* We shall show that  $p_i = i$  for  $i = 1, \dots, n$  by induction on  $i$ . To begin with, note that the linear independence of  $u_1, \dots, u_n$  implies that  $p_{i_0} = n$  for some  $1 \leq i_0 \leq n$ . By the fact that  $u_1^2, u_1 \otimes u_{i_0} \in \mathcal{ST}_n(\mathbb{F})$ , we conclude that  $p_1 = 1$ . Suppose that the inductive hypothesis holds, i.e.,  $p_j = j$  for  $j = 1, \dots, k$  for some  $k < n$ . We wish to claim that  $p_{k+1} = k+1$ . Since  $u_1, \dots, u_{k+1}$  are linearly independent, together with our induction hypothesis, we have  $k+1 \leq p_{k+1} \leq n$ . Since  $u_1, \dots, u_{n-k}$  are linearly independent, there exists an integer  $1 \leq i_1 \leq n-k$  such that  $n-k \leq p_{i_1} \leq n$ . Note that  $u_{k+1} \otimes u_1, \dots, u_{k+1} \otimes u_{n-k} \in \mathcal{ST}_n(\mathbb{F})$ , and also  $u_{k+1}^2 \in \mathcal{ST}_n(\mathbb{F})$  provided that  $k+1 \leq n-k$ . We consider two possibilities.

- Say  $i_1 = k+1$ . Then  $p_{i_1} = p_{k+1}$  and  $k+1 = i_1 \leq n-k$ . We thus have  $u_{k+1}^2 \in \mathcal{ST}_n(\mathbb{F})$ . Hence,  $k+1 \leq p_{k+1} \leq \frac{n+1}{2}$  and  $n-k \leq p_{i_1} \leq \frac{n+1}{2}$ . So,  $k \geq \frac{n-1}{2}$  and thus  $k = \frac{n-1}{2}$ , since  $k+1 \leq n-k$ . Hence,  $k+1 = \frac{n+1}{2}$ . Therefore,  $p_{k+1} = k+1$ .
- Say  $i_1 \neq k+1$ . Since  $i_1 \leq n-k$ , we have  $u_{k+1} \otimes u_{i_1} \in \mathcal{ST}_n(\mathbb{F})$ . Then  $k+1 \leq \frac{n+1}{2}$  or  $i_1 \leq \frac{n+1}{2}$ . To see this, if  $k+1 \leq \frac{n+1}{2}$ , then we are done. Suppose that  $k+1 > \frac{n+1}{2}$ . Then  $i_1 \leq n-k$  implies  $k+1 \leq n+1-i_1$ , and so  $n+1-i_1 > \frac{n+1}{2}$ . We thus have  $i_1 < \frac{n+1}{2}$ , as desired. In consequence,  $p_{k+1} \leq \frac{m+1}{2}$  or  $p_{i_1} \leq \frac{m+1}{2}$ . By Lemma 2.3 (b), we have  $p_{k+1} \leq n+1-p_{i_1}$ . Then since  $p_{i_1} \geq n-k$ , we have  $n-k \leq n+1-p_{k+1}$ , and so  $p_{k+1} \leq k+1$ . Together with  $p_{k+1} \geq k+1$ , we conclude that  $p_{k+1} = k+1$ .

By induction, we conclude that  $Pe_i \in \mathcal{U}_{i,n} \setminus \mathcal{U}_{i-1,n}$  for  $i = 1, \dots, n$ . It follows that  $P \in \mathcal{T}_n(\mathbb{F})$ .  $\square$

**3. Linear spaces of bounded rank-two matrices.** We recall that a linear subspace of a matrix space is a linear space of bounded rank-two matrices provided each matrix in it has rank bounded above by two. In [10], Lim classified linear spaces of bounded rank-two symmetric matrices over an infinite field of characteristic not two. Indeed, by a slight modification in the last paragraph of the proof of [10, Theorem 3, p. 49], the result holds for any field of characteristic not two. More recently, [5, Theorem 2.6] completes the work on characterization of spaces of bounded rank-two symmetric matrices over a field of characteristic two.

In this section, using the structural results of [10, Theorem 3] and [5, Theorem 2.6], we classify spaces of bounded rank-two per-symmetric triangular matrices over an arbitrary field. By treating the symmetricity on the minor diagonal, we can now rephrase [10, Theorem 3] and [5, Theorem 2.6] as follows.

LEMMA 3.1. *Let  $\mathbb{F}$  be a field and  $n$  be an integer such that  $n \geq 2$ . Let  $\mathcal{S}$  be a linear subspace of  $\mathcal{SM}_n(\mathbb{F})$ . Then  $\mathcal{S}$  is a linear space of bounded rank-two matrices if and only if one of the following holds:*

- (I)  $\mathcal{S} \subseteq \langle u^2, v^2, u \otimes v \rangle$  for some linearly independent vectors  $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$ .
- (II)  $\mathcal{S} \subseteq u \otimes \mathcal{M}_{n,1}(\mathbb{F})$  for some nonzero  $u \in \mathcal{M}_{n,1}(\mathbb{F})$ .
- (III)  $\mathcal{S} = u \otimes \mathcal{V} + \langle u^2 \rangle$  for some nonzero  $u \in \mathcal{M}_{n,1}(\mathbb{F})$  and some linear subspace  $\mathcal{V}$  of  $\mathcal{M}_{n,1}(\mathbb{F})$ ; and  $\mathcal{S}$  is of this form only if  $\text{char } \mathbb{F} = 2$ . Here,  $+$  denotes the sum of linear subspaces of  $\mathcal{SM}_n(\mathbb{F})$ .
- (IV)  $\mathcal{S} = \langle u \otimes v_1 + \lambda_1 u^2, \dots, u \otimes v_k + \lambda_k u^2 \rangle$  for some linearly independent vectors  $u, v_1, \dots, v_k$  in  $\mathcal{M}_{n,1}(\mathbb{F})$  and some  $\lambda_1, \dots, \lambda_k \in \mathbb{F}$  with  $(\lambda_1, \dots, \lambda_k) \neq 0$ ; and  $\mathcal{S}$  is of this form only if  $\text{char } \mathbb{F} = 2$ .
- (V)  $\mathcal{S} = \langle u \otimes v, u \otimes w, v \otimes w \rangle$  for some linearly independent vectors  $u, v, w$  in  $\mathcal{M}_{n,1}(\mathbb{F})$ ; and  $\mathcal{S}$  is of this form only if  $\text{char } \mathbb{F} = 2$ .
- (VI)  $\mathcal{S} \subseteq \langle u^2 + v^2, u^2 + w^2, (u + v) \otimes (u + w) \rangle$  for some linearly independent vectors  $u, v, w$  in  $\mathcal{M}_{n,1}(\mathbb{F})$ ; and  $\mathcal{S}$  is of this form only if  $|\mathbb{F}| = 2$ .

Let  $\mathbb{F}$  be a field of characteristic not two. As a side remark, we notice from (2.6) that  $x \otimes y + \alpha x^2 = x \otimes (y + \frac{\alpha}{2}x)$  for every  $x, y \in \mathcal{M}_{n,1}(\mathbb{F})$  and  $\alpha \in \mathbb{F}$ , and thus, any linear space of bounded rank-two of Form (III) or (IV) in Lemma 3.1 can be simplified to Form (I) or (II) in Lemma 3.1. On the other hand, for any linearly independent vectors  $u, v, w \in \mathcal{M}_{n,1}(\mathbb{F})$ ,  $\langle u \otimes v, u \otimes w, v \otimes w \rangle$  contains rank three matrices. By a direct verification,  $\text{rank}(u \otimes v + u \otimes w + v \otimes w) = 3$  since

$$\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -2 \neq 0.$$

In consequence, Form (V) in Lemma 3.1 is not a linear space of bounded rank-two when  $\text{char } \mathbb{F} \neq 2$ .

LEMMA 3.2. *Let  $\mathbb{F}$  be a field of characteristic two. Let  $\alpha \in \mathbb{F}$  be nonzero and  $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$  be linearly independent vectors. Then the following assertions hold.*

- (a)  $u \otimes v + \alpha u^2 \in \mathcal{ST}_n(\mathbb{F})$  if and only if  $u \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$  and  $v \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n+1-p$ .
- (b)  $u^2 + v^2 \in \mathcal{ST}_n(\mathbb{F})$  if and only if  $u+v \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$  and  $u, v \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n+1-p$ .

*Proof.* (a) Since  $\text{char } \mathbb{F} = 2$ , the minor diagonal of  $u \otimes v$  is zero. Then  $u \otimes v + \alpha u^2 \in \mathcal{ST}_n(\mathbb{F})$  with  $\alpha \neq 0$  if and only if  $u^2, u \otimes v \in \mathcal{ST}_n(\mathbb{F})$  if and only if  $u \in \mathcal{U}_p$  for some integer  $1 \leq p \leq \frac{n+1}{2}$ , and  $v \in \mathcal{U}_q$  for some integer  $1 \leq q \leq n+1-p$  by Lemma 2.3.

(b) By noting  $u^2 + v^2 = (u+v) \otimes v + (u+v)^2$  and  $(u+v) \otimes v = (u+v) \otimes u$ , the conclusion follows immediately from part (a).  $\square$

We are now in a position to provide a characterization of spaces of bounded rank-two per-symmetric triangular matrices over an arbitrary field.

THEOREM 3.3. *Let  $\mathbb{F}$  be a field and  $n$  be an integer such that  $n \geq 2$ . Let  $\mathcal{S}$  be a linear subspace of  $\mathcal{ST}_n(\mathbb{F})$ . Then  $\mathcal{S}$  is a linear space of bounded rank-two matrices if and only if one of the following holds:*

- (a)  $\mathcal{S} \subseteq \langle u^2, v^2, u \otimes v \rangle$  for some linearly independent vectors  $u, v \in \mathcal{U}_p$  with  $1 \leq p \leq \frac{n+1}{2}$ .
- (b)  $\mathcal{S} = u \otimes \mathcal{V}$  for some nonzero  $u \in \mathcal{U}_p$  and some linear subspace  $\mathcal{V}$  of  $\mathcal{U}_q$  with  $1 \leq p \leq n+1-q \leq n$ .
- (c)  $\mathcal{S} = u \otimes \mathcal{V} + \langle u^2 \rangle$  for some nonzero  $u \in \mathcal{U}_p$  with  $1 \leq p \leq \frac{n+1}{2}$  and some linear subspace  $\mathcal{V}$  of  $\mathcal{U}_q$  with  $1 \leq q \leq n+1-p \leq n$ ; and  $\mathcal{S}$  is of this form only if  $\text{char } \mathbb{F} = 2$ .
- (d)  $\mathcal{S} = \langle u \otimes v_1 + \lambda_1 u^2, \dots, u \otimes v_k + \lambda_k u^2 \rangle$  for some scalars  $\lambda_1, \dots, \lambda_k \in \mathbb{F}$  with  $(\lambda_1, \dots, \lambda_k) \neq 0$ , and some linearly independent vectors  $u, v_1, \dots, v_k$  such that  $u \in \mathcal{U}_p, v_1, \dots, v_k \in \mathcal{U}_q$  with  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n+1-p \leq n$ ; and  $\mathcal{S}$  is of this form only if  $\text{char } \mathbb{F} = 2$ .
- (e)  $\mathcal{S} = \langle u \otimes v, u \otimes w, v \otimes w \rangle$  for some linearly independent vectors  $u \in \mathcal{U}_p, v \in \mathcal{U}_q$  and  $w \in \mathcal{U}_r$  such that  $1 \leq p, q \leq n+1-r \leq n$  and  $p \leq n+1-q$ ; and  $\mathcal{S}$  is of this form only if  $\text{char } \mathbb{F} = 2$ .
- (f) *There exist linearly independent vectors  $u, v, w \in \mathcal{M}_{n,1}(\mathbb{F})$  such that*
  - $\circ \mathcal{S} = \langle u^2 + v^2, u^2 + w^2, (u+v) \otimes (u+w) \rangle$ , or  $\mathcal{S} = \langle u^2 + v^2, u^2 + w^2 \rangle$ , or  $\mathcal{S} = \langle x^2 + y^2, (x+z) \otimes y + (x+z)^2 \rangle$  with  $\{x, y, z\} = \{u, v, w\}$ , where  $u+v, u+w \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n+1-p$ , or

- $\mathcal{S} = \langle x^2 + y^2, (u + v) \circ (u + w) \rangle$  for a pair of distinct vectors  $x, y \in \{u, v, w\}$  with  $x + y \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n + 1 - p$ ;  
 and  $\mathcal{S}$  is of this form only if  $|\mathbb{F}| = 2$ .

*Proof.* If  $\mathcal{S}$  satisfies one of the statements (a) - (f) in Theorem 3.3, then  $\mathcal{S}$  is a linear space of bounded rank-two matrices of  $\mathcal{SM}_n(\mathbb{F})$ . Moreover, by Lemmas 2.3 and 3.2, we have  $\mathcal{S} \subseteq \mathcal{ST}_n(\mathbb{F})$ . This proves sufficiency.

We now consider necessity. Suppose that  $\mathcal{S} \neq \{0\}$ . Since  $\mathcal{S}$  is a linear space of bounded rank-two matrices of  $\mathcal{SM}_n(\mathbb{F})$ , we see that  $\mathcal{S}$  satisfies one of the statements (I) - (VI) as described in Lemma 3.1. We use the notation that have been employed in Lemma 3.1 and divide our argument in the following cases.

