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A NOTE ON THE CONVEXITY OF THE INDEFINITE JOINT NUMERICAL RANGE

HIROSHI NAKAZATO†, NATÁLIA BEBIANO‡, AND JOÃO DA PROVIDÊNCIA§

Abstract. This note investigates the convexity of the indefinite joint numerical range of a tuple of Hermitian matrices in the setting of Krein spaces. Its main result is a necessary and sufficient condition for convexity of this set. A new notion of “quasi-convexity” is introduced as a refinement of pseudo-convexity.

Key words. Krein space, Joint numerical range, Convexity.

AMS subject classifications. 46C20, 47A12.

1. Introduction. The concept of numerical range of linear operators on a Hilbert space was introduced by Toeplitz ([16]) and has been generalized in several directions. The theory of numerical ranges of operators on a Krein space has also been considered by some authors (e.g., see [2, 3, 5, 6, 12, 14] and the references therein).

An n-dimensional Krein space is a complex vector space $K$ with an indefinite inner product $\langle \cdot, \cdot \rangle$. In this space, we can choose a basis $\{\xi_1, \xi_2, \ldots, \xi_r, \xi_{r+1}, \ldots, \xi_n\}$ satisfying

$$\langle \xi_i, \xi_j \rangle = 0$$

for $1 \leq i < j \leq n$, $\langle \xi_i, \xi_i \rangle > 0$ for $1 \leq i \leq r$ and $\langle \xi_i, \xi_i \rangle < 0$ for $r + 1 \leq i \leq n$. If $r = n$, then the space $K$ is a Hilbert space. If $r = 0$, then the metric $\langle \cdot, \cdot \rangle$ introduces a Hilbert space structure in $K$. So we assume that $0 < r < n$. We identify the space $K$ with the vector space $\mathbb{C}^n$ via the isomorphism $x_1 \xi_1 + \cdots + x_n \xi_n \mapsto (x_1, \ldots, x_n)^T$. Assuming that $\langle \xi_i, \xi_j \rangle = \delta_{ij}$, the standard inner product $\langle \cdot, \cdot \rangle$ is given by

$$\langle (u_1, \ldots, u_n)^T, (v_1, \ldots, v_n)^T \rangle = u_1 v_1^* + \cdots + u_n v_n^*.$$ 

We set the real invertible matrix $J = \text{diag}(a_1, \ldots, a_n)$ with signature $(r, n-r)$. Then
the metric $[\cdot, \cdot]$ is associated with the inner product $\langle \cdot, \cdot \rangle$ by

$$[\xi, \eta] = \langle J\xi, \eta \rangle$$

for $\xi, \eta$ in $K = \mathbb{C}^n$.

Let $H_n$ denote the real vector space of $n \times n$ Hermitian matrices. Suppose $JH = (JH_1, \ldots, JH_k) \in H_n^k$. The $J$-joint numerical range $W_J(H)$ of $(H_1, \ldots, H_k)$, was introduced in [12] and is defined as

$$W_J(H_1, \ldots, H_k) = \{(\langle H_1x, x \rangle, \ldots, \langle H_kx, x \rangle) : x \in \mathbb{C}^n, \langle x, x \rangle \neq 0\} \subseteq \mathbb{R}^{k \times 1}.$$ 

Given a linear operator $T$ acting on $K$, the set

$$W_J(T) = \{\langle Tx, x \rangle : x \in K, \langle x, x \rangle \neq 0\},$$

is called the $J$-numerical range of $T$. Its boundary lies on an algebraic curve. The method to compute the equation of the boundary is given in [15]. The $J$-numerical range can be viewed as one of generalized numerical ranges (cf. [8, 11]). In view of the representation $T = T_1 + iT_2$, where $JT_1 = \frac{JT + T^*J}{2}$ and $JT_2 = \frac{JT - T^*J}{2i}$ are Hermitian matrices, $W_J(T)$ is the $J$-joint numerical range of $(T_1, T_2)$.