*Case I:* Suppose that  $\mathcal{S}$  satisfies (I) in Lemma 3.1. Let  $u \in \mathcal{U}_s \setminus \mathcal{U}_{s-1}$  and  $v \in \mathcal{U}_t \setminus \mathcal{U}_{t-1}$  for some integers  $1 \leq s, t \leq n$ . We divide our argument into the following three subcases:

- *Case I-i:* If  $1 \leq s, t \leq \frac{n+1}{2}$ , then  $u, v \in \mathcal{U}_p$  with  $p = \max\{s, t\}$ . So  $\mathcal{S}$  satisfies statement (a).
- *Case I-ii:* If  $1 \leq s \leq n + 1 - t$  and  $\frac{n+1}{2} < t \leq n$ , then  $u^2, u \circ v \in \mathcal{ST}_n(\mathbb{F})$  and  $v^2 \notin \mathcal{ST}_n(\mathbb{F})$ . If  $\mathcal{S}$  has no rank two matrices, then  $\mathcal{S} = \langle u^2 \rangle$  and it satisfies statement (a). Suppose that  $\mathcal{S}$  has a rank two matrix. Then  $\mathcal{S} \subseteq \langle u^2, u \circ v \rangle$  and it is of one of the following forms:
  - $\mathcal{S} = \langle u \circ v \rangle = u \circ \langle v \rangle$  and it satisfies statement (b);
  - $\mathcal{S} = \langle u \circ v + au^2 \rangle$  with  $a \in \mathbb{F} \setminus \{0\}$ . When  $\text{char } \mathbb{F} = 2$ ,  $\mathcal{S}$  satisfies statement (d); and when  $\text{char } \mathbb{F} \neq 2$ , we get  $\mathcal{S} = u \circ \langle v + \frac{a}{2}u \rangle$  and it satisfies statement (b);
  - $\mathcal{S} = \langle u^2, u \circ v \rangle$ . When  $\text{char } \mathbb{F} = 2$ , we obtain  $\mathcal{S} = u \circ \langle v \rangle + \langle u^2 \rangle$  and it satisfies statement (c); when  $\text{char } \mathbb{F} \neq 2$ , we see that  $\mathcal{S} = u \circ \langle v, 2^{-1}u \rangle$  and it satisfies statement (b).
- *Case I-iii:* Suppose that  $\frac{n+1}{2} < s, t \leq n$ . If  $\mathcal{S}$  contains no rank two matrices, then  $\dim \mathcal{S} = 1$ . By Lemma 2.1, we have  $\mathcal{S} = \langle x^2 \rangle$  for some nonzero vector  $x \in \mathcal{U}_p$  with  $1 \leq p \leq \frac{n+1}{2}$ . Thus,  $\mathcal{S}$  satisfies statement (a). Suppose now that  $\mathcal{S}$  has a rank two matrix, say  $A$ . Then  $A = au^2 + bv^2 + cu \circ v$  for some  $a, b, c \in \mathbb{F}$  with  $c^2 - ab \neq 0$  by (P5). On the other hand, by Lemma 2.1, there exist linearly independent vectors  $x, y$  such that either  $A = \alpha x^2 + \beta y^2$  for some  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$  and  $x, y \in \mathcal{U}_p$  with  $1 \leq p \leq \frac{n+1}{2}$ ; or  $A = x \circ y + \alpha x^2$  for some  $\alpha \in \mathbb{F} \setminus \{0\}$  and some  $x \in \mathcal{U}_p$  and  $y \in \mathcal{U}_q$  with  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n + 1 - p$ . Then

$$au^2 + bv^2 + cu \circ v = \alpha x^2 + \beta y^2 \quad \text{or} \quad au^2 + bv^2 + cu \circ v = x \circ y + \alpha x^2.$$

In both cases,  $\langle x, y \rangle = \langle u, v \rangle$  by Lemma 2.2 (a). Thus,  $\langle x^2, y^2, x \circ y \rangle =$

$\langle u^2, v^2, u \circ v \rangle$ , and so  $\mathcal{S} \subseteq \langle x^2, y^2, x \circ y \rangle$ . The result follows by a similar argument as in Cases I-i and I-ii.

*Case II:* If  $\mathcal{S}$  satisfies (II) in Lemma 3.1, then for each nonzero  $A \in \mathcal{S}$ , there exists a nonzero  $v_A \in \mathcal{M}_{n,1}(\mathbb{F})$  such that  $A = u \circ v_A$ . Since  $A \in \mathcal{ST}_n(\mathbb{F})$ , it follows from Lemma 2.3 that

- (i)  $u \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$  and  $v_A \in \mathcal{U}_{r_A} \setminus \mathcal{U}_{r_A-1}$  for some integers  $1 \leq p \leq n+1-r_A$ , or
- (ii)  $u \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$  and  $v_A = \alpha_A u + z_A \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$  for some  $\alpha_A \in \mathbb{F} \setminus \{0\}$  and  $z_A \in \mathcal{U}_{r_A} \setminus \mathcal{U}_{r_A-1}$ , where  $1 \leq r_A \leq n+1-p < \frac{n+1}{2}$  and  $u, z_A$  are linearly independent, and in addition, this case holds only if  $\text{char } \mathbb{F} = 2$ .

Notice that if (ii) holds, then  $\text{char } \mathbb{F} = 2$  and  $A$  can be rewritten as

$$A = u \circ (\alpha_A u + z_A) = u \circ z_A.$$

Consequently, in view of (i) and (ii), for each  $A \in \mathcal{S}$ , there exists  $v_A \in \mathcal{U}_{r_A} \setminus \mathcal{U}_{r_A-1}$  with  $1 \leq r_A \leq n+1-p$  such that  $A = u \circ v_A$ . Accordingly, there exists a linear subspace  $\mathcal{V}$  of  $\mathcal{U}_q$  with  $1 \leq q \leq n+1-p$  such that  $\mathcal{S} = u \circ \mathcal{V}$ . Thus,  $\mathcal{S}$  satisfies (b).

*Case III:* If  $\mathcal{S}$  satisfies (III) in Lemma 3.1, then  $u^2, u \circ v \in \mathcal{ST}_n(\mathbb{F})$  for every  $v \in \mathcal{V}$ . It follows from Lemma 2.3 that  $u \in \mathcal{U}_p$  for some  $1 \leq p \leq \frac{n+1}{2}$ , and for each  $v \in \mathcal{V}$ , there exists an integer  $1 \leq r_v \leq n+1-p$  such that  $v \in \mathcal{U}_{r_v}$ . Consequently,  $\mathcal{V}$  is a subspace of  $\mathcal{U}_q$  for some integer  $1 \leq q \leq n+1-p \leq n$ . Hence,  $\mathcal{S}$  satisfies (c).

*Case IV:* If  $\mathcal{S}$  satisfies (IV) in Lemma 3.1, then  $u \circ v_i + \lambda_i u^2 \in \mathcal{ST}_n(\mathbb{F})$  for every  $i = 1, \dots, k$ . Since  $u, v_1, \dots, v_k$  are linearly independent and  $(\lambda_1, \dots, \lambda_k) \neq 0$ , the result follows directly from Lemma 3.2 (a) and  $\mathcal{S}$  satisfies (d).

*Case V:* If  $\mathcal{S}$  satisfies (V) of Lemma 3.1, then  $u \circ v, u \circ w, v \circ w \in \mathcal{ST}_n(\mathbb{F})$ . In view of Lemma 2.3, each pair of elements of  $\{u, v, w\}$  satisfies either (b)(i) or (b)(ii) of Lemma 2.3. If all pairs of elements of  $\{u, v, w\}$  satisfy (b)(i) of Lemma 2.3, then  $\mathcal{S}$  is readily seen to satisfy (e). Suppose not. We shall show that  $\{u, v, w\}$  can be replaced by some other  $\{x, y, z\}$  such that  $\mathcal{S} = \langle x \circ y, x \circ z, y \circ z \rangle$  satisfies (e). With no loss of generality, say  $\{u, v\}$  satisfies (b)(ii) of Lemma 2.3. Then  $u \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$  and  $v = \alpha u + y \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$ , where  $y \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$ , for some  $\alpha \in \mathbb{F} \setminus \{0\}$  and integers  $p, q$  such that  $1 \leq p \leq n+1-q < \frac{n+1}{2}$ . Note that  $\langle u, y, w \rangle = \langle u, v, w \rangle$ , from which, together with Lemma 2.5, follows that  $\mathcal{S} = \langle u \circ y, u \circ w, y \circ w \rangle$ . If  $\{u, w\}$  satisfies (b)(i) of Lemma 2.3, we are done by setting  $x = u$  and  $z = w$ . Otherwise, say  $\{u, w\}$  satisfies (b)(ii) of Lemma 2.3. Then  $u \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$  and  $w = \beta u + z \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$ , where  $z \in \mathcal{U}_r \setminus \mathcal{U}_{r-1}$ , for some  $\beta \in \mathbb{F} \setminus \{0\}$  and an integer  $r$  such that  $1 \leq r \leq n+1-q < \frac{n+1}{2}$ . As before, we get that  $\langle u, y, z \rangle = \langle u, y, w \rangle$ , from which follows that  $\mathcal{S} = \langle u \circ y, u \circ z, y \circ z \rangle$  satisfies (e). Setting  $u = x$ , we are done.

Case VI: Suppose  $\mathcal{S}$  satisfies (VI) in Lemma 3.1. Note that

$$\begin{aligned} &\langle u^2 + v^2, u^2 + w^2, (u + v) \circ (u + w) \rangle \\ &= \{0, u^2 + v^2, u^2 + w^2, v^2 + w^2, (u + v) \circ (u + w), (u + v) \circ w + (u + v)^2, \\ &\quad (u + w) \circ v + (u + w)^2, (v + w) \circ u + (v + w)^2\} \end{aligned}$$

with

$$\begin{aligned} u^2 + v^2 + (u + v) \circ (u + w) &= (u + v) \circ w + (u + v)^2, \\ u^2 + w^2 + (u + v) \circ (u + w) &= (u + w) \circ v + (u + w)^2, \\ v^2 + w^2 + (u + v) \circ (u + w) &= (v + w) \circ u + (v + w)^2, \end{aligned}$$

and each nonzero matrix in  $\mathcal{S}$  is of rank two. We argue in the following three cases:

- If  $\dim \mathcal{S} = 1$ , then  $\mathcal{S} = \langle A \rangle$  for some nonzero per-symmetric upper triangular matrix  $A \in \langle u^2 + v^2, u^2 + w^2, (u + v) \circ (u + w) \rangle$ . By Lemma 2.1, there exist linearly independent vectors  $x, y$  such that either (i)  $A = \alpha x^2 + \beta y^2$  with  $x, y \in \mathcal{U}_p$  for some integer  $1 \leq p \leq \frac{n+1}{2}$  and  $\alpha, \beta \in \mathbb{F} \setminus \{0\}$ ; or (ii)  $A = x \circ y + \gamma x^2$  with  $x \in \mathcal{U}_p$  and  $y \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n + 1 - p$ , and  $\gamma \in \mathbb{F}$ . Then  $\mathcal{S}$  satisfies (a) when (i) holds,  $\mathcal{S}$  satisfies (b) when (ii) holds with  $\gamma = 0$ , or  $\mathcal{S}$  satisfies (d) when (ii) holds with  $\gamma \neq 0$ .
- If  $\dim \mathcal{S} = 3$ , then  $\mathcal{S} = \langle u^2 + v^2, u^2 + w^2, (u + v) \circ (u + w) \rangle$ . Since  $u^2 + v^2$  and  $u^2 + w^2$  are in  $\mathcal{ST}_n(\mathbb{F})$ , Lemma 3.2 (b) implies that  $u + v \in \mathcal{U}_{p_1} \setminus \mathcal{U}_{p_1-1}$ ,  $u + w \in \mathcal{U}_{p_2} \setminus \mathcal{U}_{p_2-1}$ ,  $u, v \in \mathcal{U}_{q_1} \setminus \mathcal{U}_{q_1-1}$ , and  $u, w \in \mathcal{U}_{q_2} \setminus \mathcal{U}_{q_2-1}$  for some integers  $p_1, p_2, q_1, q_2$  such that  $1 \leq p_i \leq \frac{n+1}{2}$  and  $1 \leq q_i \leq n + 1 - p_i$  for  $i = 1, 2$ . Since  $u \in (\mathcal{U}_{q_1} \setminus \mathcal{U}_{q_1-1}) \cap (\mathcal{U}_{q_2} \setminus \mathcal{U}_{q_2-1})$ , it is necessary that  $q_1 = q_2 = q$  for a common  $q$ . Setting  $p = \max\{p_1, p_2\}$ , we note that  $\mathcal{S}$  satisfies (f).
- If  $\dim \mathcal{S} = 2$ , then one of the following holds:
  - $\mathcal{S} = \{0, u^2 + v^2, u^2 + w^2, v^2 + w^2\} = \langle u^2 + v^2, u^2 + w^2 \rangle$ , where  $u + v, u + w \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n + 1 - p$ ;
  - $\mathcal{S} = \{0, u^2 + v^2, (u + w) \circ v + (u + w)^2, (v + w) \circ u + (v + w)^2\} = \langle u^2 + v^2, (u + w) \circ v + (u + w)^2 \rangle$ , where  $u + v, u + w \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n + 1 - p$ ;
  - $\mathcal{S} = \{0, u^2 + w^2, (v + w) \circ u + (v + w)^2, (u + v) \circ w + (u + v)^2\} = \langle u^2 + w^2, (u + v) \circ w + (u + v)^2 \rangle$ , where  $u + v, u + w \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n + 1 - p$ ;
  - $\mathcal{S} = \{0, v^2 + w^2, (u + v) \circ w + (u + v)^2, (u + w) \circ v + (u + w)^2\} = \langle v^2 + w^2, (v + u) \circ w + (v + u)^2 \rangle$ , where  $v + w, v + u \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n + 1 - p$ ;
  - $\mathcal{S} = \{0, u^2 + v^2, (u + v) \circ (u + w), (u + v) \circ w + (u + v)^2\} = \langle u^2 + v^2, (u + v) \circ (u + w) \rangle$ , where  $u + v \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n + 1 - p$ ;

- $\mathcal{S} = \{0, u^2 + w^2, (u + v) \otimes (u + w), (u + w) \otimes v + (u + w)^2\} = \langle u^2 + w^2, (u + v) \otimes (u + w) \rangle$ , where  $u + w \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n + 1 - p$ ;
- $\mathcal{S} = \{0, v^2 + w^2, (u + v) \otimes (u + w), (v + w) \otimes u + (v + w)^2\} = \langle v^2 + w^2, (u + v) \otimes (u + w) \rangle$ , where  $v + w \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n + 1 - p$ .

Hence,  $\mathcal{S}$  satisfies (f).  $\square$

We now continue our investigation of 2-spaces of  $\mathcal{ST}_n(\mathbb{F})$ . We first study some examples of 2-spaces of  $\mathcal{ST}_n(\mathbb{F})$ .

EXAMPLE 3.4. Let  $\mathbb{F}$  be a field and  $n$  be an integer such that  $n \geq 2$ . Recall that  $\{e_1, \dots, e_n\}$  denotes the standard basis of  $\mathcal{M}_{n,1}(\mathbb{F})$ .

- (a) Let  $n \geq 2$ . Then  $\langle e_1 \otimes e_2 + \alpha e_1^2 \rangle$  is a 1-dimensional 2-space of  $\mathcal{ST}_n(\mathbb{F})$  for any  $\alpha \in \mathbb{F}$ .
- (b) Let  $n \geq 3$  and  $\alpha, \beta, \gamma \in \mathbb{F}$  be such that  $\gamma^2 \neq \alpha\beta$ . Then  $\langle \alpha e_1^2 + \beta e_2^2 + \gamma e_1 \otimes e_2 \rangle$  is a 1-dimensional 2-space of  $\mathcal{ST}_n(\mathbb{F})$ .
- (c) Let  $n \geq 3$  and  $\mathbb{F} = \mathbb{R}$ . Then  $\langle e_1 \otimes e_2, e_1 \otimes e_2 + e_1^2 - e_2^2 \rangle$  is a 2-dimensional 2-space of  $\mathcal{ST}_n(\mathbb{R})$ . Let  $A = a(e_1 \otimes e_2) + b(e_1 \otimes e_2 + e_1^2 - e_2^2) \in \langle e_1 \otimes e_2, e_1 \otimes e_2 + e_1^2 - e_2^2 \rangle$  for some  $a, b \in \mathbb{R}$  with  $(a, b) \neq 0$ . We see that  $A$  is of rank two since

$$\det \begin{bmatrix} a+b & b \\ -b & a+b \end{bmatrix} = (a+b)^2 + b^2 \neq 0.$$

- (d) Let  $\mathbb{F}$  be a field with four elements. Then  $\text{char } \mathbb{F} = 2$  and the multiplicative group of  $\mathbb{F}$  is cyclic. We set  $\mathbb{F} = \{0, 1, \alpha, \alpha^2\}$ , where  $\alpha$  is a primitive element of  $\mathbb{F}$ . We see that  $\langle e_1 \otimes e_2 + e_1^2, e_1 \otimes e_2 + \alpha e_2^2 \rangle$  is a 2-dimensional 2-space of  $\mathcal{ST}_n(\mathbb{F})$ . To proof this, let  $A = \lambda_1(e_1 \otimes e_2 + e_1^2) + \lambda_2(e_1 \otimes e_2 + \alpha e_2^2) \in \langle e_1 \otimes e_2 + e_1^2, e_1 \otimes e_2 + \alpha e_2^2 \rangle$  for some  $\lambda_1, \lambda_2 \in \mathbb{F}$  with  $(\lambda_1, \lambda_2) \neq 0$ . By a direct verification, we have

$$\det \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 \\ \lambda_2 \alpha & \lambda_1 + \lambda_2 \end{bmatrix} = \lambda_1^2 + \lambda_2^2 + \alpha \lambda_1 \lambda_2 \neq 0.$$

Hence,  $A$  is of rank two.

EXAMPLE 3.5. Let  $\mathbb{F}$  be a field of characteristic two. Let  $u \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$ ,  $v \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$  and  $w \in \mathcal{U}_r \setminus \mathcal{U}_{r-1}$  be linearly independent vectors such that  $1 \leq p, q \leq n + 1 - r$  and  $p \leq n + 1 - q$ . Then  $u \otimes v, u \otimes w, v \otimes w$  are linearly independent elements in  $\mathcal{ST}_n(\mathbb{F})$  and each nonzero element in  $\langle u \otimes v, u \otimes w, v \otimes w \rangle$  has rank two. Thus,  $\langle u \otimes v, u \otimes w, v \otimes w \rangle$  is a 3-dimensional 2-space of  $\mathcal{ST}_n(\mathbb{F})$ . Note also that each element in  $\langle u \otimes v, u \otimes w, v \otimes w \rangle$  has a zero minor diagonal.

EXAMPLE 3.6. Let  $\mathbb{F}$  be a field of characteristic two. Let  $u \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$  and



$v_1, \dots, v_k \in \mathcal{U}_q \setminus \mathcal{U}_{q-1}$  be linearly independent vectors such that  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n+1-p$ , and  $\lambda_1, \dots, \lambda_k \in \mathbb{F}$  be such that  $(\lambda_1, \dots, \lambda_k) \neq 0$ . It is easily checked that  $u \otimes v_1 + \lambda_1 u^2, \dots, u \otimes v_k + \lambda_k u^2$  are linearly independent. Let  $A \in \langle u \otimes v_1 + \lambda_1 u^2, \dots, u \otimes v_k + \lambda_k u^2 \rangle$  be nonzero. Then there exist  $\beta_1, \dots, \beta_k \in \mathbb{F}$  not all of which are zero such that

$$\begin{aligned} A &= \beta_1(u \otimes v_1 + \lambda_1 u^2) + \dots + \beta_k(u \otimes v_k + \lambda_k u^2) \\ &= u \otimes (\beta_1 v_1 + \dots + \beta_k v_k) + (\beta_1 \lambda_1 + \dots + \beta_k \lambda_k) u^2. \end{aligned}$$

Since  $u, v_1, \dots, v_k$  are linearly independent and  $(\beta_1, \dots, \beta_k) \neq 0$ , we get  $\beta_1 v_1 + \dots + \beta_k v_k, u$  are linearly independent, and so  $\text{rank } A = 2$ . Then  $\langle u \otimes v_1 + \lambda_1 u^2, \dots, u \otimes v_k + \lambda_k u^2 \rangle$  is a  $k$ -dimensional 2-space of  $\mathcal{ST}_n(\mathbb{F})$ .

As an immediate consequence of Theorem 3.3, we obtain a complete description of 2-spaces of  $\mathcal{ST}_n(\mathbb{F})$  over an arbitrary field  $\mathbb{F}$ .

**COROLLARY 3.7.** *Let  $\mathbb{F}$  be a field and  $n$  be an integer such that  $n \geq 2$ . Then  $\mathcal{S}$  is a 2-space of  $\mathcal{ST}_n(\mathbb{F})$  if and only if one of the following holds:*

- (a)  $\mathcal{S} = \langle a_i u \otimes v + b_i u^2 + c_i v^2 \mid i = 1, 2 \rangle$  for some linearly independent vectors  $u, v \in \mathcal{U}_p$  with  $1 \leq p \leq \frac{n+1}{2}$ , and some fixed scalars  $a_i, b_i, c_i \in \mathbb{F}$  for  $i = 1, 2$  such that

$$(\lambda_1 a_1 + \lambda_2 a_2)^2 \neq (\lambda_1 b_1 + \lambda_2 b_2)(\lambda_1 c_1 + \lambda_2 c_2)$$

for every  $\lambda_1, \lambda_2 \in \mathbb{F}$  with  $(\lambda_1, \lambda_2) \neq 0$ .

- (b)  $\mathcal{S} = u \otimes \mathcal{V}$  for some nonzero vector  $u \in \mathcal{U}_p$  and some subspace  $\mathcal{V}$  of  $\mathcal{U}_q$  with  $1 \leq p \leq n+1-q \leq n$ , and  $\mathcal{V} \cap \langle u \rangle = \{0\}$  when  $\text{char } \mathbb{F} \neq 2$ .
- (c)  $\mathcal{S} = \langle u \otimes v_1 + \lambda_1 u^2, \dots, u \otimes v_k + \lambda_k u^2 \rangle$  for some scalars  $\lambda_1, \dots, \lambda_k \in \mathbb{F}$  with  $(\lambda_1, \dots, \lambda_k) \neq 0$ , and some linearly independent vectors  $u, v_1, \dots, v_k$  such that  $u \in \mathcal{U}_p$  with  $1 \leq p \leq \frac{n+1}{2}$  and  $v_1, \dots, v_k \in \mathcal{U}_q$  with  $1 \leq q \leq n+1-p$ ; and  $\mathcal{S}$  is of this form only if  $\text{char } \mathbb{F} = 2$ .
- (d)  $\mathcal{S} = \langle u \otimes v, u \otimes w, v \otimes w \rangle$  for some linearly independent vectors  $u \in \mathcal{U}_p, v \in \mathcal{U}_q$  and  $w \in \mathcal{U}_r$  such that  $1 \leq p, q \leq n+1-r \leq n$  and  $p \leq n+1-q$ ; and  $\mathcal{S}$  is of this form only if  $\text{char } \mathbb{F} = 2$ .
- (e) There exist linearly independent vectors  $u, v, w \in \mathcal{M}_{n,1}(\mathbb{F})$  such that
- $\mathcal{S} = \langle u^2 + v^2, u^2 + w^2, (u+v) \otimes (u+w) \rangle$ , or  $\mathcal{S} = \langle u^2 + v^2, u^2 + w^2 \rangle$ , or  $\mathcal{S} = \langle x^2 + y^2, (x+z) \otimes y + (x+z)^2 \rangle$  with  $\{x, y, z\} = \{u, v, w\}$ , where  $u+v, u+w \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n+1-p$ ; or
  - $\mathcal{S} = \langle x^2 + y^2, (u+v) \otimes (u+w) \rangle$  for a pair of distinct vectors  $x, y \in \{u, v, w\}$  with  $x+y \in \mathcal{U}_p$  and  $u, v, w \in \mathcal{U}_q$  for some integers  $1 \leq p \leq \frac{n+1}{2}$  and  $1 \leq q \leq n+1-p$ ,
- and  $\mathcal{S}$  is of this form only if  $|\mathbb{F}| = 2$ .

**4. Bounded rank-two linear preservers.** In this section, we characterize bounded rank-two linear preservers  $\psi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{SM}_m(\mathbb{F})$ , with  $m, n \geq 3$  and  $\text{char } \mathbb{F} \neq 2$ . We then obtain a classification of bounded rank-two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two.

We start with the following lemma whose proof is straightforward and omitted.

LEMMA 4.1. *Let  $\mathbb{F}$  be a field and  $u, v, x, y, z \in \mathcal{M}_{n,1}(\mathbb{F})$ . Then the following statements hold.*

- (a) *If  $x, y$  are linearly independent, then the following are equivalent.*
- (i)  $ax^2 + by^2 + cx \circ y \in \langle u^2, v^2, u \circ v \rangle$  for some  $a, b, c \in \mathbb{F}$  with  $ab \neq c^2$ .
  - (ii)  $\langle x, y \rangle = \langle u, v \rangle$ .
  - (iii)  $\langle x^2, y^2, x \circ y \rangle = \langle u^2, v^2, u \circ v \rangle$ .
- (b) *If  $y, z$  are linearly independent and  $x \circ y, x \circ z \in \langle u^2, v^2, u \circ v \rangle$ , then  $x \in \langle y, z \rangle$ .*

LEMMA 4.2. *Let  $\mathbb{F}$  be a field of characteristic not two and  $n$  be an integer such that  $n \geq 2$ . Let  $A = u \circ v$  and  $B = w \circ z$  be nonzero matrices for some  $u, v, w, z \in \mathcal{M}_{n,1}(\mathbb{F})$  such that  $u, v, w$  are linearly independent. If  $\text{rank}(A + \lambda B) \leq 2$  for all  $\lambda \in \mathbb{F}$ , then either  $z \in \langle u \rangle$  or  $z \in \langle v \rangle$ .*

*Proof.* Since  $u, v, w$  are linearly independent and  $\text{rank}(A + B) \leq 2$ , we have  $z \in \langle u, v, w \rangle$ . Let  $z = au + bv + cw$  for some  $a, b, c \in \mathbb{F}$ . Since  $A + \lambda B = 2\lambda cw^2 + u \circ v + \lambda a(u \circ w) + \lambda b(w \circ v)$  has rank bounded above by two, it follows that

$$0 = \det \begin{bmatrix} 1 & \lambda a & 0 \\ \lambda b & 2\lambda c & \lambda a \\ 0 & \lambda b & 1 \end{bmatrix} = -2ab\lambda^2 + 2c\lambda \quad \text{for every } \lambda \in \mathbb{F}.$$

Since  $|\mathbb{F}| \geq 3$ , we obtain  $c = 0$  and  $ab = 0$ .  $\square$

THEOREM 4.3. *Let  $\mathbb{F}$  be a field of characteristic not two and  $m, n$  be integers such that  $m, n \geq 3$ . Then  $\psi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{SM}_m(\mathbb{F})$  is a bounded rank-two linear preserver if and only if  $m \geq n$  and  $\psi$  is of one of the following forms:*

- (i) *There exist a nonzero vector  $u \in \mathcal{M}_{m,1}(\mathbb{F})$  and a linear mapping  $\varphi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{M}_{m,1}(\mathbb{F})$  such that*

$$(4.1) \quad \psi(A) = u \circ \varphi(A) \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

*where  $\varphi(A) \neq 0$  for every nonzero bounded rank-two matrix  $A \in \mathcal{ST}_n(\mathbb{F})$ .*

- (ii) *There exist a full rank matrix  $P \in \mathcal{M}_{m,n}(\mathbb{F})$  and a nonzero  $\lambda \in \mathbb{F}$  such that*

$$\psi(A) = \lambda PAP^+ \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}).$$

(iii) When  $n = 4$ , in addition to (i) and (ii),  $\psi$  also takes the form

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{12} & \alpha a_{13} + \theta(a_{14} - a_{23}) & \beta a_{14} \\ 0 & a_{22} & (2\alpha - \beta)a_{23} & \alpha a_{13} + \theta(a_{14} - a_{23}) \\ 0 & 0 & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_4(\mathbb{F})$ , where  $P \in \mathcal{M}_{m,4}(\mathbb{F})$  is a full rank matrix,  $\alpha, \beta \in \mathbb{F}$  are nonzero with  $\beta \neq 2\alpha$ , and  $\theta \in \mathbb{F}$  is nonzero only if  $|\mathbb{F}| = 3$ .