In the sequel, we also consider the sets

$$W^+_J(H) = W_J^+(H_1, \ldots, H_n) = \{\pm(\langle H_1x, x \rangle, \ldots, \langle H_kx, x \rangle) : x \in \mathbb{C}^n, \langle x, x \rangle = \pm1\} \subseteq \mathbb{R}^{k \times 1}.$$ 

When $J = I$, $W_J(H) = W_J^+(H)$ reduces to the usual joint numerical range

$$W(H) = \{x^*H_1x, \ldots, x^*H_kx : x \in \mathbb{C}^n, x^*x = 1\} \subseteq \mathbb{R}^{k \times 1},$$

which has attracted the attention of several researchers (e.g., see [4, 7] and references therein). This concept is useful in theoretical and applied subjects, in particular, control systems (e.g. see [9] and their references).

For $JH \in H_n^k$, the following properties can be easily checked.

(P1) $W_J(H) = W_J^+(H) \cup W_J^-(H)$.

(P2) $W_{(J^{-1})}^-(H) = W_J^-(H)$.

Having in mind property (P2), we can focus our study on $W_J^+(H)$ and translate the results on this set to $W_J^-(H)$ and $W_J(H)$. We observe that $W_J^+(H)$ need not be closed or bounded, however it is always connected. In fact, since $\{x \in \mathbb{C}^n : x^*Jx = 1\}$ is connected, the same happens with the set

$$V_J = \{xx^* : x \in \mathbb{C}^n, x^*Jx = 1\}.$$
Thus, $W^+_J(H)$ is the image of $V_J$ under the linear map $\phi_H : \mathcal{H}_n \to \mathbb{R}^{k \times 1}$ defined by

$$\phi_H(X) = (\text{tr} (JH_1X), \ldots, \text{tr} (JH_kX)), \ X \in \mathcal{H}_n,$$

where $\text{tr} Y$ denotes the trace of the matrix $Y$.

We recall that a subset $S$ in $\mathbb{R}^{k \times 1}$ has affine dimension $m$ if there exists $v \in S$ such that the linear span of $S - v$ has dimension $m$. Having in mind that $\text{rank}(\phi_H)$ equals the dimension of $\text{span}\{H_1, \ldots, H_k\}$ and $V_H \subseteq \mathcal{H}_2$ has real affine dimension 4, we conclude that $W^+_J(H)$ has affine dimension at most 4.

In studies of the numerical range and its generalizations, a useful technique is performing a convenient reduction to the $2 \times 2$ case. For instance, in results of convexity [1], this approach is useful, as well as when devising an effective procedure for generating the numerical range of an arbitrary $n$ by $n$ complex matrix reducing the general case to the bi-dimensional one. The following theorem was obtained in [13] and treats this case.

**Theorem 1.1.** Suppose $J, JH_1, \ldots, JH_k \in \mathcal{H}_2$, $H = (H_1, \ldots, H_k)$ and $\text{span}\{J, JH_1, \ldots, JH_k\}$ has dimension $m \leq 4$. Then $W^+_J(H)$ has affine dimension $m - 1$ and $W^+_J(H)$ is: a singleton if $m = 1$; a open or closed half line or a straight line if $m = 2$; a closed one-component hyperbola with interior, an open two-dimensional half plane, or a two-dimensional plane if $m = 3$; a closed one-component hyperboloid without interior if $m = 4$.

As a simple consequence of this result, we can conclude that for $J, JH_1, \ldots, JH_k \in \mathcal{H}_2$ and $H = (H_1, \ldots, H_k)$, the set $W^+_J(H)$ is convex if and only if $\text{span}\{J, JH_1, \ldots, JH_k\}$ has dimension less than 4.

Since the works of Toeplitz [15], it is known that the joint numerical range is not, in general, convex. There exists a vast literature on the (non)convexity of $W(H)$. The joint numerical range of a triple of $2 \times 2$ Hermitian matrices is typically not convex. The situation is analogous if $k \geq 4$ with the non convexity of $W(H_1, \ldots, H_k)$ (cf. [10]). As a consequence, there has been great emphasis on the study of sufficient conditions for the convexity of $W(H_1, \ldots, H_k)$ with $k > 3$. Interestingly, if $H_1, H_2, H_3 \in \mathcal{H}_n$ where $n > 2$, then $W(H_1, H_2, H_3)$ is convex (cf. [1] [3]). In view of this fact, it seems natural to ask whether $W_J(H_1, H_2, H_3)$ is convex.