(iv) When  $n = 3$ , in addition to (i) and (ii),  $\psi$  also takes one of the following forms:

(a) There exist a surjective linear mapping  $\phi : \mathcal{ST}_3(\mathbb{F}) \rightarrow \mathbb{F}^3$  and a full rank matrix  $P \in \mathcal{M}_{m,2}(\mathbb{F})$  such that

$$\psi(A) = P \begin{bmatrix} \phi(A)_3 & \phi(A)_1 \\ \phi(A)_2 & \phi(A)_3 \end{bmatrix} P^+ \quad \text{for all } A \in \mathcal{ST}_3(\mathbb{F}),$$

where  $\phi(A)_i$  denotes the  $i$ -th component of  $\phi(A) \in \mathbb{F}^3$  and  $\phi(A) \neq 0$  for every nonzero bounded rank-two matrix  $A \in \mathcal{ST}_3(\mathbb{F})$ .

(b) There exist a full rank matrix  $P \in \mathcal{M}_{m,3}(\mathbb{F})$  and  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$  with  $\lambda_3 \neq 0$  such that either

$$\psi(A) = P \begin{bmatrix} a_{pp} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} & \eta_1 a_{12} + \lambda_1 a_{qq} \\ 0 & \lambda_3 a_{qq} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} \\ 0 & 0 & a_{pp} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$ , where  $\eta_1, \eta_2 \in \mathbb{F}$  are nonzero and  $\{p, q\} = \{1, 2\}$ , or

$$\psi(A) = P \begin{bmatrix} a_{pp} & a_{1s} + \lambda_2 a_{qq} & \eta a_{1t} + \lambda_1 a_{qq} \\ 0 & \lambda_3 a_{qq} & a_{1s} + \lambda_2 a_{qq} \\ 0 & 0 & a_{pp} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$ , where  $\eta \in \mathbb{F}$  is nonzero and  $\{p, q\} = \{s, t\} = \{1, 2\}$ .

*Proof.* Sufficiency is clear. We now consider necessity. Let  $\mathcal{X}_1 = e_1 \otimes \langle e_1, \dots, e_n \rangle$  and  $\mathcal{X}_2 = e_2 \otimes \langle e_1, \dots, e_{n-1} \rangle$ . By Lemma 3.1 and Theorem 3.3, together with the assumption of  $\psi$ , we see that  $\psi(\mathcal{X}_1)$  and  $\psi(\mathcal{X}_2)$  are spaces of bounded rank-two matrices of  $\mathcal{SM}_m(\mathbb{F})$  containing linearly independent sets  $\{\psi(e_1 \otimes e_1), \dots, \psi(e_1 \otimes e_n)\}$  and  $\{\psi(e_2 \otimes e_1), \dots, \psi(e_2 \otimes e_{n-1})\}$ , respectively. Thus,  $m \geq n$ . We now divide our proof into three main cases:

*Case I:*  $n \geq 5$ . By Lemma 3.1, we have

$$(4.2) \quad \psi(e_1^2) = u \otimes v_1 \quad \text{and} \quad \psi(e_1 \otimes e_i) = u \otimes v_i \quad \text{for } i = 2, \dots, n,$$

for some nonzero vector  $u \in \mathcal{M}_{m,1}(\mathbb{F})$  and linearly independent vectors  $v_1, \dots, v_n \in \mathcal{M}_{m,1}(\mathbb{F})$ , and

$$(4.3) \quad \psi(e_2^2) = x \otimes y_2 \quad \text{and} \quad \psi(e_2 \otimes e_i) = x \otimes y_i \quad \text{for } i = 1, 3, 4, \dots, n-1,$$

for some nonzero vector  $x \in \mathcal{M}_{m,1}(\mathbb{F})$  and linearly independent vectors  $y_1, \dots, y_{n-1} \in \mathcal{M}_{m,1}(\mathbb{F})$ . We consider the following two subcases:

*Case 1-A:*  $\langle x \rangle = \langle u \rangle$ . There is no loss of generality in assuming  $x = u$ . For each  $3 \leq i \leq \frac{n+1}{2}$ , let  $\mathcal{X}_i = e_i \otimes \langle e_1, e_2, e_i \rangle$ . Clearly,  $\psi(\mathcal{X}_i)$  is a 3-dimensional linear space of bounded rank-two matrices of  $\mathcal{SM}_m(\mathbb{F})$ . Then each  $\psi(\mathcal{X}_i)$  can be expressed in either of the forms (I) and (II) in Lemma 3.1. Suppose that there exists  $3 \leq i_0 \leq \frac{n+1}{2}$  such that  $\psi(\mathcal{X}_{i_0})$  satisfies (I). Since  $\psi(e_{i_0} \otimes e_1) = u \otimes v_{i_0}$ ,  $\psi(e_{i_0} \otimes e_2) = u \otimes y_{i_0}$  and  $\psi(e_{i_0}^2)$  are linearly independent elements in  $\psi(\mathcal{X}_{i_0})$ ,  $v_{i_0}, y_{i_0}$  are linearly independent, and together with Lemma 4.1 (a), we have  $\langle u^2, v_{i_0}^2, u \otimes v_{i_0} \rangle = \psi(\mathcal{X}_{i_0}) = \langle u^2, y_{i_0}^2, u \otimes y_{i_0} \rangle$ . Again, by Lemma 4.1,  $\langle u, v_{i_0} \rangle = \langle u, y_{i_0} \rangle$ . In particular,  $y_{i_0} \in \langle u, v_{i_0} \rangle$ . Then  $\psi(e_{i_0} \otimes e_2) = \eta_1 u^2 + \eta_2 u \otimes v_{i_0}$  and  $\psi(e_{i_0}^2) = \alpha u^2 + \beta v_{i_0}^2 + \gamma u \otimes v_{i_0}$  for some  $\eta_1, \eta_2, \alpha, \beta, \gamma \in \mathbb{F}$  with  $\eta_1, \beta \neq 0$ . By (4.2), note that

- if  $u, v_1$  are linearly dependent, then  $\psi(e_1^2) = \lambda_1 u^2$  for some  $\lambda_1 \in \mathbb{F} \setminus \{0\}$ ;
- if  $u, v_1$  are linearly independent, then  $v_{i_0} \in \langle u, v_1 \rangle$ . For, if not, then  $v_{i_0}, u, v_1$  are linearly independent, and so  $\psi(e_{i_0}^2 + e_1^2) = \alpha u^2 + \beta v_{i_0}^2 + \gamma u \otimes v_{i_0} + u \otimes v_1$  is of rank three, a contradiction. Therefore,  $v_1 \in \langle u, v_{i_0} \rangle$  since  $\{u, v_{i_0}\}$  is linearly independent. Thus,  $\psi(e_1^2) = \varsigma_1 (u \otimes v_{i_0}) + \lambda_1 u^2$  for some scalars  $\varsigma_1, \lambda_1 \in \mathbb{F}$  with  $\lambda_1 \neq 0$ .

Accordingly, we may write generally that

$$(4.4) \quad \psi(e_1^2) = \varsigma_1 (u \otimes v_{i_0}) + \lambda_1 u^2$$

for some  $\varsigma_1, \lambda_1 \in \mathbb{F}$  with  $(\varsigma_1, \lambda_1) \neq 0$ . We apply this argument again, with (4.2) and  $v_1$  replaced by (4.3) and  $y_2$ , to obtain

$$(4.5) \quad \psi(e_2^2) = \varsigma_2 (u \otimes v_{i_0}) + \lambda_2 u^2$$

for some  $\varsigma_2, \lambda_2 \in \mathbb{F}$  with  $(\varsigma_2, \lambda_2) \neq 0$ . Furthermore, since  $\psi(\langle e_1^2, e_2^2, e_1 \otimes e_2 \rangle)$  has dimension three, it follows from (4.2), (4.4) and (4.5) that  $u, v_{i_0}, v_2$  are linearly independent. Then

$$\psi(e_{i_0}^2 + (e_1^2 + e_2^2 + e_1 \otimes e_2)) = \beta v_{i_0}^2 + u \otimes v_2 + (\alpha + \lambda_1 + \lambda_2) u^2 + (\gamma + \varsigma_1 + \varsigma_2) u \otimes v_{i_0}$$

is of rank three, a contradiction. Thus,  $\psi(\mathcal{X}_i)$  satisfies (II) in Lemma 3.1 for every  $3 \leq i \leq \frac{n+1}{2}$ . Consequently, by Lemma 2.4, for each  $3 \leq i \leq \frac{n+1}{2}$ , there exists a nonzero vector  $z_i \in \mathcal{M}_{m,1}(\mathbb{F})$  such that

$$(4.6) \quad \psi(e_i^2) = u \otimes z_i.$$

For  $n \geq 6$ , we will consider  $\mathcal{X}_{ij} = e_i \otimes \langle e_1, e_2, e_i, e_j \rangle$  for any  $3 \leq i \leq \frac{n+1}{2}$  and  $i < j \leq n + 1 - i$ . Clearly,  $\psi(\mathcal{X}_{ij})$  is a linear space of bounded rank-two matrices of  $\mathcal{SM}_m(\mathbb{F})$  containing linearly independent elements  $\psi(e_i \otimes e_j)$ ,  $\psi(e_i^2)$ ,  $\psi(e_1 \otimes e_i)$ ,  $\psi(e_2 \otimes e_i)$ . By Lemmas 3.1 and 2.4, we obtain  $\psi(\mathcal{X}_{ij}) \subseteq u \otimes \mathcal{M}_{m,1}(\mathbb{F})$ . Then for each  $3 \leq i \leq \frac{n+1}{2}$  and  $i < j \leq n + 1 - i$ , there exists a nonzero vector  $v_{ij} \in \mathcal{M}_{m,1}(\mathbb{F})$  such that

$$(4.7) \quad \psi(e_i \otimes e_j) = u \otimes v_{ij}.$$

Consequently, by (4.2), (4.3), (4.6), (4.7) and the linearity of  $\psi$ , we conclude, for  $n \geq 5$ , that there exists a linear mapping  $\varphi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{M}_{m,1}(\mathbb{F})$  such that

$$\psi(A) = u \otimes \varphi(A) \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

where  $\varphi(A) \neq 0$  for every nonzero bounded rank-two matrix  $A \in \mathcal{ST}_n(\mathbb{F})$ . Hence, (4.1) holds.

*Case 1-B:*  $\langle x \rangle \neq \langle u \rangle$ . By (4.2) and (4.3), we see that  $u \otimes v_2 = \psi(e_1 \otimes e_2) = x \otimes y_1$ . It follows from Lemma 2.2 (b) that  $y_1 = \varsigma u$  and  $x = \varsigma^{-1}v_2$  for some nonzero scalar  $\varsigma \in \mathbb{F}$ , because  $u, x$  are linearly independent. Then

$$(4.8) \quad \psi(e_1 \otimes e_2) = \varsigma u \otimes x.$$

Our next claim is that

$$(4.9) \quad \{u, x, v_3, \dots, v_n\} \text{ is linearly independent.}$$

We first show that  $v_1 \in \langle u, x \rangle$ . Suppose that  $v_1 \notin \langle u, x \rangle$ . Since  $\text{rank } \psi(e_1^2 + \gamma e_2^2) \leq 2$  for all  $\gamma \in \mathbb{F}$ , we have either  $y_2 \in \langle u \rangle$  or  $y_2 \in \langle v_1 \rangle$  by Lemma 4.2. Note that  $\langle y_1 \rangle = \langle u \rangle$  and  $\langle y_1 \rangle \neq \langle y_2 \rangle$  implies  $y_2 \in \langle v_1 \rangle$ , and so  $y_2 \notin \langle u, x \rangle$ . Let  $v_1 = \epsilon y_2$  for some nonzero scalar  $\epsilon \in \mathbb{F}$ . It follows that  $\psi((e_1 + e_2)^2) = \epsilon u \otimes y_2 + x \otimes y_2 + \varsigma u \otimes x$  is of rank three, a contradiction. Hence,  $v_1 \in \langle u, x \rangle$ . Similarly, we obtain  $y_2 \in \langle u, x \rangle$ . By (4.2) and (4.3), since  $\psi(e_1^2), \psi(e_2^2), \psi(e_1 \otimes e_2)$  are linearly independent, we obtain

$$(4.10) \quad \psi(e_1^2) = u \otimes (\theta_1 x + \vartheta_1 u) \quad \text{and} \quad \psi(e_2^2) = x \otimes (\theta_2 u + \vartheta_2 x)$$

for some scalars  $\theta_1, \theta_2, \vartheta_1, \vartheta_2 \in \mathbb{F}$  with  $\vartheta_1, \vartheta_2 \neq 0$ . Since  $\psi(e_1^2), \psi(e_1 \otimes e_2), \dots, \psi(e_1 \otimes e_n)$  are linearly independent, it follows from (4.2), (4.8) and (4.10) that  $\{\theta_1 x + \vartheta_1 u, \varsigma x, v_3, \dots, v_n\}$  is a linearly independent set, and hence, Claim (4.9) is proved.

Let  $3 \leq i \leq n - 1$ . Since  $\text{rank } \psi((e_1 + \gamma e_2) \otimes e_i) \leq 2$  for every  $\gamma \in \mathbb{F}$ , it follows from (4.9) and Lemma 4.2 that either  $y_i \in \langle v_i \rangle$  or  $y_i \in \langle u \rangle$ . Since  $u \in \langle y_1 \rangle$ , we have  $y_i \in \langle v_i \rangle$ . Setting  $w_1 = u$ ,  $w_2 = x$ , and  $w_i = v_i$  for  $i = 3, \dots, n$ , we thus have  $\{w_1, \dots, w_n\}$  is linearly independent by (4.9). In view of (4.2), (4.3) and (4.8), we have

$$(4.11) \quad \psi(e_1 \otimes e_2) = \varsigma w_1 \otimes w_2 \quad \text{and} \quad \psi(e_1 \otimes e_n) = w_1 \otimes w_n,$$

and for each  $3 \leq i \leq n-1$ , there exists a nonzero scalar  $\zeta_i \in \mathbb{F}$  such that  $\psi(e_1 \otimes e_i) = w_1 \otimes w_i$  and  $\psi(e_2 \otimes e_i) = \zeta_i w_2 \otimes w_i$ . Moreover, since  $1 \leq \text{rank } \psi((e_1 + e_2) \otimes (e_i + e_j)) \leq 2$  for every distinct pair  $3 \leq i, j \leq n-1$ , we have  $\zeta_i = \zeta_j$  for any distinct integers  $3 \leq i, j \leq n-1$ . Consequently, there exists a nonzero scalar  $\zeta \in \mathbb{F}$  such that

$$(4.12) \quad \psi(e_1 \otimes e_i) = w_1 \otimes w_i \quad \text{and} \quad \psi(e_2 \otimes e_i) = \zeta w_2 \otimes w_i$$

for all  $i = 3, \dots, n-1$ .

We next claim that for each  $1 \leq i \leq \frac{n+1}{2}$ , there exists a nonzero scalar  $\mu_i \in \mathbb{F}$  such that

$$(4.13) \quad \psi(e_i^2) = \mu_i w_i^2.$$

Recall that  $\mathcal{X}_i = e_i \otimes \langle e_1, e_2, e_i \rangle$  for  $3 \leq i \leq \frac{n+1}{2}$ . Then  $\psi(\mathcal{X}_i)$  is a 3-dimensional linear space of bounded rank-two matrices of  $\mathcal{SM}_m(\mathbb{F})$ . In view (4.12), since  $w_1, w_2, w_i$  are linearly independent, it follows from Lemma 4.1 (b) that  $\psi(\mathcal{X}_i)$  is of Form (II) in Lemma 3.1. Thus,  $\psi(\mathcal{X}_i) \subseteq w_i \otimes \mathcal{M}_{m,1}(\mathbb{F})$  by Lemma 2.4. For each  $3 \leq i \leq \frac{n+1}{2}$ , there exists a nonzero vector  $z_i \in \mathcal{M}_{m,1}(\mathbb{F})$  such that  $\psi(e_i^2) = w_i \otimes z_i$ . We shall show that  $\theta_1 = 0$ . Suppose not. In view of (4.9), we have  $\{w_1, \theta_1 w_2 + \vartheta_1 w_1, w_i\}$  and  $\{w_1, 2\theta_1 w_2 + 2\vartheta_1 w_1 + w_i, w_i\}$  are linearly independent sets. Since  $\text{rank } \psi(e_1^2 + \gamma e_i^2) \leq 2$  and  $\text{rank } \psi(e_1 \otimes (e_1 + e_i) + \gamma e_i^2) \leq 2$  for all  $\gamma \in \mathbb{F}$ , it follows from (4.10), (4.12) and Lemma 4.2 that

$$z_i \in \langle w_1 \rangle \quad \text{or} \quad z_i \in \langle \theta_1 w_2 + \vartheta_1 w_1 \rangle,$$

and

$$z_i \in \langle w_1 \rangle \quad \text{or} \quad z_i \in \langle 2\theta_1 w_2 + 2\vartheta_1 w_1 + w_i \rangle.$$

We thus have  $z_i \in \langle w_1 \rangle$ . Therefore,  $\psi(e_i^2), \psi(e_1 \otimes e_i)$  are linearly dependent, a contradiction. Hence,  $\theta_1 = 0$ . Thus,  $\psi(e_1^2) \in \langle w_1^2 \rangle$  by (4.10). Similarly, we can show that  $\theta_2 = 0$  in (4.10). Consequently, Claim (4.13) holds for  $i = 1, 2$ . We now consider  $3 \leq i \leq \frac{n+1}{2}$ . Since  $\text{rank } \psi(e_1 \otimes (e_1 + e_i) + \gamma e_i^2) \leq 2$  for every  $\gamma \in \mathbb{F}$ , we have  $\text{rank}(w_1 \otimes (\mu_1 w_1 + w_i) + \gamma z_i \otimes w_i) \leq 2$  for every  $\gamma \in \mathbb{F}$ . If  $z_i \notin \langle w_1, w_i \rangle$ , then, by Lemma 4.2, we have either  $w_i \in \langle w_1 \rangle$  or  $w_i \in \langle \mu_1 w_1 + w_i \rangle$ . Since  $w_1, w_i$  are linearly independent, we obtain  $\mu_1 = 0$ , a contradiction. Therefore,  $z_i \in \langle w_1, w_i \rangle$ . Furthermore, since  $\text{rank } \psi(e_2 \otimes (e_2 + e_i) + \gamma e_i^2) \leq 2$  for all  $\gamma \in \mathbb{F}$ , in the same manner we can show that  $z_i \in \langle w_2, w_i \rangle$ . Hence,  $z_i \in \langle w_1, w_i \rangle \cap \langle w_2, w_i \rangle = \langle w_i \rangle$ . Accordingly, Claim (4.13) is proved.

Next, we consider  $n \geq 6$ . Let  $3 \leq i \leq \frac{n+1}{2}$  and  $i+1 \leq j \leq n+1-i$ . Recall that  $\mathcal{X}_{ij} = e_i \otimes \langle e_1, e_2, e_i, e_j \rangle$ . Since  $\psi(\mathcal{X}_{ij})$  is a linear space of bounded rank-two matrices of  $\mathcal{SM}_m(\mathbb{F})$  containing linearly independent elements  $\psi(e_i \otimes e_j), \psi(e_i^2), \psi(e_1 \otimes e_i)$ ,

$\psi(e_2 \otimes e_i)$ , it follows from Lemmas 3.1 and 2.4 that  $\psi(\mathcal{X}_{ij}) \subseteq w_i \otimes \mathcal{M}_{m,1}(\mathbb{F})$ . Then there exists a nonzero vector  $z_{ij} \in \mathcal{M}_{m,1}(\mathbb{F})$  such that  $\psi(e_i \otimes e_j) = w_i \otimes z_{ij}$ . On the other hand,  $\psi(e_j \otimes \langle e_1, e_2, e_i \rangle)$  is a linear space of bounded rank-two matrices of  $\mathcal{SM}_m(\mathbb{F})$  containing linearly independent elements  $\psi(e_i \otimes e_j)$ ,  $\psi(e_1 \otimes e_j)$ ,  $\psi(e_2 \otimes e_j)$ . Since  $w_1, w_2, w_j$  are linearly independent, it follows from Lemmas 3.1, 4.1 (b) and 2.4 that  $\psi(e_j \otimes \langle e_1, e_2, e_i \rangle) \subseteq w_j \otimes \mathcal{M}_{m,1}(\mathbb{F})$ . Then  $\psi(e_i \otimes e_j) = w_j \otimes y_{ij}$  for some nonzero vector  $y_{ij} \in \mathcal{M}_{m,1}(\mathbb{F})$ . Therefore,  $w_i \otimes z_{ij} = \psi(e_i \otimes e_j) = w_j \otimes y_{ij}$ , and so  $\langle z_{ij} \rangle = \langle w_j \rangle$  and  $\langle y_{ij} \rangle = \langle w_i \rangle$  by Lemma 2.2 (b). Consequently, for each  $3 \leq i \leq \frac{n+1}{2}$  and  $i+1 \leq j \leq n+1-i$ , there exists a nonzero scalar  $\eta_{ij} \in \mathbb{F}$  such that

$$(4.14) \quad \psi(e_i \otimes e_j) = \eta_{ij} w_i \otimes w_j.$$

After composing the map:  $A \mapsto \mu_1^{-1}A$  for  $A \in \mathcal{SM}_m(\mathbb{F})$ , if necessary, we have

$$(4.15) \quad \psi(e_1^2) = w_1^2 \quad \text{and} \quad \psi(e_1 \otimes e_i) = \mu_1^{-1} w_1 \otimes w_i$$

for  $i = 3, \dots, n$ , and for simplicity of notation, we abbreviate  $\mu_1^{-1}\zeta$  to  $\zeta$  in (4.11),  $\mu_1^{-1}\zeta$  to  $\zeta$  in (4.12),  $\mu_1^{-1}\mu_i$  to  $\mu_i$  in (4.13) for  $2 \leq i \leq \frac{n+1}{2}$ , and  $\mu_1^{-1}\eta_{ij}$  to  $\eta_{ij}$  in (4.14) for  $3 \leq i \leq \frac{n+1}{2}$  and  $i+1 \leq j \leq n+1-i$ . Since  $\text{rank } \psi((e_1 + e_2)^2 + e_k^2) \leq 2$  and  $\text{rank } \psi((e_i + e_k)^2 + e_j^2) \leq 2$  for any distinct integers  $1 \leq i, j \leq 2$  and  $3 \leq k \leq \frac{n+1}{2}$ , it follows from (4.11), (4.12), (4.13) and (4.15) that  $\mu_2 = \zeta^2$ ,  $\zeta^2 = (\mu_1^{-1}\zeta)^2$  and  $\mu_i = (\mu_1^{-1})^2$  for  $3 \leq i \leq \frac{n+1}{2}$ . Moreover, in view of (4.12), (4.13), (4.14) and (4.15), we have  $\psi((e_1 + e_i) \otimes e_j + (e_1 + e_i)^2) = \mu_1^{-1} w_1 \otimes w_j + \eta_{ij} w_i \otimes w_j + w_1^2 + (\mu_1^{-1})^2 w_i^2 + \mu_1^{-1} w_1 \otimes w_i$  is of rank bounded above by two for every  $3 \leq i \leq \frac{n+1}{2}$  and  $i < j \leq n+1-i$ , and hence,

$$0 = \det \begin{bmatrix} \mu_1^{-1} & \mu_1^{-1} & 1 \\ \eta_{ij} & (\mu_1^{-1})^2 & \mu_1^{-1} \\ 0 & \eta_{ij} & \mu_1^{-1} \end{bmatrix} = ((\mu_1^{-1})^2 - \eta_{ij})^2 \Rightarrow \eta_{ij} = (\mu_1^{-1})^2$$

for every  $3 \leq i \leq \frac{n+1}{2}$  and  $i < j \leq n+1-i$ . Also, since  $\zeta^2 = (\mu_1^{-1}\zeta)^2$ , we have either  $\zeta = \mu_1^{-1}\zeta$  or  $\zeta = -\mu_1^{-1}\zeta$ . Suppose that  $\zeta = -\mu_1^{-1}\zeta$ . Then

$$\psi((e_1 + e_2 + e_3)^2 - e_3^2) = w_1^2 + \zeta^2 w_2^2 + \zeta w_1 \otimes w_2 + \mu_1^{-1} w_1 \otimes w_3 + (-\mu_1^{-1}\zeta) w_2 \otimes w_3$$

is of rank three, a contradiction. So  $\zeta = \mu_1^{-1}\zeta$ . Consequently, by (4.11), (4.12), (4.13), (4.14) and (4.15) that  $\psi(e_i^2) = (\alpha_i e_i)^2$  for all  $1 \leq i \leq \frac{n+1}{2}$ , and  $\psi(e_i \otimes e_j) = (\alpha_i w_i) \otimes (\alpha_j w_j)$  for all  $1 \leq i \leq \frac{n+1}{2}$  and  $i < j \leq n+1-i$ , where  $\alpha_1 = 1$ ,  $\alpha_2 = \zeta$  and  $\alpha_i = \mu_1^{-1}$  for  $i = 3, \dots, n$ . Let  $P \in \mathcal{M}_{m,n}(\mathbb{F})$  be the matrix defined by  $P e_i = \alpha_i w_i$  for every  $i = 1, \dots, n$ . Evidently,  $P$  is of rank  $n$  since  $\{w_1, \dots, w_n\}$  is linearly independent. By the linearity of  $\psi$ , we conclude that

$$\psi(A) = \lambda P A P^+ \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

where  $\lambda = \mu_1^{-1} \in \mathbb{F}$  is nonzero. We are done.