Besides this introductory section, this note consists of two more sections. In Section 2, an example answers negatively the above question. In Section 3, a necessary and sufficient condition for convexity of the $J$-joint numerical range is obtained.
2. Preliminary examples of non-convexity. Let us consider

\[
J = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix},
H_1 = JH_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
H_2 = JH_2 = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
H_3 = JH_3 = \begin{bmatrix}
0 & i & 0 \\
-i & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

We recall that

\[
W_j^+(H_1, H_2, H_3) = \left\{ \left( \frac{\langle JH_1\xi, \xi \rangle}{\langle J\xi, \xi \rangle}, \frac{\langle JH_2\xi, \xi \rangle}{\langle J\xi, \xi \rangle}, \frac{\langle JH_3\xi, \xi \rangle}{\langle J\xi, \xi \rangle} \right) : \xi \in \mathbb{C}^3, \langle \xi, \xi \rangle = 1, (J\xi, \xi) > 0 \right\},
\]

where \( \langle \xi, \eta \rangle = \eta^* \xi \) for \( \xi, \eta \in \mathbb{C}^3 \). Since the equation

\[
\langle A(c\xi), (c\xi) \rangle = \langle A\xi, \xi \rangle
\]

holds for any linear operator \( A \), any unit vector \( \xi \) and any complex number \( c \) with modulus 1, we conclude that the set \( W(J, H_1, H_2, H_3) \) coincides with the following set

\[
\{(J\xi, \xi), (H_1\xi, \xi), (H_2\xi, \xi), (H_3\xi, \xi) : \xi \in \Omega\}
\]

for

\[
\Omega = \{ (\cos \phi \cos \psi \exp(i\theta), \cos \phi \sin \psi \exp(i\eta), \sin \phi)^T : 0 \leq \phi < \pi/4, 0 \leq \psi \leq 2\pi, 0 \leq \theta \leq 2\pi, 0 \leq \eta \leq 2\pi \}.
\]

For the vectors \( \xi \in \Omega \), the following relations hold

\[
\langle J\xi, \xi \rangle = \cos(2\phi),
\]

\[
\langle H_1\xi, \xi \rangle = \frac{\cos(2\phi) + 1}{2} \cos(2\psi),
\]

\[
\langle H_2\xi, \xi \rangle = \frac{\cos(2\phi) + 1}{2} \sin(2\psi) \cos(\theta - \eta),
\]

\[
\langle H_3\xi, \xi \rangle = \frac{\cos(2\phi) + 1}{2} \sin(2\psi) \sin(\theta - \eta).
\]

Thus, we may assume that \( 0 \leq \theta \leq 2\pi \) and \( \eta = 0 \) for the determination of the set \( W_j^+(H_1, H_2, H_3) \). Since

\[
\left\{ \frac{1}{2} \cos(2\phi) + \frac{1}{2} : 0 \leq \phi < \pi/4 \right\} = [1, +\infty),
\]

\[
\langle JH_1\xi, \xi \rangle = \langle J\xi, \xi \rangle, \langle JH_2\xi, \xi \rangle = \langle J\xi, \xi \rangle, \langle JH_3\xi, \xi \rangle = \langle J\xi, \xi \rangle.
\]

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we obtain

\[ W^+_J(H_1, H_2, H_3) = \{ (t \cos(2\psi), t \sin(2\psi) \cos \theta, t \sin(2\psi) \sin \theta) : 1 \leq t, 0 \leq \psi \leq 2\pi, 0 \leq \theta \leq 2\pi \} = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 1 \}. \]

Hence, the connected set \( W^+_J(H_1, H_2, H_3) \) is not convex, since the two points \((0, 0, 1), (0, 0, -1)\) belong to it but their midpoint \((0, 0, 0)\) does not belong to \( W^+_J(H_1, H_2, H_3) \).

By a similar process, we can prove that \( W^-_J(H_1, H_2, H_3) = \mathbb{R}^3 \), and so \( W_J(H_1, H_2, H_3) = \mathbb{R}^3 \). We can also construct a 3-tuple of Hermitian matrices \((H_1, H_2, H_3)\) on a 4-dimensional Krein space of type \((2, 2)\) for which \( W^+_J(H_1, H_2, H_3) \) is not convex.