*Case II:  $n = 4$ .* Let  $\mathcal{S}_1 = e_1 \circ \langle e_1, e_2, e_3, e_4 \rangle$  and  $\mathcal{S}_2 = e_2 \circ \langle e_1, e_2, e_3 \rangle$ . Then  $\psi(\mathcal{S}_1)$  and  $\psi(\mathcal{S}_2)$  are 4-dimensional and 3-dimensional spaces of bounded rank-two matrices of  $\mathcal{SM}_m(\mathbb{F})$ , respectively. By Lemma 3.1, there exist a nonzero vector  $u \in \mathcal{M}_{m,1}(\mathbb{F})$  and linearly independent vectors  $v_1, v_2, v_3, v_4 \in \mathcal{M}_{m,1}(\mathbb{F})$  such that

$$(4.16) \quad \psi(e_1^2) = u \circ v_1 \quad \text{and} \quad \psi(e_1 \circ e_i) = u \circ v_i \quad \text{for } i = 2, 3, 4,$$

and  $\psi(\mathcal{S}_2)$  is either of Form (I) or Form (II) in Lemma 3.1. We claim that  $\psi(\mathcal{S}_2)$  is of Form (II). Suppose to the contrary that  $\psi(\mathcal{S}_2)$  is of Form (I). We argue in the following two cases:

*Case II-1:  $\langle v_2 \rangle \neq \langle u \rangle$ .* By Lemma 4.1 (a), we obtain  $\psi(\mathcal{S}_2) = \langle u^2, v_2^2, u \circ v_2 \rangle$ . Let  $\psi(e_2^2) = \mu_1 u \circ v_2 + \mu_2 u^2 + \mu_3 v_2^2$  and  $\psi(e_2 \circ e_3) = \eta_1 u \circ v_2 + \eta_2 u^2 + \eta_3 v_2^2$  for some  $\mu_i, \eta_i \in \mathbb{F}$ ,  $i = 1, 2, 3$ . Suppose that  $\mu_3 \neq 0$ . Note that  $\text{rank} \psi(e_1^2 + e_2^2) \leq 2$  implies  $v_1 \in \langle u, v_2 \rangle$ . Since  $v_1, v_2, v_3$  are linearly independent, it follows that  $u, v_2, v_3$  are linearly independent. In view of (4.16), we have  $\psi(e_1^2) = \lambda_1 u \circ v_2 + \lambda_2 u^2$  for some scalars  $\lambda_1, \lambda_2 \in \mathbb{F}$  with  $\lambda_2 \neq 0$ . We set

$$\zeta = \begin{cases} 1 & \text{if } \eta_3 = 0, \\ \eta_3^{-1} \mu_3 & \text{if } \eta_3 \neq 0. \end{cases}$$

Then  $\zeta \neq 0$  and  $\zeta \eta_3 + \mu_3 \neq 0$ , and

$$\begin{aligned} \psi(\zeta(e_1 + e_2) \circ e_3 + (e_1 + e_2)^2) &= \zeta u \circ v_3 + (\zeta \eta_3 + \mu_3) v_2^2 \\ &\quad + (\zeta \eta_1 + \lambda_1 + \mu_1 + 1) u \circ v_2 + (\zeta \eta_2 + \lambda_2 + \mu_2) u^2 \end{aligned}$$

is of rank three, a contradiction. Hence,  $\mu_3 = 0$ . Since  $\psi(e_2 \circ e_1), \psi(e_2^2), \psi(e_2 \circ e_3)$  are linearly independent, it follows that  $\eta_3 \neq 0$ . By a similar argument, with  $\psi(e_1^2)$  replaced by  $\psi(e_2 \circ e_3)$ , to obtain  $\psi(\zeta'(e_1 + e_2) \circ e_3 + (e_1 + e_2)^2)$  is of rank three for  $\zeta' \in \mathbb{F}$ , which is impossible.

*Case II-2:  $\langle v_2 \rangle = \langle u \rangle$ .* By (4.16), we have  $\psi(e_1 \circ e_2) = \alpha u^2$  for some  $\alpha \in \mathbb{F} \setminus \{0\}$  and  $v_1, u, v_3, v_4$  are linearly independent. Let  $\psi(\mathcal{S}_2) = \langle x^2, v^2, x \circ v \rangle$  for some linearly independent vectors  $x, v \in \mathcal{M}_{m,1}(\mathbb{F})$ . Since  $\psi(e_1 \circ e_2) \in \psi(\mathcal{S}_2)$ , it follows that  $\alpha u^2 = \theta_1 x^2 + \theta_2 v^2 + \theta_3 x \circ v$  for some  $\theta_1, \theta_2, \theta_3 \in \mathbb{F}$  with  $(\theta_1, \theta_2, \theta_3) \neq 0$ . We now show that  $u \in \langle x, v \rangle$ . Suppose to the contrary that  $u \notin \langle x, v \rangle$ . If  $\theta_3 = 0$ , then  $\alpha u^2 - \theta_1 x^2 - \theta_2 v^2 = 0$  implies that  $\alpha = \theta_1 = \theta_2 = 0$ , a contradiction. Thus,  $\theta_3 \neq 0$ , and so  $\alpha u^2 - \theta_3 x \circ v = \theta_1 x^2 + \theta_2 v^2$ , which is an impossibility. We thus have  $u \in \langle x, v \rangle$ . Since  $x, v$  are linearly independent, we may assume without loss of generality that  $u, v$  are linearly independent. Then  $\langle u, v \rangle = \langle x, v \rangle$ , so  $\psi(\mathcal{S}_2) = \langle u^2, v^2, u \circ v \rangle$  by Lemma 4.1 (a). Let  $\psi(e_2^2) = a_1 u \circ v + a_2 u^2 + a_3 v^2$  for some  $a_1, a_2, a_3 \in \mathbb{F}$ . Suppose that  $a_3 \neq 0$ . Since  $\psi(e_1^2 + e_2^2) = u \circ v_1 + a_1 u \circ v + a_2 u^2 + a_3 v^2$  has rank bounded



above by two, it follows that  $v \in \langle u, v_1 \rangle$ . Then  $\langle u^2, v^2, u \otimes v \rangle = \langle u^2, v_1^2, u \otimes v_1 \rangle$ , and therefore,  $\psi(e_2^2) = b_1 u \otimes v_1 + b_2 u^2 + b_3 v_1^2$  for some  $b_1, b_2, b_3 \in \mathbb{F}$  with  $b_3 \neq 0$ , because  $u, v$  are linearly independent. Let  $\psi(e_2 \otimes e_3) = c_1 u \otimes v_1 + c_2 u^2 + c_3 v_1^2$  for some scalars  $c_1, c_2, c_3 \in \mathbb{F}$ , and let

$$\beta = \begin{cases} 1 & \text{if } c_3 = 0, \\ c_3^{-1} b_3 & \text{if } c_3 \neq 0. \end{cases}$$

Then  $\beta \neq 0$  and  $\beta c_3 + b_3 \neq 0$ , and

$$\begin{aligned} \psi(\beta(e_1 + e_2) \otimes e_3 + (e_1 + e_2)^2) &= \beta u \otimes v_3 + (\beta c_3 + b_3) v_1^2 \\ &\quad + (\beta c_1 + b_1 + 1) u \otimes v_1 + (\beta c_2 + b_2 + \alpha) u^2 \end{aligned}$$

is of rank three, a contradiction. Then  $a_3 = 0$ . Since  $\psi(e_2^2), \psi(e_2 \otimes e_1), \psi(e_2 \otimes e_3)$  are linearly independent, it follows that  $\psi(e_2 \otimes e_3) = d_1 u \otimes v + d_2 u^2 + d_3 v^2$  for some  $d_1, d_2, d_3 \in \mathbb{F}$  with  $d_3 \neq 0$ . Note that  $\psi((e_1 + e_2) \otimes e_3) = u \otimes v_3 + d_1 u \otimes v + d_2 u^2 + d_3 v^2$  has rank bounded above by two implies  $v \in \langle u, v_3 \rangle$ . So  $\langle u^2, v^2, u \otimes v \rangle = \langle u^2, v_3^2, u \otimes v_3 \rangle$ . We now apply a similar argument as above, with  $v_1$  replaced by  $v_3$ , to obtain  $\psi(\beta'(e_1 + e_2) \otimes e_3 + (e_1 + e_2)^2)$  is of rank three for some  $\beta' \in \mathbb{F} \setminus \{0\}$ . This leads to a contradiction.

Accordingly,  $\psi(\mathcal{S}_2)$  is of Form (II). Then there exists a nonzero vector  $x \in \mathcal{M}_{m,1}(\mathbb{F})$  such that

$$(4.17) \quad \psi(e_2^2) = x \otimes y_2 \quad \text{and} \quad \psi(e_2 \otimes e_i) = x \otimes y_i \quad \text{for } i = 1, 3$$

for some linearly independent vectors  $y_1, y_2, y_3 \in \mathcal{M}_{m,1}(\mathbb{F})$ . We divide into two subcases:

*Case A:*  $\langle x \rangle = \langle u \rangle$ . It follows from (4.16) and (4.17), together with the linearity of  $\psi$ , that there exists a linear mapping  $\varphi : \mathcal{ST}_4(\mathbb{F}) \rightarrow \mathcal{M}_{m,1}(\mathbb{F})$  such that

$$\psi(A) = u \otimes \varphi(A) \quad \text{for all } A \in \mathcal{ST}_4(\mathbb{F}),$$

where  $\varphi(A) \neq 0$  for every nonzero bounded rank-two matrix  $A \in \mathcal{ST}_4(\mathbb{F})$ . So (4.1) holds true.

*Case B:*  $\langle x \rangle \neq \langle u \rangle$ . Note that  $x \otimes y_1 = \psi(e_1 \otimes e_2) = u \otimes v_2$  implies  $v_2 = \varsigma x$  and  $y_1 = \varsigma u$  for some nonzero scalar  $\varsigma \in \mathbb{F}$ . Thus,

$$(4.18) \quad \psi(e_1 \otimes e_2) = \varsigma u \otimes x.$$

By a similar argument as in (4.9), we show that  $\{u, x, v_3, v_4\}$  are linearly independent. Setting  $w_1 = u, w_2 = \varsigma x, w_3 = v_3$  and  $w_4 = v_4$ , we thus have  $\{w_1, w_2, w_3, w_4\}$  is linearly independent. In view of (4.16), (4.17) and (4.18), we have

$$(4.19) \quad \psi(e_1^2) = w_1 \otimes v_1 \quad \text{and} \quad \psi(e_1 \otimes e_i) = w_1 \otimes w_i, \quad i = 2, 3, 4,$$

and

$$(4.20) \quad \psi(e_2^2) = w_2 \otimes z_2 \quad \text{and} \quad \psi(e_2 \otimes e_3) = w_2 \otimes z_3,$$

where  $z_i = \varsigma^{-1}y_i$  for  $i = 2, 3$ . We claim that

$$(4.21) \quad v_1 \in \langle w_1, w_2 \rangle \quad \text{and} \quad z_2 \in \langle w_1, w_2 \rangle.$$

We will only verify  $v_1 \in \langle w_1, w_2 \rangle$  as the second statement can be proved similarly. Suppose, contrary to our claim, that  $v_1 \notin \langle w_1, w_2 \rangle$ . Since  $\text{rank } \psi(e_1^2 + \gamma e_2^2) \leq 2$  for all  $\gamma \in \mathbb{F}$ , it follows from (4.19), (4.20) and Lemma 4.2 that  $z_2 \in \langle w_1 \rangle$  or  $z_2 \in \langle v_1 \rangle$ . Since  $y_1, y_2$  are linearly independent and  $w_1 \in \langle y_1 \rangle$ , we conclude that  $z_2 = \lambda v_1$  for some nonzero  $\lambda \in \mathbb{F}$ . Consequently,  $\psi((e_1 + e_2)^2) = w_1 \otimes v_1 + w_1 \otimes w_2 + \lambda w_2 \otimes v_1$  is of rank three, a contradiction. Claim (4.21) is proved. By (4.19) and (4.20),

$$(4.22) \quad \psi(e_1^2) = \lambda_1 w_1^2 + \lambda_2 w_1 \otimes w_2 \quad \text{and} \quad \psi(e_2^2) = \lambda_3 w_2^2 + \lambda_4 w_1 \otimes w_2$$

for some scalars  $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{F}$  with  $\lambda_1, \lambda_3 \neq 0$ . Moreover, since  $\text{rank } \psi(e_1 \otimes e_3 + \gamma e_2 \otimes e_3) \leq 2$  for all  $\gamma \in \mathbb{F}$ , it follows from (4.19), (4.20) and Lemma 4.2 that  $z_3 \in \langle w_1 \rangle$  or  $z_3 \in \langle w_3 \rangle$ . Since  $y_1, y_3$  are linearly independent, we have  $z_3 = \xi w_3$  for some nonzero scalar  $\xi \in \mathbb{F}$ . By (4.20), we have

$$(4.23) \quad \psi(e_2 \otimes e_3) = \xi w_2 \otimes w_3.$$

In view of (4.19), (4.20), (4.22) and (4.23), we see that

$$\begin{aligned} \psi((\gamma e_1 + e_2)^2 + (\gamma e_1 + e_2) \otimes e_3) &= \gamma^2 \lambda_1 w_1^2 + \lambda_3 w_2^2 + (\gamma^2 \lambda_2 + \gamma + \lambda_4) w_1 \otimes w_2 \\ &\quad + \gamma w_1 \otimes w_3 + \xi w_2 \otimes w_3 \end{aligned}$$

has rank bounded above by two for all  $\gamma \in \mathbb{F}$ . It follows that

$$(4.24) \quad \begin{aligned} 0 &= \det \begin{bmatrix} \gamma & \gamma^2 \lambda_2 + \gamma + \lambda_4 & \gamma^2 \lambda_1 \\ \xi & \lambda_3 & \gamma^2 \lambda_2 + \gamma + \lambda_4 \\ 0 & \xi & \gamma \end{bmatrix} \\ &= -\gamma(2\lambda_2 \xi \gamma^2 - (\lambda_3 - \xi(2 - \xi \lambda_1))\gamma + 2\lambda_4 \xi) \end{aligned}$$

for all  $\gamma \in \mathbb{F}$ . Since  $\mathbb{F}$  is a field of characteristic not two, we conclude immediately from (4.24) that  $\lambda_3 = \xi(2 - \xi \lambda_1)$  with  $\xi \lambda_1 \neq 2$ , and  $\lambda_4 = -\lambda_2$ . Moreover, if  $|\mathbb{F}| \geq 4$ , then we can deduce from (4.24) that  $\lambda_2 = 0 = \lambda_4$ .

Let  $P \in \mathcal{M}_{m,4}(\mathbb{F})$  be the matrix defined by  $Pe_i = w_i$  for  $i = 1, 3, 4$ , and  $Pe_2 = \xi w_2$ . Clearly,  $P$  is of full rank. Denote  $\alpha = \xi^{-1}$ ,  $\beta = \lambda_1$  and  $\theta = \lambda_2 \xi^{-1}$ . Then  $\alpha, \beta \neq 0$  and  $2\alpha - \beta = \lambda_3 \xi^{-2} \neq 0$ . By (4.19), (4.20), (4.22), (4.23) and the linearity

of  $\psi$ , we obtain

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{12} & \alpha a_{13} + \theta(a_{14} - a_{23}) & \beta a_{14} \\ 0 & a_{22} & (2\alpha - \beta)a_{23} & \alpha a_{13} + \theta(a_{14} - a_{23}) \\ 0 & 0 & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_4(\mathbb{F})$ , where  $\theta$  is nonzero only if  $|\mathbb{F}| = 3$ . We are done.

*Case III:  $n = 3$ .* Let  $\mathcal{W} = e_1 \otimes \langle e_1, e_2, e_3 \rangle$ . Then  $\psi(\mathcal{W})$  is a 3-dimensional linear space of bounded rank-two matrices of  $\mathcal{SM}_m(\mathbb{F})$ . By Lemma 3.1,  $\psi(\mathcal{W})$  is either of Form (I) or Form (II) in Lemma 3.1. Then either

$$(4.25) \quad \psi(\mathcal{W}) \subseteq u \otimes \mathcal{M}_{m,1}(\mathbb{F})$$

for some nonzero vector  $u \in \mathcal{M}_{m,1}(\mathbb{F})$ ; or

$$(4.26) \quad \psi(\mathcal{W}) = \langle u^2, v^2, u \otimes v \rangle$$

for some linearly independent vectors  $u, v \in \mathcal{M}_{m,1}(\mathbb{F})$ . We argue in the following two cases:

*Case III-1:  $\psi(e_2^2) \in \psi(\mathcal{W})$ .* We consider the following two subcases.

If (4.25) holds, then  $\text{Im } \psi \subseteq u \otimes \mathcal{M}_{m,1}(\mathbb{F})$ . We thus obtain a linear mapping  $\varphi : \mathcal{ST}_3(\mathbb{F}) \rightarrow \mathcal{M}_{m,1}(\mathbb{F})$  such that

$$\psi(A) = u \otimes \varphi(A) \quad \text{for all } A \in \mathcal{ST}_3(\mathbb{F}),$$

where  $\varphi(A) \neq 0$  for every nonzero bounded rank-two matrix  $A \in \mathcal{ST}_3(\mathbb{F})$ . Hence, (4.1) holds.

If (4.26) holds, then  $\text{Im } \psi = \langle u^2, v^2, u \otimes v \rangle$ . So, for each  $A \in \mathcal{ST}_3(\mathbb{F})$ , there exists a unique ordered triple  $(\alpha_A, \beta_A, \gamma_A) \in \mathbb{F}^3$  such that  $\psi(A) = \alpha_A u^2 + \beta_A v^2 + \gamma_A u \otimes v$ . We define the linear mapping  $\phi : \mathcal{ST}_3(\mathbb{F}) \rightarrow \mathbb{F}^3$  such that

$$\phi(A) = (\alpha_A, \beta_A, \gamma_A) \quad \text{for all } A \in \mathcal{ST}_3(\mathbb{F}).$$

Note that  $\text{Im } \psi = \langle u^2, v^2, u \otimes v \rangle$  and  $\psi$  preserves nonzero bounded rank-two matrices implies  $\phi$  is surjective and  $\phi(A) \neq 0$  for every nonzero bounded rank-two matrix  $A \in \mathcal{ST}_3(\mathbb{F})$ . Let  $P \in \mathcal{M}_{m,2}(\mathbb{F})$  be the matrix defined by  $P e_1 = u$  and  $P e_2 = v$ . Then  $P$  is of full rank and

$$\psi(A) = P \begin{bmatrix} \phi(A)_3 & \phi(A)_1 \\ \phi(A)_2 & \phi(A)_3 \end{bmatrix} P^+ \quad \text{for all } A \in \mathcal{ST}_3(\mathbb{F}),$$

where  $\phi(A)_i$  denotes the  $i$ -th component of  $\phi(A) \in \mathbb{F}^3$ . We are done.

*Case III-2:*  $\psi(e_2^2) \notin \psi(\mathcal{W})$ . Let  $\mathcal{W}_1 = \langle e_1^2, e_2^2, e_1 \otimes e_2 \rangle$ . Note that  $\psi(\mathcal{W}_1)$  is a 3-dimensional linear space of bounded rank-two matrices of  $\mathcal{SM}_m(\mathbb{F})$ . By Lemma 3.1, we have either

$$(4.27) \quad \psi(\mathcal{W}_1) \subseteq x \otimes \mathcal{M}_{m,1}(\mathbb{F})$$

for some nonzero vector  $x \in \mathcal{M}_{m,1}(\mathbb{F})$ ; or

$$(4.28) \quad \psi(\mathcal{W}_1) = \langle x^2, y^2, x \otimes y \rangle$$

for some linearly independent vectors  $x, y \in \mathcal{M}_{m,1}(\mathbb{F})$ . We need to consider the following four subcases:

*Case III-2-A:* (4.25) and (4.27) hold. Since  $\psi(\mathcal{W}) \subseteq u \otimes \mathcal{M}_{m,1}(\mathbb{F})$  contains two linearly independent elements  $\psi(e_1^2) = x \otimes y_1$  and  $\psi(e_1 \otimes e_2) = x \otimes y_2$  for some  $y_1, y_2 \in \mathcal{M}_{m,1}(\mathbb{F})$ , it follows from Lemma 2.4 that  $\langle x \rangle = \langle u \rangle$ . Thus,  $\text{Im } \psi \subseteq u \otimes \mathcal{M}_{m,1}(\mathbb{F})$ , and hence, (4.1) holds true.

*Case III-2-B:* (4.26) and (4.28) hold. By (4.26) and (4.28), we see that

$$a_1 u^2 + a_2 v^2 + a_3 u \otimes v = \psi(e_1^2) = b_1 x^2 + b_2 y^2 + b_3 x \otimes y$$

is of rank one or rank two for some nonzero elements  $(a_i), (b_i) \in \mathbb{F}^3$ , and

$$c_1 u^2 + c_2 v^2 + c_3 u \otimes v = \psi(e_1 \otimes e_2) = d_1 x^2 + d_2 y^2 + d_3 x \otimes y$$

is of rank one or rank two for some nonzero elements  $(c_i), (d_i) \in \mathbb{F}^3$ . Therefore,

$$u \cdot (a_1 u^+ + a_3 v^+) + v \cdot (a_2 v^+ + a_3 u^+) = x \cdot (b_1 x^+ + b_3 y^+) + y \cdot (b_2 y^+ + b_3 x^+),$$

$$u \cdot (c_1 u^+ + c_3 v^+) + v \cdot (c_2 v^+ + c_3 u^+) = x \cdot (d_1 x^+ + d_3 y^+) + y \cdot (d_2 y^+ + d_3 x^+).$$

Since  $\psi(e_1^2), \psi(e_1 \otimes e_2)$  are linearly independent, it follows that, in each case, we obtain  $\langle u, v \rangle = \langle x, y \rangle$ . By Lemma 4.1 (a),  $\langle u^2, v^2, u \otimes v \rangle = \langle x^2, y^2, x \otimes y \rangle$ , and so  $\psi(e_2^2) \in \psi(\mathcal{W})$ , a contradiction.

*Case III-2-C:* (4.25) and (4.28) hold. Let  $\psi(e_1^2) = u \otimes z_1, \psi(e_1 \otimes e_2) = u \otimes z_2$  and  $\psi(e_1 \otimes e_3) = u \otimes z_3$  for some linearly independent vectors  $z_1, z_2, z_3 \in \mathcal{M}_{m,1}(\mathbb{F})$ . By (4.28), we get

$$u \otimes z_1 = a_1 x^2 + a_2 y^2 + a_3 x \otimes y,$$

$$u \otimes z_2 = b_1 x^2 + b_2 y^2 + b_3 x \otimes y$$

for some nonzero elements  $(a_i), (b_i) \in \mathbb{F}^3$ . Thus,

$$(4.29) \quad u \cdot z_1^+ + z_1 \cdot u^+ = x \cdot (a_1 x^+ + a_3 y^+) + y \cdot (a_2 y^+ + a_3 x^+),$$

$$(4.30) \quad u \cdot z_2^+ + z_2 \cdot u^+ = x \cdot (b_1x^+ + b_3y^+) + y \cdot (b_2y^+ + b_3x^+).$$

We consider the following four subcases:

*Subcase III-2-C-1:*  $\text{rank } \psi(e_1^2) = \text{rank } \psi(e_1 \otimes e_2) = 1$ . Then  $\langle z_1 \rangle = \langle u \rangle = \langle z_2 \rangle$ . This contradicts the fact that  $z_1, z_2$  are linearly independent.

*Subcase III-2-C-2:*  $\text{rank } \psi(e_1^2) = \text{rank } \psi(e_1 \otimes e_2) = 2$ . Then  $\{u, z_i\}$  is linearly independent for  $i = 1, 2$ . It follows from (4.29) and (4.30) that  $\langle u, z_1 \rangle = \langle x, y \rangle = \langle u, z_2 \rangle$ . Since  $\text{rank } \psi(e_1 \otimes e_2) = 2$  and  $\{z_1, z_2\}$  is linearly independent,  $z_2 = \mu_1 u + \mu_2 z_1$  for some nonzero scalars  $\mu_1, \mu_2 \in \mathbb{F}$ . We thus have  $\{z_1, u, z_3\}$  is linearly independent and  $\psi(e_1 \otimes e_2) = \eta_1 u^2 + \eta_2 u \otimes z_1$ , with  $\eta_1 = 2\mu_1$  and  $\eta_2 = \mu_2$  nonzero. Since  $\langle x^2, y^2, x \otimes y \rangle = \langle u^2, z_1^2, u \otimes z_1 \rangle$ , we have  $\psi(e_2^2) = \lambda_1 u^2 + \lambda_2 u \otimes z_1 + \lambda_3 z_1^2$  for some  $(\lambda_i) \in \mathbb{F}^3$  with  $\lambda_3 \neq 0$ . Let  $P \in \mathcal{M}_{3,m}(\mathbb{F})$  be the matrix defined by  $Pe_1 = u$ ,  $Pe_2 = z_1$  and  $Pe_3 = z_3$ . Then  $P$  is of full rank, and  $\psi(e_1^2) = P(e_1 \otimes e_2)P^+$ ,  $\psi(e_1 \otimes e_2) = P(\eta_1 e_1^2 + \eta_2 e_1 \otimes e_2)P^+$ ,  $\psi(e_1 \otimes e_3) = P(e_1 \otimes e_3)P^+$  and  $\psi(e_2^2) = P(\lambda_1 e_1^2 + \lambda_2 e_1 \otimes e_2 + \lambda_3 e_2^2)P^+$ . By the linearity of  $\psi$ , we obtain

$$\psi(A) = P \begin{bmatrix} a_{11} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{22} & \eta_1 a_{12} + \lambda_1 a_{22} \\ 0 & \lambda_3 a_{22} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{22} \\ 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$ . We are done.

*Subcase III-2-C-3:*  $\text{rank } \psi(e_1^2) = 1$  and  $\text{rank } \psi(e_1 \otimes e_2) = 2$ . Then  $\langle z_1 \rangle = \langle u \rangle$ , and so  $\psi(e_1^2) = \eta u^2$  for some nonzero scalar  $\eta \in \mathbb{F}$ . Note that  $\{u, z_2, z_3\}$  is linearly independent. By (4.30), we have  $\langle u, z_2 \rangle = \langle x, y \rangle$ , and so  $\langle u^2, z_2^2, u \otimes z_2 \rangle = \langle x^2, y^2, x \otimes y \rangle$  by Lemma 4.1 (a). Thus,  $\psi(e_2^2) = \lambda_1 u^2 + \lambda_2 u \otimes z_2 + \lambda_3 z_2^2$  for some  $(\lambda_i) \in \mathbb{F}^3$  with  $\lambda_3 \neq 0$ . Let  $P \in \mathcal{M}_{3,m}(\mathbb{F})$  be the matrix defined by  $Pe_1 = u$ ,  $Pe_2 = z_2$  and  $Pe_3 = z_3$ . Then  $P$  is of full rank and

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{12} + \lambda_2 a_{22} & \eta a_{13} + \lambda_1 a_{22} \\ 0 & \lambda_3 a_{22} & a_{12} + \lambda_2 a_{22} \\ 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$ . We are done.

*Subcase III-2-C-4:*  $\text{rank } \psi(e_1^2) = 2$  and  $\text{rank } \psi(e_1 \otimes e_2) = 1$ . Then  $\langle z_2 \rangle = \langle u \rangle$  and  $\psi(e_1 \otimes e_2) = \eta u^2$  for some nonzero scalar  $\eta \in \mathbb{F}$ . So  $\{z_1, u, z_3\}$  is linearly independent. By (4.29), we conclude that  $\langle u^2, z_1^2, u \otimes z_1 \rangle = \langle x^2, y^2, x \otimes y \rangle$ . Thus,  $\psi(e_2^2) = \lambda_1 u^2 + \lambda_2 u \otimes z_1 + \lambda_3 z_1^2$  for some  $(\lambda_i) \in \mathbb{F}^3$  with  $\lambda_3 \neq 0$ . Let  $P \in \mathcal{M}_{3,m}(\mathbb{F})$  be the matrix

defined by  $Pe_1 = u$ ,  $Pe_2 = z_1$  and  $Pe_3 = z_3$ . Then  $P$  is of full rank and

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{13} + \lambda_2 a_{22} & \eta a_{12} + \lambda_1 a_{22} \\ 0 & \lambda_3 a_{22} & a_{13} + \lambda_2 a_{22} \\ 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$ . We are done.