Indeed, let us now consider \( \tilde{J} = I_2 \oplus -I_2 \) and the direct sums of the Pauli matrices with \( 2 \times 2 \) zero blocks,

\[
\tilde{H}_1 = \tilde{J} \tilde{H}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\tilde{H}_2 = \tilde{J} \tilde{H}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\tilde{H}_3 = \tilde{J} \tilde{H}_3 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Let

\[
\tilde{\xi} = \begin{bmatrix} \cos \theta \cos \phi \cos \psi e^{i\alpha} \\ \cos \theta \cos \phi \sin \psi e^{i\beta} \\ \cos \theta \sin \phi e^{i\gamma} \\ \sin \theta \end{bmatrix},
\]

for \( \theta, \phi, \psi, \alpha, \beta, \gamma \in [0, 2\pi] \). We find

\[
\tilde{\xi}^* \tilde{J} \tilde{\xi} = \frac{1}{4} (-2 + 2 \cos(2\phi) + 4 \cos(2\phi) \cos(2\theta) + 2 \cos(2\theta)),
\]

\[
\tilde{\xi}^* \tilde{H}_1 \tilde{\xi} = \frac{1}{4} \cos(2\phi)(1 + \cos(2\phi) + \cos(2\theta) + \cos(2\phi) \cos(2\theta)),
\]

\[
\tilde{\xi}^* \tilde{H}_2 \tilde{\xi} = \frac{1}{4} \cos(\alpha - \beta) \sin(2\psi)(1 + \cos(2\phi) + \cos(2\theta) + \cos(2\phi) \cos(2\theta)),
\]

\[
\tilde{\xi}^* \tilde{H}_3 \tilde{\xi} = \frac{1}{4} \sin(\beta - \alpha) \sin(2\psi)(1 + \cos(2\phi) + \cos(2\theta) + \cos(2\phi) \cos(2\theta)).
\]

For

\[
R(\theta, \phi) = \begin{vmatrix} 1 + \cos(2\phi) + \cos(2\theta) + \cos(2\phi) \cos(2\theta) \\ -2 + 2 \cos(2\phi) + 2 \cos(2\theta) + 4 \cos(2\phi) \cos(2\theta) \end{vmatrix},
\]
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and

\[ D(\theta, \phi) = (1 + \cos(2\phi) + \cos(2\theta) + \cos(2\phi) \cos(2\theta)), \]

we have

\[ \{ R(\theta, \phi) : \theta, \phi \in [0, 2\pi], D(\theta, \phi) < 0 \} = [0, +\infty[. \]

Thus, we get

\[ W^+_J(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3) = \left\{ (x, y, z) : x^2 + y^2 + z^2 \geq \frac{2}{3} \right\}, \quad W^-_J(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3) = \mathbb{R}^3. \]

3. Convexity. A linear operator \( H \) on \( \mathcal{K} \) is said to be \( J \)-Hermitian if it satisfies

\[ [H\xi, \eta] = [\xi, H\eta] \]

for \( \xi, \eta \in \mathcal{K} \).

Theorem 3.1. Assume that \( \mathcal{K} \) is a Krein space with dimension \( n \geq 3 \) and \( (H_1, H_2, H_3) \) is an ordered triple of \( J \)-Hermitian operators on \( \mathcal{K} \). If the joint numerical range \( W(J, JH_1, JH_2, JH_3) \) is convex, then:

1. The \( J \)-joint numerical ranges

\[ W^+_J(H_1, H_2, H_3) = \left\{ \left( \frac{[H_1\xi, \xi]}{[\xi, \xi]}, \frac{[H_2\xi, \xi]}{[\xi, \xi]}, \frac{[H_3\xi, \xi]}{[\xi, \xi]} \right) : \xi \in \mathcal{K}, [\xi, \xi] > 0 \right\}, \]

\[ W^-_J(H_1, H_2, H_3) = \left\{ \left( \frac{[H_1\xi, \xi]}{[\xi, \xi]}, \frac{[H_2\xi, \xi]}{[\xi, \xi]}, \frac{[H_3\xi, \xi]}{[\xi, \xi]} \right) : \xi \in \mathcal{K}, [\xi, \xi] < 0 \right\}, \]

are convex.

2. For \((x_1, y_1, z_1) \in W^+_J(H_1, H_2, H_3)\) and \((x_2, y_2, z_2) \in W^-_J(H_1, H_2, H_3)\), we have

\[ \{ \lambda(x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2) : 1 \leq \lambda \} \subset W^+_J(H_1, H_2, H_3), \]

\[ \{ \lambda(x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2) : \lambda \leq 0 \} \subset W^-_J(H_1, H_2, H_3). \]
Proof. (1) We consider the Hermitian matrices $K_j$ defined by

$$K_j = JH_j$$

for $j = 1, 2, 3$. We assume that $W(J, K_1, K_2, K_3)$ is convex. Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in W^+_J(H_1, H_2, H_3)$ and $0 < \lambda < 1$. There exist $\xi, \eta \in \mathbb{C}^n$ such that $\xi^*\xi = 1$, $\eta^*\eta = 1$ and

$$(t, X_1, Y_1, Z_1) = (\xi^*J\xi, \xi^*K_1\xi, \xi^*K_2\xi, \xi^*K_3\xi),$$

$$(s, X_2, Y_2, Z_2) = (\eta^*J\eta, \eta^*K_1\eta, \eta^*K_2\eta, \eta^*K_3\eta),$$

with

$$(x_1, y_1, z_1) = \left( \frac{X_1}{t}, \frac{Y_1}{t}, \frac{Z_1}{t} \right), \quad (x_2, y_2, z_2) = \left( \frac{X_2}{s}, \frac{Y_2}{s}, \frac{Z_2}{s} \right).$$

We claim that $(x_3, y_3, z_3) = \lambda(x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2) \in W^+_J(H_1, H_2, H_3)$. For

$$\mu = \frac{\lambda s}{(1 - \lambda)t + \lambda s},$$

which satisfies $0 < \mu < 1$, let consider $(u, X_3, Y_3, Z_3) = \mu(t, X_1, Y_1, Z_1) + (1 - \mu)(s, X_2, Y_2, Z_2)$, which is obviously in $W(J, K_1, K_2, K_3)$. It follows that

$$\left( \frac{1}{u}, \frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u} \right) = \frac{\mu t}{\mu t + (1 - \mu)s}(1, x_1, y_1, z_1) + \frac{(1 - \mu)s}{\mu t + (1 - \mu)s}(1, x_2, y_2, z_2),$$

or, since $\lambda = \frac{\mu t}{\mu t + (1 - \mu)s}$,

$$\left( \frac{1}{u}, \frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u} \right) = \lambda(1, x_1, y_1, z_1) + (1 - \lambda)(1, x_2, y_2, z_2).$$

Thus,

$$(x_3, y_3, z_3) = \left( \frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u} \right).$$

Clearly, there exists $\zeta \in \mathbb{C}^n$ such that $\zeta^*\zeta = 1$ and

$$(u, X_3, Y_3, Z_3) = (\zeta^*J\zeta, \zeta^*K_1\zeta, \zeta^*K_2\zeta, \zeta^*K_3\zeta),$$

which proves the claim. Thus, the convexity of $W^+_J(H_1, H_2, H_3)$ is proved under the assumption that $W(J, K_1, K_2, K_3)$ is convex. Similarly, if $W(J, K_1, K_2, K_3)$ is convex, then $W^-_J(H_1, H_2, H_3)$ is also convex.
(2) Let \((x_1, y_1, z_1) \in W_j^+(H_1, H_2, H_3)\), \((x_2, y_2, z_2) \in W_j^+(H_1, H_2, H_3)\) and \(1 \leq \lambda < +\infty\). There exist \(\xi, \eta \in \mathbb{C}^n\) such that \(\xi^*\xi = 1\), \(\eta^*\eta = 1\) and

\[
(t, X_1, Y_1, Z_1) = (\xi^* J\xi, \xi^* K_1\xi, \xi^* K_2\xi, \xi^* K_3\xi),
\]

\[
(s, X_2, Y_2, Z_2) = (\eta^* J\eta, \eta^* K_1\eta, \eta^* K_2\eta, \eta^* K_3\eta),
\]

with

\[
(x_1, y_1, z_1) = \left(\frac{X_1}{t}, \frac{Y_1}{t}, \frac{Z_1}{t}\right), \quad (x_2, y_2, z_2) = \left(\frac{X_2}{s}, \frac{Y_2}{s}, \frac{Z_2}{s}\right).
\]