*Case III-2-D:* (4.26) and (4.27) hold. Let  $\tau : \mathcal{ST}_3(\mathbb{F}) \rightarrow \mathcal{ST}_3(\mathbb{F})$  be the bijective linear mapping defined by

$$\tau(A) = \begin{bmatrix} a_{22} & a_{12} & a_{13} \\ 0 & a_{11} & a_{12} \\ 0 & 0 & a_{22} \end{bmatrix} \quad \text{for all } A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F}).$$

It is easily seen that  $\tau$  is a bounded rank-two linear preserver such that  $\tau(\mathcal{W}) = \mathcal{W}_1$  and  $\tau(\mathcal{W}_1) = \mathcal{W}$ . It follows from (4.26) and (4.27) that

$$(\psi \circ \tau)(\mathcal{W}) = \psi(\mathcal{W}_1) \subseteq x \otimes \mathcal{M}_{m,1}(\mathbb{F}) \quad \text{and} \quad (\psi \circ \tau)(\mathcal{W}_1) = \psi(\mathcal{W}) = \langle u^2, v^2, u \otimes v \rangle.$$

We then infer by similar arguments as in Subcase III-2-C and conclude that  $\psi$  takes one of the following forms: there exists a full rank matrix  $P \in \mathcal{M}_{3,m}(\mathbb{F})$  such that

$$\psi(A) = P \begin{bmatrix} a_{22} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{11} & \eta_1 a_{12} + \lambda_1 a_{11} \\ 0 & \lambda_3 a_{11} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{11} \\ 0 & 0 & a_{22} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$ , where  $\lambda_1, \lambda_2, \lambda_3, \eta_1, \eta_2 \in \mathbb{F}$  with  $\lambda_3, \eta_1, \eta_2 \neq 0$ ; or

$$\psi(A) = P \begin{bmatrix} a_{22} & a_{12} + \lambda_2 a_{11} & \eta a_{13} + \lambda_1 a_{11} \\ 0 & \lambda_3 a_{11} & a_{12} + \lambda_2 a_{11} \\ 0 & 0 & a_{22} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$ , where  $\lambda_1, \lambda_2, \lambda_3, \eta \in \mathbb{F}$  with  $\lambda_3, \eta \neq 0$ ; or

$$\psi(A) = P \begin{bmatrix} a_{22} & a_{13} + \lambda_2 a_{11} & \eta a_{12} + \lambda_1 a_{11} \\ 0 & \lambda_3 a_{11} & a_{13} + \lambda_2 a_{11} \\ 0 & 0 & a_{22} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$ , where  $\lambda_1, \lambda_2, \lambda_3, \eta \in \mathbb{F}$  with  $\lambda_3, \eta \neq 0$ .  $\square$

By Theorem 4.3, Lemma 2.3 (a) and (b) (i), and Lemma 2.6, we obtain a classification of bounded rank-two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two.

**COROLLARY 4.4.** *Let  $\mathbb{F}$  be a field of characteristic not two and  $m, n$  be integers such that  $m, n \geq 3$ . Then  $\psi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{ST}_m(\mathbb{F})$  is a bounded rank-two linear preserver if and only if  $m \geq n$  and  $\psi$  is of one of the following forms:*

- (i) There exist a nonzero vector  $u \in \mathcal{U}_{p,m}$  and a linear mapping  $\varphi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{U}_{q,m}$ , with  $1 \leq p \leq m+1-q \leq m$ , such that

$$\psi(A) = u \otimes \varphi(A) \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

where  $\varphi(A) \neq 0$  for every nonzero bounded rank-two matrix  $A \in \mathcal{ST}_n(\mathbb{F})$ .

- (ii) There exist a full rank matrix  $P \in \mathcal{M}_{m,n}(\mathbb{F})$  and a nonzero  $\lambda \in \mathbb{F}$  such that

$$\psi(A) = \lambda PAP^+ \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

where  $Pe_i \in \mathcal{U}_{p_i,m} \setminus \mathcal{U}_{p_i-1,m}$  for  $i = 1, \dots, n$  such that  $1 \leq p_i \leq \frac{m+1}{2}$  for every  $1 \leq i \leq \frac{n+1}{2}$ , and  $p_i \leq m+1-p_j$  for every  $1 \leq i < j \leq n+1-i$ . In particular,  $P \in \mathcal{T}_n(\mathbb{F})$  when  $m = n$ .

- (iii) When  $n = 4$ , in addition to (i) and (ii),  $\psi$  also takes the form

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{12} & \alpha a_{13} + \theta(a_{14} - a_{23}) & \beta a_{14} \\ 0 & a_{22} & (2\alpha - \beta)a_{23} & \alpha a_{13} + \theta(a_{14} - a_{23}) \\ 0 & 0 & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_4(\mathbb{F})$ , where  $\alpha, \beta, \theta \in \mathbb{F}$  are scalars such that  $\alpha, \beta$  are nonzero with  $\beta \neq 2\alpha$ , and  $\theta$  is nonzero only if  $|\mathbb{F}| = 3$ , and  $P \in \mathcal{M}_{m,4}(\mathbb{F})$  is a full rank matrix in which  $Pe_i \in \mathcal{U}_{p_i,m}$  for  $1 \leq i \leq 4$  with  $1 \leq p_i \leq \frac{m+1}{2}$  for every  $1 \leq i \leq 2$ , and  $p_i \leq m+1-p_j$  for every  $1 \leq i < j \leq 5-i$ . In particular,  $P \in \mathcal{T}_4(\mathbb{F})$  when  $m = 4$ .

- (iv) When  $n = 3$ , in addition to (i) and (ii),  $\psi$  also takes one of the following forms:

- (a) There exist a surjective linear mapping  $\phi : \mathcal{ST}_3(\mathbb{F}) \rightarrow \mathbb{F}^3$  and a full rank matrix  $P \in \mathcal{M}_{m,2}(\mathbb{F})$  such that

$$\psi(A) = P \begin{bmatrix} \phi(A)_3 & \phi(A)_1 \\ \phi(A)_2 & \phi(A)_3 \end{bmatrix} P^+ \quad \text{for all } A \in \mathcal{ST}_3(\mathbb{F}),$$

where  $Pe_1, Pe_2 \in \mathcal{U}_{p,m}$  for some integer  $1 \leq p \leq \frac{m+1}{2}$ ,  $\phi(A)_i$  is the  $i$ -th component of  $\phi(A) \in \mathbb{F}^3$ , and  $\phi(A) \neq 0$  for every nonzero bounded rank-two matrix  $A \in \mathcal{ST}_3(\mathbb{F})$ .

- (b) There exist scalars  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$  with  $\lambda_3 \neq 0$  such that either

$$\psi(A) = P \begin{bmatrix} a_{pp} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} & \eta_1 a_{12} + \lambda_1 a_{qq} \\ 0 & \lambda_3 a_{qq} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} \\ 0 & 0 & a_{pp} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$ , where  $\eta_1, \eta_2 \in \mathbb{F}$  are nonzero and  $\{p, q\} = \{1, 2\}$ ; or

$$\psi(A) = P \begin{bmatrix} a_{pp} & a_{1s} + \lambda_2 a_{qq} & \eta_1 a_{1t} + \lambda_1 a_{qq} \\ 0 & \lambda_3 a_{qq} & a_{1s} + \lambda_2 a_{qq} \\ 0 & 0 & a_{pp} \end{bmatrix} P^+$$

for all  $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$ , where  $\eta \in \mathbb{F}$  is nonzero and  $\{p, q\} = \{s, t\} = \{1, 2\}$ . Here,  $P \in \mathcal{M}_{m,3}(\mathbb{F})$  is a full rank matrix such that  $Pe_1, Pe_2 \in \mathcal{U}_{p,m}$  with  $1 \leq p \leq \frac{m+1}{2}$  and  $Pe_3 \in \mathcal{U}_{q,m}$  with  $1 \leq q \leq m+1-p$ . In particular,  $P \in \mathcal{T}_3(\mathbb{F})$  when  $m = 3$ .

We end this section by giving an example of rank-one linear preserver / rank-one non-increasing linear mapping and some examples of rank-two non-increasing linear mappings on per-symmetric triangular matrices.

EXAMPLE 4.5. Let  $\mathbb{F}$  be a field and  $m, n$  be integers  $\geq 2$ . Let  $p := \lfloor \frac{n+1}{2} \rfloor$ , where  $\lfloor \cdot \rfloor$  is the floor function. Let  $\psi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{SM}_m(\mathbb{F})$  be the linear mapping defined by

$$\psi(A) = \lambda P \begin{bmatrix} \phi(A_1) & \varphi(A_2) \\ 0 & \phi(A_1)^+ \end{bmatrix} P^+ \quad \text{for every } A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1^+ \end{bmatrix} \in \mathcal{ST}_n(\mathbb{F})$$

with  $A_1 \in \mathcal{T}_{p,n-p}(\mathbb{F})$  and  $A_2 \in \mathcal{SM}_p(\mathbb{F})$ , where  $\lambda \in \mathbb{F} \setminus \{0\}$ ,  $P \in \mathcal{M}_{m,n}(\mathbb{F})$  is of full rank, and  $\phi : \mathcal{T}_{p,n-p}(\mathbb{F}) \rightarrow \mathcal{M}_{p,n-p}(\mathbb{F})$  and  $\varphi : \mathcal{SM}_p(\mathbb{F}) \rightarrow \mathcal{SM}_p(\mathbb{F})$  are linear mappings. Here  $\mathcal{T}_{p,n-p}(\mathbb{F}) = \mathcal{T}_p(\mathbb{F})$  when  $n-p = p$ , and

$$\mathcal{T}_{p,n-p}(\mathbb{F}) = \left\{ \begin{bmatrix} T \\ 0 \end{bmatrix} \in \mathcal{M}_{p,n-p}(\mathbb{F}) \mid T \in \mathcal{T}_{p-1}(\mathbb{F}) \right\} \quad \text{when } n-p = p-1.$$

It is easily verified that

- $\psi$  is a rank-one linear preserver whenever  $\varphi$  is a rank-one linear preserver on  $\mathcal{SM}_p(\mathbb{F})$ , and
- $\psi$  is rank-one non-increasing whenever  $\varphi$  is a rank-one non-increasing linear mapping on  $\mathcal{SM}_p(\mathbb{F})$ .

By the structural results of rank-one linear preservers and rank-one non-increasing linear mappings on symmetric matrices (see a complete result under a more general setting in [8], [12]), the structure of  $\psi$  can be established immediately.

EXAMPLE 4.6. Let  $\mathbb{F}$  be a field and  $m, n$  be integers  $\geq 2$ . Let  $\psi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{SM}_m(\mathbb{F})$  be the linear mapping defined by

$$\psi(A) = \lambda PAP^+$$

for every  $A \in \mathcal{ST}_n(\mathbb{F})$ , where  $\lambda \in \mathbb{F}$  and  $P \in \mathcal{M}_{m,n}(\mathbb{F})$ . Clearly,  $\psi$  is rank-two non-increasing.

EXAMPLE 4.7. Let  $\mathbb{F}$  be a field and  $n$  be an integer  $\geq 2$ . Let  $\psi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{SM}_n(\mathbb{F})$  be the linear mapping defined by

$$\psi(A) = \text{diag}(a_{11}, \dots, a_{nn})$$



for every  $A = (a_{ij}) \in \mathcal{ST}_n(\mathbb{F})$ . It is immediate to see that  $\text{rank } \psi(A) \leq 2$  whenever  $\text{rank } A \leq 2$ .

EXAMPLE 4.8. Let  $\mathbb{F}$  be a field and  $n$  be an integer  $\geq 2$ . Let  $\psi : \mathcal{ST}_n(\mathbb{F}) \rightarrow \mathcal{SM}_n(\mathbb{F})$  be the linear mapping defined by

$$\psi(A) = \begin{bmatrix} \lambda_1 A_{11} & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 A_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_2 A_{22} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_1 A_{11} \end{bmatrix}$$

for every  $A = (A_{ij}) \in \mathcal{ST}_n(\mathbb{F})$  with  $A_{ij} \in \mathcal{M}_{n_i, n_j}(\mathbb{F})$  for  $1 \leq i \leq j \leq k$ . Here  $\lambda_i \in \mathbb{F}$  with  $\lambda_{k+1-i} = \lambda_i$  for  $i = 1, \dots, k$ , and  $n_1 + \dots + n_k = n$  with  $n_{k+1-i} = n_i$  for  $i = 1, \dots, k$ . It is easily verified that is rank-two non-increasing.

EXAMPLE 4.9. Let  $\mathbb{F}$  be a field. We define the linear mapping  $\psi : \mathcal{ST}_5(\mathbb{F}) \rightarrow \mathcal{SM}_5(\mathbb{F})$  such that

$$\psi(A) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} & 0 \\ 0 & 0 & a_{33} & a_{23} & 0 \\ 0 & 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{12} & a_{11} \end{bmatrix}$$

for every  $A = (a_{ij}) \in \mathcal{ST}_5(\mathbb{F})$ . A direct verification shows that  $\psi$  satisfies  $\text{rank } \psi(A) \leq 2$  whenever  $\text{rank } A \leq 2$ .

EXAMPLE 4.10. Let  $\mathbb{F}$  be a field and  $\psi : \mathcal{ST}_5(\mathbb{F}) \rightarrow \mathcal{SM}_5(\mathbb{F})$  be the linear mapping defined by

$$\psi(A) = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & a_{23} & a_{33} & 0 & 0 \\ 0 & 0 & a_{23} & a_{22} & a_{12} \\ 0 & 0 & 0 & 0 & a_{11} \end{bmatrix}$$

for every  $A = (a_{ij}) \in \mathcal{ST}_5(\mathbb{F})$ . Then  $\psi$  satisfies  $\text{rank } \psi(A) \leq 2$  whenever  $\text{rank } A \leq 2$ . Nevertheless, we note that  $\psi$  is not rank-one non-increasing. For example,  $\psi(E_{23} + E_{24} + E_{33} + E_{34}) = E_{32} + E_{33} + E_{43}$  is of rank two.

Examples 4.6–4.10 demonstrate that the structure of rank-two non-increasing linear mappings on per-symmetric triangular matrices is complicated. This shows that condition (1.1) is a relevant assumption in our study.

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