Notice that \(t > 0\), \(s < 0\). We claim that \((x_3, y_3, z_3) = \lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2) \in W_j^+(H_1, H_2, H_3)\). For

\[
\mu = \frac{\lambda^s}{(1-\lambda)t + \lambda s},
\]

which satisfies \(0 < \mu < 1\), let consider \((u, X_3, Y_3, Z_3) = \mu(t, X_1, Y_1, Z_1) + (1 - \mu)(s, X_2, Y_2, Z_2)\), which is in \(W(J, K_1, K_2, K_3)\) by the assumption of the convexity of this set. It follows that

\[
\left(1, \frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u}\right) = \frac{\mu t}{\mu t + (1-\mu)s}(1, x_1, y_1, z_1) + \frac{(1-\mu)s}{\mu t + (1-\mu)s}(1, x_2, y_2, z_2),
\]

or, since \(\lambda = \frac{\mu t}{\mu t + (1-\mu)s}\),

\[
\left(1, \frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u}\right) = \lambda(1, x_1, y_1, z_1) + (1-\lambda)(1, x_2, y_2, z_2).
\]

Thus,

\[
(x_3, y_3, z_3) = \left(\frac{X_3}{u}, \frac{Y_3}{u}, \frac{Z_3}{u}\right).
\]

Clearly, there exists \(\zeta \in \mathbb{C}^n\) such that \(\zeta^*\zeta = 1\) and

\[
(u, X_3, Y_3, Z_3) = (\zeta^* J\zeta, \zeta^* K_1\zeta, \zeta^* K_2\zeta, \zeta^* K_3\zeta),
\]

which proves the claim. We have shown that \(\{\lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2) : 1 \leq \lambda \leq \lambda\} \subset W_j^+(H_1, H_2, H_3)\). Similarly we can show that \(\{\lambda(x_1, y_1, z_1) + (1-\lambda)(x_2, y_2, z_2) : \lambda \leq 0\} \subset W_j^+(H_1, H_2, H_3)\).

**Definition 3.2.** The set \(W_J(H_1, H_2, H_3)\) is said to be quasi-convex if the following conditions hold together:
(1) If \((x_1, y_1, z_1), (x_2, y_2, z_2) \in W_J^+(H_1, H_2, H_3)\), then
\[
\{\lambda(x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2) : 0 \leq \lambda \leq 1\} \subset W_J^+(H_1, H_2, H_3);
\]

(2) If \((x_1, y_1, z_1), (x_2, y_2, z_2) \in W_J^-(H_1, H_2, H_3)\), then
\[
\{\lambda(x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2) : 0 \leq \lambda \leq 1\} \subset W_J^-(H_1, H_2, H_3);
\]

(3) If \((x_1, y_1, z_1) \in W_J^+(H_1, H_2, H_3)\) and \((x_2, y_2, z_2) \in W_J^-(H_1, H_2, H_3)\), then
\[
\{\lambda(x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2) : 1 \leq \lambda \leq 0\} \subset W_J^+(H_1, H_2, H_3),
\]
and
\[
\{\lambda(x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2) : \lambda \leq 0\} \subset W_J^-(H_1, H_2, H_3).
\]

Next we show that the converse of Theorem 3.1 holds.

**Theorem 3.3.** Assume that \(K\) is a Krein space with dimension \(n \geq 3\) and \((H_1, H_2, H_3)\) is an ordered triple of \(J\)-Hermitian operators on \(K\). If the \(J\)-joint numerical range \(W_J(H_1, H_2, H_3)\) is quasi-convex, then the joint numerical range \(W(J, JH_1, JH_2, JH_3)\) is convex.

**Proof.** Suppose that \(W_J(H_1, H_2, H_3)\) is quasi-convex. Assume that \(s(1, x_1, y_1, z_1), t(1, x_2, y_2, z_2) \in W(J, JH_1, JH_2, JH_3)\), \(s, t \neq 0\). First consider the case \(s, t > 0\), so that consider \((x_1, y_1, z_1), (x_2, y_2, z_2) \in W_J^+(H_1, H_2, H_3)\). Then there exist \(\xi, \eta \in \mathbb{C}^n\) such that \(\xi^*\xi = 1, \eta^*\eta = 1\) satisfying
\[
\begin{align*}
&\xi^*J\xi, \xi^*K_1\xi, \xi^*K_2\xi, \xi^*K_3\xi, \\
&\eta^*J\eta, \eta^*K_1\eta, \eta^*K_2\eta, \eta^*K_3\eta,
\end{align*}
\]
For \((x_3, y_3, z_3) = \lambda(x_1, y_1, z_1) + (1 - \lambda)(x_2, y_2, z_2)\), let \(\zeta \in \mathbb{C}^n\) be such that \(\zeta^*\zeta = 1\) and satisfy the following relation
\[
\begin{align*}
u(1, x_3, y_3, z_3) &= (\zeta^*J\zeta, \zeta^*K_1\zeta, \zeta^*K_2\zeta, \zeta^*K_3\zeta).
\end{align*}
\]
Clearly,
\[
u(1, x_3, y_3, z_3) = \mu s(1, x_1, y_1, z_1) + (1 - \mu) t(1, x_2, y_2, z_2),
\]
for \(\mu = \lambda t/(\lambda t + (1 - \lambda)s)\).
The cases \( s, t < 0 \) and \( s < 0, t > 0 \) are treated similarly except for the case 
\[ \lambda s + (1 - \lambda) t = 0. \]
Since \( W(J, JH_1, JH_2, JH_3) \) is compact, the result follows by using a limiting process.

Let \( (K_0, K_1, K_2, K_3) \) be a 4-tuple of \( 3 \times 3 \) Hermitian matrices. The following question arises in a natural way: Is the joint numerical range \( W(K_0, K_1, K_2, K_3) \) convex? Following [10], we can provide an example of non convexity. Let

\[
K_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
K_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Then \( W(K_0, K_1, K_2, K_3) \) is expressed as

\[
W(K_0, K_1, K_2, K_3) = \{ (t, (t+1)/2x, (t+1)/2y, (1+t)/2z) : -1 \leq t \leq 1, \ x^2 + y^2 + z^2 = 1 \},
\]

and so \( W(K_0, K_1, K_2, K_3) \) lies on the hyper surface

\[
x^2 + y^2 + z^2 - \frac{(t + 1)^2}{4} = 0.
\]

Gutkin, Jonckheere and Karow’s [10] Theorems 5.1 and 5.5 provide a sufficient condition for the joint range \( W(K_0, K_1, K_2, K_3) \) to be convex. We consider the characteristic equation

\[
det(\lambda I_3 - (tK_0 + xK_1 + yK_2 + zK_3)) = 0
\]

for every point \( (t, x, y, z) \in \mathbb{R}^4 \) on the unit sphere \( t^2 + x^2 + y^2 + z^2 = 1 \). If the eigenvalues \( \lambda_1(t, x, y, z), \lambda_2(t, x, y, z), \lambda_3(t, x, y, z) \) are simple for every point \( (t, x, y, z) \in \mathbb{S}^3 \), then the additional condition concerning the eigenspace of the first eigenvalue of \( tK_0 + xK_1 + yK_2 + zK_3 \) in Theorem 5.1 in [10] is automatically satisfied, and so \( W(K_0, K_1, K_2, K_3) \) is convex.

We shall provide an example of a 4-tuple of Hermitian matrices satisfying this condition. Let

\[
K_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -1 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}.
\]
A Note on the Convexity of the Indefinite Joint Numerical Range

\[
K_2 = \begin{bmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & i/\sqrt{2} & 0 \\ -i/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & -i/\sqrt{2} & 0 \end{bmatrix}.
\]

Then we have

\[
\det(\lambda I_3 - (tK_0 + xK_1 + yK_2 + zK_3)) = \lambda(\lambda^2 - (t^2 + x^2 + y^2 + z^2)).
\]

Thus, the eigenvalues of \( tK_0 + xK_1 + yK_2 + zK_3 \) are \( \{1, 0, -1\} \) for every point \((t, x, y, z)\) of the unit sphere \( S^3 \). We observe that the matrices in this example belong to the basis of the real vector space of \( 3 \times 3 \) Hermitian traceless matrices known as Gell-Mann matrices, which play an important role in quark physics. We have many examples of convex numerical ranges \( W(JH_0, JH_1, JH_2, JH_3) \) by a small perturbation of Gell-Mann matrices.

REFERENCES


